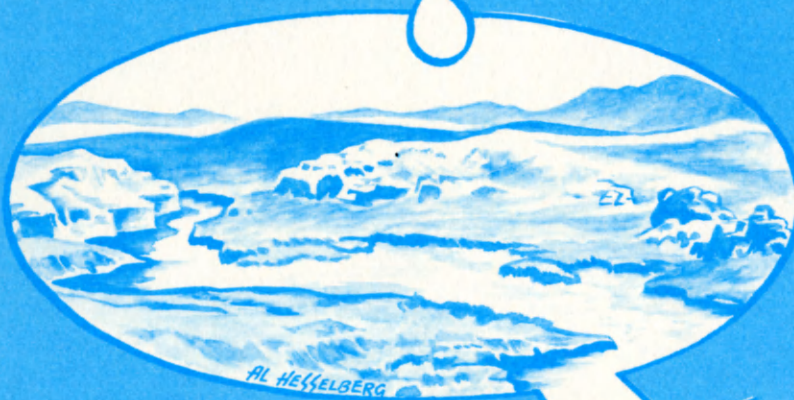


Interactive Algorithm for Multiple Objective Decision Making



by
David Monarchi

Technical Reports on
Hydrology and Water Resources

The University of Arizona
Tucson, Arizona 85721



AN INTERACTIVE ALGORITHM
FOR
MULTIOBJECTIVE DECISION MAKING

by

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PREFACE

This report constitutes the doctoral dissertation of the same title completed by the author in May, 1972 and accepted by the College of Business and Public Administration.

The report is unusual. Neither major nor minor area were officially in the resource field but in the above College whose support is gratefully acknowledged, especially, for computer time. The report is one result of a continuing and informal community of interest between faculty and students on this campus in an area that we choose to call Natural Resources Systems. Competence in aspects of this subject may be found in many Colleges on campus.

The methodology seems to be ideally suited to water and other natural resource problems. It is truly multiobjective and allows no explicit weighting. It is quite compatible and complementary to other methods developed here such as Bayesian decision theory (Technical Report #2), collective utility (Technical Report #5), and cost-effectiveness (by example in a series of papers developed under the leadership of L. Duckstein).

The case study is synthetic but realistically illustrates the methodology. It was chosen from reports, published in the Congressional Record, on the Planning, Programming and Budgeting System (PPBS).

This report series constitutes an effort to communicate to practitioners and researchers the complete research results, including economic foundations and detailed theoretical development that cannot be reproduced in professional journals. These reports are not intended to serve as

a substitute for the review and referee process exerted by the scientific and professional community in these journals. The author, of course, is solely responsible for the validity of the statements contained herein. A complete list of currently-available reports may be found in the back of this report.

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ABSTRACT

This research develops an algorithm for solving a class of multiple objective decision problems. These problems are characterized by continuous policy variables, nonlinear constraints, and nonlinear criterion functions.

Our underlying philosophy is that of the Gestalt psychologists--we cannot separate the problem and its solution from the environment in which the problem is placed. The decision maker is necessarily a part of this environment, thus implying that he, as an individual, must be part of the solution of the problem. Another central assumption in this research is that there is not an "optimal" answer to the problem, only "satisfactory" solutions. The reasons for this are based partly on the insensitivities of the body to minute changes and to the insensitivity of our preferences within certain ranges of acceptance. In addition, we assume that the individual is capable of solving decision situations involving a maximum of about 10 goals and that he operates upon them in some sort of serial manner as he searches for a satisfactory alternative. The serial manner is a reflection of his current ranking of the goals.

Based on these assumptions we have developed a cyclical interactive algorithm in which the decision maker guides a search mechanism in attempting to find a satisfactory alternative. Each cycle in the search consists of an optimization phase and an evaluation

phase, after which the decision maker can define a new direction of search or terminate the algorithm.

The optimization phase is based on a linearization technique which has been quite effective in terms of the problems we have attempted to solve. It is capable of solving general nonlinear programming problems with a large number of nonlinear constraints. Although the constraint set must be convex in order to guarantee the location of a global optimum, we can use the method on concave sets recognizing that we may find only a local optimum.

An extensive synthetic case study of a water pollution decision problem with 6 conflicting goals is provided to demonstrate the feasibility of the algorithm.

Finally, the limitations of the research are discussed. We tentatively conclude that we have developed a method applicable to our research problem and that the method can be applied to "real world" decision situations.

CHAPTER 1

INTRODUCTION

Decision making is basically choosing among alternatives. It differs from instinct in that it takes place on the conscious level and so is a characteristically human trait. Decision making also differs from habit, which can be viewed as a pattern of behavior based on past decisions. Habit is a choice that has moved from the conscious to the unconscious level of behavior (Katona 1953; Bayton 1958).

The decision process involves a model or logic structure which, by simplifying reality, enables the decision maker (DM) to impose some sort of order upon the variables under consideration. The DM requires information concerning the alternative courses of action available and evaluates their respective results. The termination of the process--a decision--is the selection of a particular alternative.

If comparison of alternatives generates a subjective measure of the worth of each, then choice is simply the selection of the alternative with the greatest value. Thus comparison reflects the value structure of the individual. This value structure, or preference system, is a result of the total environmental information input during his past and so the structure varies from individual to individual.

Most decision situations involve the same sequence of steps: recognition, development, comparison, and choice. The recognition of a decision situation and the development of a set of available alternative actions evolve from perceptual orientations which are a function of the amount of information available from past experiences and/or new data. Comparison requires the development of (at least) a partial ordering of the alternatives. Comparison can lead to a reduction of the number of alternatives to be considered in the future. Choice necessarily partitions the alternatives into two sets: the one chosen and the rest. In this sense, even previously incomparable alternatives will be compared. As a brief example of this last point, consider the woman who insists that she cannot decide which of two dresses to wear to a party. Each has individual advantages and the two cannot be compared. However, when it is time to leave -- ^{at}2 hours--she will have made a decision. The "incomparable" have been compared, unless she chooses a third dress, and a choice for that situation has been established. The situation may or may not be repeated in the future depending upon which elements of the situation the individual considers important. There are some situations, such as the choice of a grocery store, in which the decision becomes automatic--a habit. There are other situations, such as choosing a name for a baby, which are unique for each occurrence. Note however, that information may shorten the decision process, even for unique events. For example, we might consider a large number of names for

the first child and then use part of this stored information in choosing a name for a second child.

Perhaps we can clarify our point by observing that, in one interpretation, the statement "you can't step into the same river twice" is true; but in an alternate interpretation, it's false. The answer depends upon your point of view, your choice of the important elements in the situation.

Optimization theory has entered into the decision making process with regard to the development and selection of alternatives. Certain implicit assumptions are always in effect whenever an "objective" technique such as linear programming is employed. First, it is assumed in classical optimization theory that the criterion of desirability is adequately expressed by some objective measure, such as cost or profits, corresponding to a dimension of worth. This implies that a movement from the optimum as measured objectively results in a change in the desirability of the choice. Second, the development of an optimum according to some objective measure assumes that the total information content of the environment relative to this decision situation is contained in that measure. Consequently, when a decision situation is optimized and the result taken as the best available alternative, the context in which "best" is applied should meet these assumptions.

However, preferences may not be sensitive to minute changes in the alternatives; and the information used in making choices may be greater than that coded into an objective algorithm. So there is

a need for research which will develop techniques able to include the preferences of the individual and the information flow from the environment to him.

This research assumes that a decision problem or situation exists and has been recognized as such. It focuses upon the development of alternatives, their comparison, and the choice among them. We will define an algorithm as a sequential procedure for achieving an objective. Although this definition is not mathematically precise it does agree in content with that given by Markov (1954). Markov goes on to define "algorithm" with great precision and rigor but we feel there is little to be gained by introducing such formalisms here. The interested reader should refer to Markov (1954, Chapter II) for a detailed presentation. The purpose of this research is to develop an algorithm which is applicable to a multiple goal decision problem in which the individual goals are related to nonlinear criterion functions subject to nonlinear constraints. In order to accomplish this, we will employ the DM himself in a dynamic manner to define the search for a "best possible alternative." Strictly speaking, we will not be developing a method of solving the decision problem; the algorithm will generate information during a restricted search for the available alternatives so that the DM can select the "best" one.

To complete the introduction, we will briefly summarize the following chapters. Chapter 2 develops certain psychological concepts and ideas which are fundamental to our interpretation of choice. In particular, the inherently subjective nature of preferences is explored.

This leads to a definition of a solution to the decision problem which is based on finding a satisfactory alternative (satisfactum) rather than the best alternative (optimum).

Chapter 3 provides a formal statement of the research. With that statement as a guide Chapter 4 develops the overall scheme upon which our algorithm is constructed. In this chapter we review the literature in the area of multiobjective decision making. From this we select a particular orientation and examine it with respect to the concepts discussed in Chapter 2 and such psychological research findings as are appropriate to the subject. The notation for our algorithm is stated and then the algorithm itself is developed.

The problem we are considering involves repeated nonlinear optimizations, so it is important to this research to select a proper optimization technique. In Chapter 5 we briefly review and evaluate alternative methods. Then the method actually used as part of the algorithm is examined and its implementation upon the computer is discussed.

Chapter 6 contains 2 examples. The first is simple and allows us to thoroughly examine the algorithm at work. The second is more complex but also more realistic. It is presented to demonstrate the practicability of the algorithm in a real world situation.

The final chapter contains a discussion of the research limitations. Certain conclusions are drawn and possible avenues of future study suggested.

CHAPTER 2

CONCEPTS

This chapter reviews certain concepts central to this research. We begin by discussing Gestalt psychology in general because it has influenced the entire development of our work. Following this, goals and aspiration levels are discussed; criterion functions are defined and differentiated from preference functions; and optimal versus satisfactory choice-making is examined. Finally, the key points of the chapter are summarized.

Gestalt Psychology

The Gestalt school of psychology is based on the premise that the context of a situation is an essential part of that situation (see, for example, Scheerer 1967 or Köhler 1947). More concisely, the whole is different from the sum of the parts. Their philosophy is especially relevant to the question of perception: the perception of a situation. Evaluation (i.e., of alternatives) is dependent upon perception and so it too is situation dependent.

We can contrast the contextual dependency of perception with the (relative) contextual independency of physical measurements. For example, water may feel "hot" at one time and "cold" at another even though its measured temperature is the same in both instances.

Considering some further examples, a musical composition is somehow more than a simple enumeration of the notes to be played; it is the relationship of those notes to each other and also our interpretation of those relationships. We speak of words used inappropriately in writing because they do not fit with their surroundings. A quotation taken out of context may take on new meanings unintended by the author. Preference for a high powered automobile may depend on whether it is a station wagon or a sports car; and our preferences between these cars are influenced by our life style, marital status, etc.

So perception in the Gestalt framework is based on the overall interaction of the elements in the situation together with the larger context (the environment) in which the situation is placed. The context of the situation thus consists of the space-time framework which defines the situation. As we noted earlier, the specific elements of that context which the individual considers important can vary from situation to situation and individual to individual. In our research we assume that the DM's perception of a set of numbers indicative of his attainment of a corresponding set of goals is related both to that set of numbers as a whole and to the context in which the decision problem itself is located. In comparison, the stimulus-response school of psychology (see for example, Thorndike 1898) states that for a given stimulus or signal from the environment, there is a fixed response which is independent of the total context of the situation. The actual "fixedness" of the

response is dependent upon the number of times the subject has been conditioned or exposed to that stimulus.

Goals

Berelson and Steiner (1964, pp. 239-240) define motivation as ". . . all those inner striving conditions variously described as wishes, desires, needs, drives and the like . . ." and a goal as the ". . . objective, condition, or activity toward which the motive is directed; in short, that which will satisfy or reduce the striving."

In this research, we restrict our attention to decisions made to achieve explicit goals or objectives. Individuals with a goal they desire to attain are in a state of psychological imbalance or uncertainty which they attempt to correct through the manipulation of external variables called "decision variables." The manipulation is typically in the form of allocations of scarce resources, although the manipulation could also be of a system or organization. Our work is restricted to those decision situations which call for an allocation of resources. A specific allocation for all variables is called a policy vector. The individual attempts to choose some vector which will result in the attainment of his objectives.

Multiple Objectives

Do we, then, attempt to satisfy one goal and afterwards proceed to the next? Or are our actions influenced by several goals at the same time? Berelson and Steiner (1964, pp. 266-267) regard

multiple goal behavior as the more usual:

. . . Behavior is not normally under the control of one isolated motive at a time; several, often inconsistent, motives are normally in play. Beyond that, internal and external barriers stand between people and their goals even when the goals are perfectly clear and consistent. Thus, some of the most important implications and consequences of motives lie not in their direct satisfaction but rather in the adjustments and resolutions required when direct satisfaction is impossible."

Many other authors indicate that most decision problems involve multiple objectives. For example, see Ackoff and Sasieni (1968, p. 430), Barnard (1958), Dalkey (1969, p. 74), Hillier and Liberman (1967, pp. 13-14), Kimble and Garnezy (1963, p. 199 and p. 485) and March and Simon (1958, p. 118). Appendix A contains a list of additional authors who hold this opinion.

The preceding quotation is also of interest because of the adjusting process mentioned as a means of solving such conflicting situations. We will examine this further in the next section.

Although we have postulated the existence of multiple goals, we have not, as yet, put a specific number on this concept. Can we supply a number, i.e., "5," and say that an individual cannot assimilate more than 5 goals per decision situation? Unfortunately, no. The exhaustive literature review by Johnsen (1968) indicates that there certainly is a limit (which seems to be about 7 in a business environment) but the limit is largely determined by the individual and the situation. The amount of information that the individual must process relative to each goal also seems to affect this limit. Johnsen (1968, pp. 36-37 and pp. 364-365) states that an individual

in an "average situation" can coordinate about 5 goals, with 10 goals an apparent maximum, and can integrate about 7 units of information for each goal. It is difficult to define a "unit of information" in a perceptual sense because it can vary with the goal, the situation, and the individual. We can loosely think of it as an attribute of the goal which the individual considers important in judging whether or not he has attained the goal. Additionally, there seems to be a limit of about 50 on the product of the number of units and the number of goals (i.e., 7 goals with 7 units of information each). Multiple goal situations can consist of goals which are themselves multiple or vector-valued. In this research, we will focus upon multiple goal problems in which the attainment of a goal is determined by a single attribute. For example, the attainment of a production goal can be determined by counting the quantity produced, a single attribute.

In the next section, we will discuss the possibility of conflict among the various goals.

Conflict

Multiple goal behavior immediately raises the possibility of conflict. Certainly situations do exist in which all the goals can be satisfied simultaneously. For example, we might desire to go to both Rome and Paris on a vacation. If we have the time and the money to do so, then there is no conflict. But such situations are of little interest to us because the problem could then be decomposed into determining a means of achieving each goal independently. We will assume

that, due to resource limitations, or just mutually conflicting natures, multiple goal situations result in conflict. As an example, consider the problem of choosing a mode of travel between two cities so as to minimize both time and cost while maximizing both comfort and the scenic beauty of the route. A conflict in the basic nature of the goals results because minimum time may involve levels of acceleration which are uncomfortable. A resource limitation which precludes air travel may be imposed by our bank account. So in this instance the DM searches for some overall level of satisfaction, a "Gestalt optimum."

A distinction is made by psychologists between frustration and conflict (see, for example, Cofer and Appley 1964, p. 429 and p. 464). We define "conflict" as a property of a situation in which the simultaneous attainment of all goals at the present aspiration levels is impossible. Frustration is an emotional response to such a situation and reflects an inability to reconcile desires and abilities, aspirations and alternatives. All of us experience occasional moments of frustration, but continued frustration reflects an inability to adapt. In the example of the dress and the party, conflict existed because both dresses were desirable but only one could be worn.

Conflict can be resolved in two ways: innovation and adaptation. Innovation refers to the development of previously unknown alternatives so that the original goals can be attained. Information plays an important role here because it can suggest new avenues of research. Adaptation refers to changes in the current value structure

of the individual so that he is content with one of the available alternatives. Information is also important during adaptation because an awareness that there is no better alternative can facilitate the acceptance of an initially unpalatable choice. In reality, conflict is resolved by both methods simultaneously. We learn to accept that which we can attain while still striving to broaden the range of the attainable.

Aspiration Levels

An aspiration level is a degree of goal achievement which the individual consciously strives to attain. For Morris (1964, p. 96) it is a level of satisfaction which separates satisfactory and unsatisfactory alternatives. Katona (1951, p. 91) states:

There is first, an ideal level -- the perfect score -- which is known to be the best possible performance. Then there is the level of achievement, represented by the last actual score or by the average of recent scores. Finally, there is the level of aspiration -- the level which the person desires or expects to achieve in his next performance or next performances.

Johnsen (1968, p. 331) reformulates the comments of Lewin et al., (1944, p. 334):

Level of aspiration is simply a state of affairs measurable in space and time, and setting level of aspiration (i.e., goal setting) can typically be illustrated by the sequence: (1) given some knowledge of past performance, (2) setting level of aspiration = decision on how high the goal should be set, (3) execution of action and (4) the reaction to the level of attainment, such as feeling of success or failure ('disappointment'), leaving the activity altogether, or continuing the new level of aspiration.

And Radner (1964, p. 212) states that if $\{Z\}$ represents a series of

values obtained by some search procedure, then an aspiration level can be defined as some number, \hat{z} , such that the search terminates at the first $Z_i \geq \hat{z}$.

In general, the aspiration level does not remain fixed but changes as a result of the sequence of successes and failures from attempts at goal attainment. The relationship is quite complex because the aspiration level is a function of the entire value system of the individual which is also dynamically changing. The attainment or nonattainment of a goal has emotional overtones which affect the evaluation of the degree of goal attainment.

Charnes and Cooper (1959) indicate that if an aspiration level is violated, then the individual will adjust it. Lewin et al. (1944, pp. 373-374), discuss this change as a function of the strength (or degree) and relative frequency of the successes and failures. (We define a success as the attainment of at least the aspiration level for that goal.) The strength of the result affects the amount of change while the historical pattern produces cumulative effects. So general success breeds higher aspiration levels than general failure, and a failure has less effect if it occurs in a pattern of successes than if it is one of a string of failures.

This change of aspiration level affects the search pattern as the individual "learns from experience" because it alters his definition of what constitutes an acceptable alternative. We will provide a means for such changes in the algorithm we will develop.

Criterion Functions

The set of alternatives and their corresponding effects can be defined respectively as the domain and the range of a set of criterion functions. A criterion function is a mapping (rule of correspondence) from a set of resource allocations (the domain) onto a set of numbers (the range) which determines goal attainment. If production of 10 widgets/day is the goal, then simple enumeration at the end of the production line will indicate goal attainment or nonattainment. But a criterion function would relate the inputs in the production process (man-hours, material, etc.) to the output through an analytical expression which is accurate for some range of inputs. Thus

$$\text{Production} = (\text{Man-Hours}) \times (\text{Tons of Raw Material}) \times \\ (\text{Number of Units}/(\text{Man-Hours} \times \text{Tons of} \\ \text{Raw Material}))$$

is a criterion function whose range (amount produced) enables us to determine if the goal will be met for a variety of input conditions (resource allocations). There will be one criterion function associated with each goal and it will be used to predict goal attainment or nonattainment. Vector-valued problems would have multiple criterion functions for each goal.

So the term "criterion function" as used in this context is the familiar objective function of mathematical programming problems.

In the class of problems this research is dealing with, however, the values it takes on serve as an input to the decision making process. It is the worth of the decision, not the magnitude of the criterion functions, that we are intent upon optimizing. The "best" decision will result from choosing the "best" alternative with respect to the individual's current definition of "best." We must inquire into the nature of the value structure of the individual to determine if an analytical expression relating worth to the alternatives can be developed. This would supply a suitable mechanism for finding the "best" alternative.

Preference Functions

Choices among alternatives express preferences which represent the relative worth of each alternative. The ordering or ranking created by these choices can be either complete, with no incomparable alternatives, or incomplete, in which case some alternatives cannot be compared.

The nature of this choice mechanism is still much in debate (Kotler 1967, pp. 82-94), but in principle, a preference function which provides an analytical expression for the calculation of the worth of a choice to an individual can be formulated for all situations at all times. The existence of an objective preference function was postulated by utility theorists who assumed that a cardinal dimension measured in "utils" could be developed as the range of a preference function. The cardinality requirement has

been replaced by one involving ordinal relationships so that rankings can be established. An individual calculates how many utiles a good or situation is worth to him and, if he is "rational," chooses the one with the highest value. Dupnick (1971, Chapter 3) has an excellent review of the development of utility theory.

Realistic multiple goal situations are quite complex because (1) they usually involve some attributes which have no scale of measurement (i.e., comfort); (2) the desirability of the levels of attainment for a particular goal is interrelated with the level of attainment for each of the other goals; (3) comments (1) and (2) mean that the weighting of the attributes of the goals depends upon the subjective weights of the goals themselves; (4) the relationships in (2) and (3) imply that ranking rules must be formulated in a Gestalt framework which reflects both the environment of the individual and his state of mind; (5) the dynamic nature of the environment together with the points (1) through (4) implies that the ranking rules will change over time.

This increases the difficulties in developing an analytic preference function to be optimized. Shepard (1964, p. 273) observes that this dynamic situation:

. . . makes the particular form taken by the psychological rules for combining or "trading off" rather difficult to pin down. This may partly explain the fact that although economists concerned with the prediction of human behavior have placed great theoretical emphasis on what they variously call equal-preference contours, constant-utility curves, or indifference maps, actual empirical determination of the curves is notably lacking (see Edwards 1954, pp. 384-387).

If an analytical preference function is to be developed, some experimental means of measuring preference is necessary. However, the "value of something" has been an elusive quantity to measure because of definitional problems. "Value" changes. It is different from individual to individual and from moment to moment. The changes in the value structure of the individual result from an interactive Gestalt learning process between himself and his environment and also changes in the environment. The information he receives in this process together with his past experiences alter his view of the world. His perception of the information itself changes and intertwines with his system of values.

The dynamic situation mentioned above has prevented researchers from developing a cardinal scale of utility. In fact, research in this area has had only limited success in developing ordinal scales (indifference curves) for individuals in restricted situations. Results reported by MacCrimmon and Toda (1969) illustrate the type of limitations that have to be imposed upon the experiments. Their method of determining indifference curves required immediacy of the consequences of decision making. And, in addition, the preferences between bundles of goods had to be portrayed graphically, which is difficult when more than 3 goods are considered or when the goods are abstract qualities, such as beauty, with measurement problems of their own.

There is another aspect of preference functions which has frustrated experimental efforts. The diversity of information

integrated by the individual in determining preferences is much greater than is explicitly recognized in the typical research situation.

Yntema and Klem (1965, p. 4) describe part of their experiment: "In fact, a rather long list of attendant circumstances had to be specified in order to make the problem (in the experiment) a definite and meaningful one . The attendant circumstances represent the information that the man must take into account in specifying the worth function." So preference is related to the individual's total environment as he perceives it.

There is another aspect of information which is difficult to control experimentally. It arises from the fact that some objective information is only "quasi-objective." Individual perception of the concept of "2" varies little, but perception of the color green is more subjective and varies from person to person. And yet it is this "objective information" that the individual includes as input to his preference function. Rather than providing an objective measure of preference, the function transforms objective and/or subjective information into a subjective evaluation which may not even be defined. It is possible to say "I like this better than that" without being able to explain why.

The choice of a car provides an example of the utilization of both objective and subjective information to arrive at a decision. The weight, size, price, and horsepower are all objective measurements. The reaction to the color and styling are subjective interpretations.

In this research, the objective component is provided by the numerical values of the criterion functions for various combinations of resource inputs. The subjective component is the combining of these values as a group by the individual with respect to his aspiration levels, his environment and his perception of "good" vs. "bad" alternatives.

The difficulties associated with specifying a preference function make it unlikely that we can express the decision maker's interpretation and ranking of a set of values from the criterion function in an objective manner. Indeed, he himself may be unable to state a ranking relationship although he develops one subjectively. Because of this situation, approaches for working with implicit preference functions must be examined in developing an algorithm to solve the decision problem. Although there is no evidence that a dynamic analytical preference function can be constructed, we will continue to use the word "function" because of the connotation of "transform" or "mapping" associated with it.

The definition of a "solution to the decision problem" is dictated by this inability to specify the preference function. In the usual optimization problem, the search is directed toward an optimum which possesses well-defined mathematical properties (see, for example, Korn and Korn 1968, pp. 332-334). This is the result of having an analytic objective function (a criterion function in this context). But in our research we are dealing with a subjective evaluation, a value judgment expressed by an undefined preference function.

So the concept of a solution must be reexamined. We will explore the nature of both an optimal choice and a satisfactory choice. The latter, termed a "satisfactum," will be seen to be the definition of "solution" that we wish to employ.

Optimal Choices

Radner (1964, p. 80) defines an action as optimal if every alternative action in the set of available alternatives is not preferred to it. Optimizing then is " . . . the principle of choice according to which the decision-maker chooses an optimal action." However, choices are based on the set of perceived alternatives, a subset of the collection of all available alternatives. Limitations in information account in part for the differences between the 2 sets, but in addition there are bounds on our imagination which prevent us from asking the correct questions to get additional information. Shelly and Bryan (1964, p. 9) comment: "The best may simply be 'the best we can think of,' with no conviction existing that there is not something better. . . . Consequently, an optimal decision may be considered to be the selection of an action which produces a result that is in some sense 'best.'"

If the context of the problem requires the optimization of some subjective feeling such as comfort, then the meaning of optimality becomes obscured. In these cases " . . . there are no objective measures in terms of which an optimum can be determined" (Gulliksen 1964, p. 73). And so " . . . the optimum choice (out of a given set of

alternatives) is the one that leads to the highest subjective evaluation of its ensuing consequences" (Shepard 1964, p. 259).

Decision problems typically involve multiple conflicting goals so that choices must be made amongst multiattributed alternatives. But Shepard (1964) indicates that such judgments are rarely optimal even by the subjective standards of the DM himself. Or as Tucker (1964, p. 87) points out, " . . . a situation may be optimal with respect to one area of goals and not optimal with respect to others." In this case, the Simon (1957) hypothesis of satisfying behavior takes on more relevance. He defines an optimal alternative as one which is preferred to all others on a complete ordering by a set of criteria and a satisfactory alternative as one which meets or exceeds a set of criteria describing minimally satisfactory alternatives. Simon (1953, p. 141) also states: "Most human decision-making, whether individual or organizational, is concerned with the discovery and selection of satisfactory alternatives; only in exceptional cases is it concerned with the discovery and selection of optimal alternatives." And so the individual searches for an "acceptable" rather than "best" solution to the problem (see also Reitman 1964).

Acceptable Choices

Acceptability is a value judgment derived from the individual's preference function using values from the criterion functions as inputs. So the partitioning into "acceptable" and "not acceptable" is based on both objective and subjective factors. Objectively, there is the past factual knowledge that the individual possesses. Subjectively,

there are previous emotional experiences (successes and failures) and the value structure of the individual.

A satisfactum is any value within an interval of acceptability on the range of a criterion function. It is a satisfactory choice. A multiple goal satisfactum implies acceptable values of all criterion functions.

For example, if room temperature is denoted by T , then a possible preference function in relating comfort to T is shown in Figure 2.1. Denoting "comfort" by C ; then in some manner $C = P(T)$ where P is the preference function or value structure which "interprets" a temperature T in terms of some level of comfort. The domain of P is the set of values for T , an objective measure; but the range of P is a completely subjective interpretation which we have labelled comfort. The " a " represents the lower bound of acceptability to the individual. If we define a satisfactum as $C^0 \geq a$, then it is clear that there are many satisfactums because there is a whole range of temperatures within which the individual is comfortable.

If, starting from a room temperature which is "too hot," T is lowered until the individual is comfortable, the value which produces this satisfactum is likely to be different from the corresponding T value when the process has an initial condition of "too cold." So the value of T which produces a satisfactum and the corresponding degree of comfort are dependent upon the method of search and the starting point together with the psychological state of the individual.

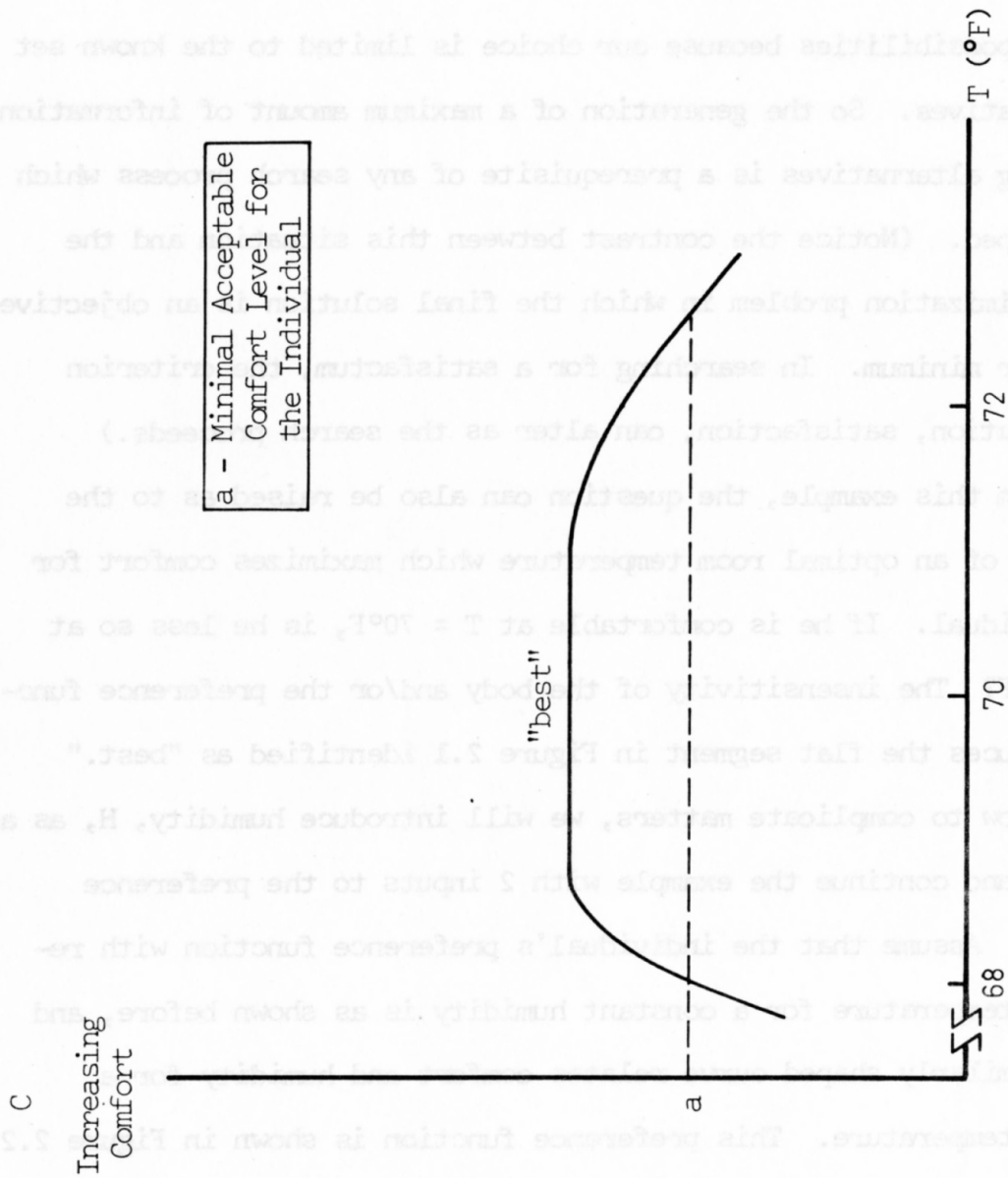


Figure 2.1 Hypothetical preference function.

This dependency upon the search process is due to the interrelationship of the region of acceptability and information. Obviously, we will be content with less if we do not know the full range of possibilities because our choice is limited to the known set of alternatives. So the generation of a maximum amount of information concerning alternatives is a prerequisite of any search process which is developed. (Notice the contrast between this situation and the usual optimization problem in which the final solution is an objective maximum or minimum. In searching for a satisfactum, the criterion for a solution, satisfaction, can alter as the search proceeds.)

In this example, the question can also be raised as to the existence of an optimal room temperature which maximizes comfort for the individual. If he is comfortable at $T = 70^{\circ}\text{F}$, is he less so at $T = 70.1^{\circ}\text{F}$? The insensitivity of the body and/or the preference function produces the flat segment in Figure 2.1 identified as "best."

Now to complicate matters, we will introduce humidity, H , as a variable and continue the example with 2 inputs to the preference function. Assume that the individual's preference function with respect to temperature for a constant humidity is as shown before, and that a similarly shaped curve relates comfort and humidity for a constant temperature. This preference function is shown in Figure 2.2. Many combinations of T and H produce the feeling "comfortable" in the individual. And, as before, there is no unique optimum.

Now add constraints to the range of T and H so that the previous range of acceptability is completely unattainable. The situation

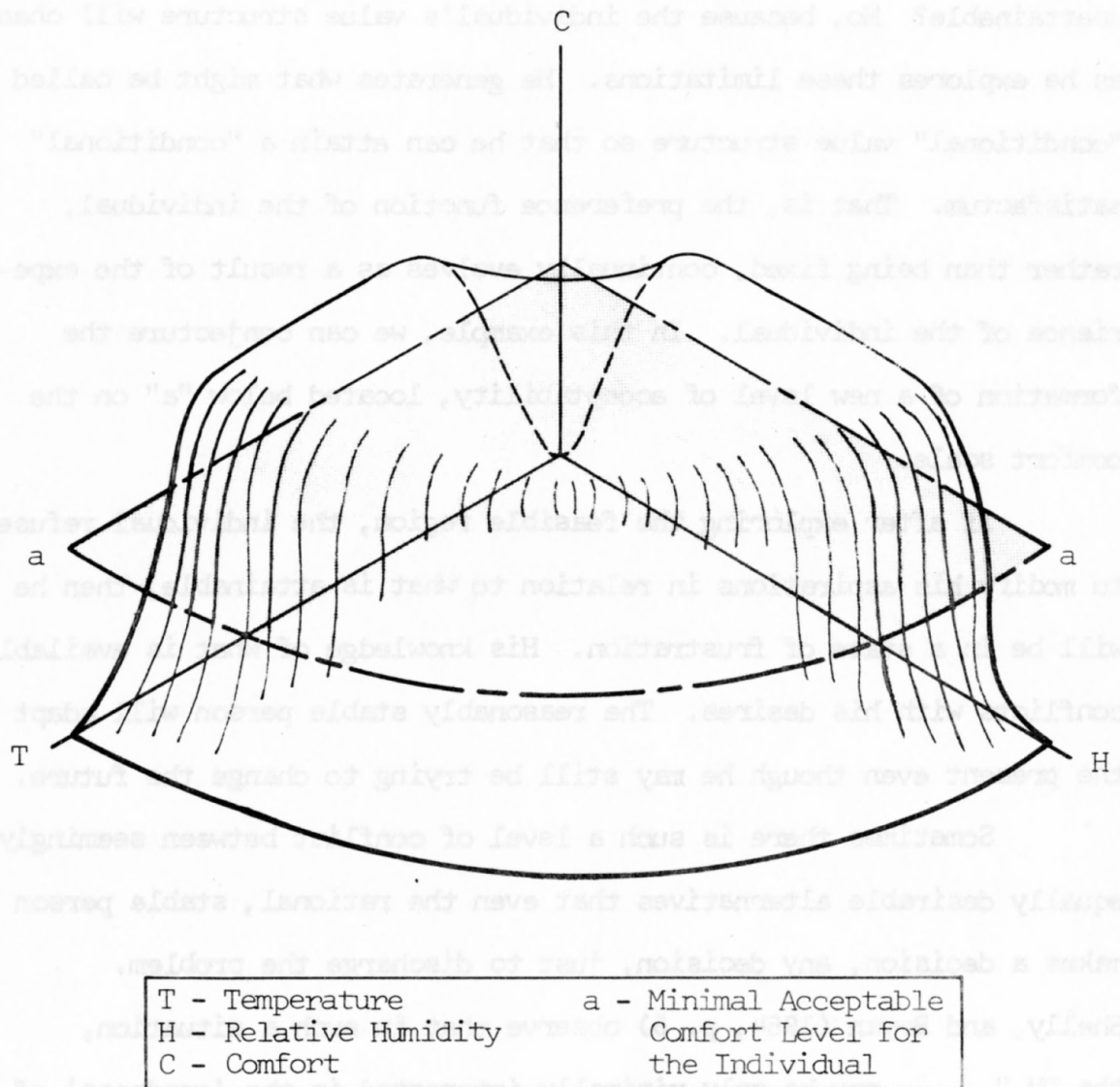


Figure 2.2 Hypothetical preference function for two criterion functions.

is depicted in Figure 2.3. Does this mean that a satisfactum is unattainable? No, because the individual's value structure will change as he explores these limitations. He generates what might be called a "conditional" value structure so that he can attain a "conditional" satisfactum. That is, the preference function of the individual, rather than being fixed, continually evolves as a result of the experience of the individual. In this example, we can conjecture the formation of a new level of acceptability, located below "a" on the comfort scale.

If after exploring the feasible region, the individual refuses to modify his aspirations in relation to what is attainable, then he will be in a state of frustration. His knowledge of what is available conflicts with his desires. The reasonably stable person will adapt to the present even though he may still be trying to change the future.

Sometimes there is such a level of conflict between seemingly equally desirable alternatives that even the rational, stable person makes a decision, any decision, just to discharge the problem. Shelly, and Bryan (1964, p. 5) observe that in such a situation, the DM " . . . may be only minimally interested in the 'goodness' of this solution." Is this a satisfactum? Yes; the weighting on the original goals has been reduced (or alternatively, their aspiration levels lowered) so that the only important consideration is reaching a decision.

In our example, the individual finds an "acceptable" combination of T and H within the constraints. This does not mean that he

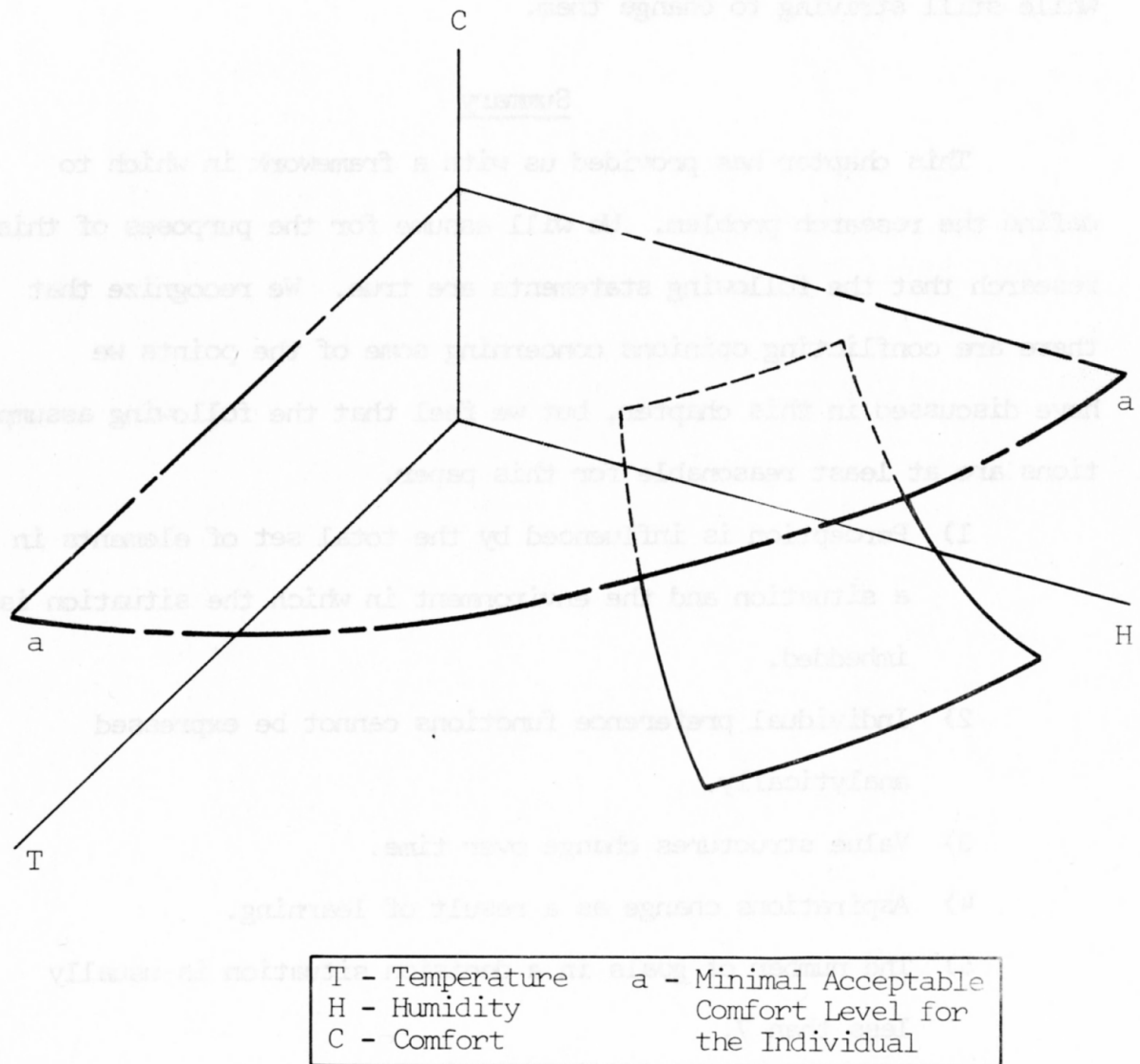


Figure 2.3 Constrained hypothetical preference function for two criterion functions.

is as comfortable as he could be, only that he can live with the situation--he adapts. As a further note of clarification, "adaptation" is not "surrender." An individual can adjust to constraints while still striving to change them.

Summary

This chapter has provided us with a framework in which to define the research problem. We will assume for the purposes of this research that the following statements are true. We recognize that there are conflicting opinions concerning some of the points we have discussed in this chapter, but we feel that the following assumptions are at least reasonable for this paper.

- 1) Perception is influenced by the total set of elements in a situation and the environment in which the situation is imbedded.
- 2) Individual preference functions cannot be expressed analytically.
- 3) Value structures change over time.
- 4) Aspirations change as a result of learning.
- 5) The number of goals in a decision situation is usually less than 7.
- 6) The DM normally satisfices rather than optimizes.
- 7) A solution to a decision problem is any acceptable course of action.
- 8) "Acceptability" is a learned perception.

In the next chapter, we will formally state the research problem.

CHAPTER 3

STATEMENT OF THE PROBLEM

In this chapter, we will explicitly formulate the research problem. Our objective is to develop an algorithm which will enable the DM to solve a decision situation. A decision situation (DS) is a need for the DM to make a choice among alternatives. A solution is defined as either the selection of a feasible acceptable alternative or the replacement of the original problem by the need to make any choice and consequently the selection of any feasible alternative.

Alternatives are composed of 2 parts: (1) an allocation of resources and (2) the results of that allocation as defined by a set of criterion functions. The DM must have a set of results to evaluate and also know how to attain those results, so information concerning the alternatives in the DS is a necessary ingredient in the decision-making process. Feasible alternatives are those whose allocation of resources satisfy any objective constraint imposed by the DS.

The individual makes a decision based on his evaluation of the alternatives. Our assumption is that he will choose an "acceptable" one and that the "acceptability" of an alternative is a result of (1) his value structure and (2) his set of aspirations. These are in turn affected by information both in the form of results from past decisions and from the environment as a whole. The aspiration

levels relate to "benchmarks" on the criterion functions and indicate the desired level of output for each one. The value structure is dynamic and subjective by nature. It determines the strength of the desires above.

We assume that the value structure and its relationship to the alternatives and the final choice can be conceptualized by the term "preference function." A preference function denotes the manner in which the various alternatives are weighted and ranked. This research assumes that the preference function provides a complete ordering of all alternatives that are known to the DM; there are no comparisons which cannot be made. This assumption is necessary because the algorithm we are going to develop will contain an iterative search mechanism to allow for a progressive definition of the DM's preferences. This requires the DM to define a new direction of search at each iteration on the basis of his evaluation of all the previously generated alternatives. If a comparison of alternatives is not possible, then the algorithm would break down because a new direction of search would not be defined. So the algorithm must allow for a changing, undefined value structure and shifting aspiration levels while generating information relevant to the solution of the DS.

We will assume that each of the goals in the DS is operationally defined. That is, its attainment or nonattainment is objectively measurable upon some suitable response surface or criterion function. So associated with each goal is a single criterion function. This response surface may be nonlinear but we will require that it be

differentiable. It is not the goal itself but rather the criterion function against whose value the goal is compared which is nonlinear. Goals are a statement of desires rather than a relationship or function.

A short example may clarify this point. A production goal of at least 1000 widgets/day is objectively determinable because it satisfies the above condition of measurability. The criterion function is the relationship, usually nonlinear, between production and such resources as capital equipment, hours worked, employee capabilities, etc. The value of the criterion function for a set of resource inputs is compared to 1000 to determine if the goal has been met.

The previous example also illustrates that the criterion function is defined on a set of resources. The designation of values for all input variables (decision variables) is a policy (allocation policy, policy vector). We will assume that the values taken on by these variables are all positive and continuous. The constraints upon these variables may be expressed as nonlinear equations subject to the condition that they be differentiable.

Also for each goal, we will define a goal level as that value of the criterion function which has been imposed as an objective by conditions and/or policies external to the decision situation. The imposition of goals by upper management levels upon lower management is a familiar example of this.

Aspiration levels, on the other hand, are a function of the individual's past as discussed in a previous chapter. They are subject

to modification over time as the result of a learning process. Returning to the production example, the goal level was 1000 widget/day. But the DM may have an aspiration level greater than or less than the goal level. We will assume that, in the absence of other information, the DM initially equates his aspiration levels with the goal levels.

Stating the problem formally, we assume the existence of a DS in which there are N resources measured as continuous variables to be allocated so that T goals may be attained. A policy vector consisting of those allocations is denoted

$$\underline{x} = (x_1, \dots, x_n)$$

The values which \underline{x} can take on are restricted by a constraint set consisting of

L equality constraints $\underline{h} = \underline{H}(\underline{x}) = \underline{0}$ \underline{H} differentiable
(i.e., all L functions $H_1(\underline{x}), \dots, H_L(\underline{x})$ are differentiable)

M inequality constraints $\underline{g} = \underline{G}(\underline{x}) \geq \underline{0}$ \underline{G} differentiable

and N bounds $0 \leq \underline{b}_L \leq \underline{x} \leq \underline{b}_U < \infty$

Associated with each goal is a criterion function

$$z_t = Z_t(\underline{x}) \quad t = 1, \dots, T$$

which defines the results of alternate allocation policies. So we have

$$\underline{z} = \underline{Z}(\underline{x}) \quad \underline{Z} \text{ differentiable}$$

Goal attainment or nonattainment can be determined by the value of z_t .

The objective of this research is to develop an algorithm which will generate information leading to the location of a satisfactum for the DM in this DS. A satisfactum, p^o , is defined by

$$p^o = p \geq a$$

where "a" is the lower bound of a region of acceptability on the range of the individual's preference function, $P(\underline{z})$. We assume that there is no upper bound on the region of acceptability. The preference function operates on \underline{z} to determine the value of p from

$$p = P(\underline{z})$$

More precisely, p is the value of the results from an allocation policy \underline{x}

$$p = P[\underline{Z}(\underline{x})]$$

The preference function is itself a function of the total perceived stimulus input to the individual from his environment and his past history in the form of memory. That is

$$P = f[E(t), M(t)]$$

where $E(t)$ is the current environmental input, $M(t)$ is the individual's current memory of his past successes and failures.

Because of its subjective nature, we assume that the preference function cannot be defined nor p measured, but we do assume that P provides a complete ordering of the results for all allocation vectors which satisfy the constraints. That is, no incomparable vectors of results are allowed. For two such vectors, \underline{z} and \underline{z}' , the possible choices are \underline{z} preferred to \underline{z}' , \underline{z}' preferred to \underline{z} , or indifference between them.

CHAPTER 4

DEVELOPMENT OF THE ALGORITHM

After reviewing the literature concerned with multiobjective decision making, we will select those aspects relevant to our research and develop an algorithm capable of solving the problem posed in Chapter 3.

Literature Review

In general, models used in decision making have been concerned with the optimization of a single objective. Even in this case " . . . it has been only in those situations in which the criterion function to be optimized is sufficiently simple (e.g., profits) that conspicuous progress has been made in the development of optimization techniques" (Shelly and Bryan 1964, p. 8).

In his comprehensive survey of the area of multiple decision making, Johnsen (1968, pp. 342-393) observes that:

The aim of most normative models in management science is to optimize one single goal, for example to maximize profit per unit of time or minimize costs per unit of time.

Optimization according to a goal in terms of (perhaps transformed into) one single dimension of measurement is performed by all classical microeconomic models of the firm. The same is true of the great majority of operational microeconomics and operations research where partial optimization has been dominating, cf., for instance, linear programming models, inventory models, queueing models, risk models and uncertainty models, to use an often applied classification of OR models.

Also the great majority of normative behavioral science models, apart from those dominated by economic-science, aim at an optimum. In small group theory we found attempts to optimize group behavior, in organization theory attempts to optimize organization behavior, and in psychology attempts to optimize individual behavior. Such research aims are clearly inconsistent when one acknowledges that the decision-making units, with whom these disciplines are concerned, aim at several goals at the same time. To some extent the conclusion is drawn that multi-goals have to be taken into account; therefore non-optimization models are developed. It is mainly from these few attempts that we find some useful material for multigoal models, for example systems models and value models.

System models are many-headed monsters. Systems models aim partly at optimizing the whole system and partly at sub-optimizing subsets of the system. In some cases there exists a multigoal formulation for a system model. The same may be said of simulation models.

Value models in the sense of utility models deal explicitly with more than one goal. However, they usually convert these into one yardstick, and furthermore, no activities are related to the value expressions.

Geoffrion (1970) notes that for only 2 or 3 criteria it is possible to compute and graph the trade-off curves for the criterion functions and simply allow the decision maker to choose the point of highest "value" to him. The individual's preference function has remained unspecified. The chief difficulties here are the calculation of the constrained criterion functions and the usual problems of accuracy with a graphical solution.

Multiple Objective Models

Although multigoal models are less numerous than their single objective counterparts, there is a growing literature on the subject.

Roy (1970) suggests the following classification:

- 1) Models which aggregate multiple objective functions into a unique function defining preference.
- 2) Models which provide for a progressive definition of preferences together with exploration of the feasible set.
- 3) Models which aim at formulating a partial ordering stronger than the partial order formed by the product of n complete orders associated with the n objective functions.
- 4) Models whose goal is the reduction of uncertainty and incomparability.

Aggregation Models. These models convert several objectives into a single utility function, generally a profit or cost measure, which specifies the preference order. The objectives must all be measurable on some common dimension. The additivity requirement for the conversion necessitates a complete ordering of all the outcomes.

A study reported by Ackoff (1962, Chap. 3) showed two objectives expressed as one objective function after careful formulation of the costs. Balderston (1960) pointed out that this approach can be viewed in terms of individuals' aspiration levels.

Cost-benefit analysis (Prest and Turvey 1965) is an aggregation scheme which assumes that all benefits can be expressed in monetary terms. However, the method is limited to situations in which a set of alternatives already exists. Collective utility is also an aggregation framework which assumes that a set of alternatives is

available. But the mathematical framework available to the analyst is quite explicit and powerful as compared to cost-benefit analysis.

The development of goal programming has been based on this technique. Pioneer work in this area was done by Charnes and Cooper (1961, v. 1, pp. 215-249) who developed an algorithm which leads toward a satisfactum (a satisfying rather than optimizing condition). Charnes and Stedry (1964) have developed a class of models in which the probability of goal attainment is an explicit function of search activity. A similar idea was also discussed by Brooks (1958) in an article on stratified random sampling methods for seeking maxima. Ijiri (1965) develops the concept of operational subgoals as the means to attaining actual planning goals, although the attainment of a subgoal is not necessarily indicative of attainment of the original goal. Other authors associated with the development of these models are Boldur (1970), Terry (1963), and Raiffa (1969).

Sequential Models. In this class of methods, the concept of an optimal solution is replaced by that of a satisfactum or "acceptable" solution. Sequential search approaches are developed to formulate a better knowledge of the preference structure of the DM. This is integrated into a systematic exploration of the set of alternate activities. Since the solution does not have to be optimal, the search procedure can be terminated after arbitrarily many solutions have been explored. The sequential nature of the method, however, does require that at any iteration a complete ordering exists among

all alternatives examined up through that iteration. This ensures that a new direction of search can be defined.

Cost-effectiveness is a technique applicable to multiple criteria decision problems. The method employs an array which lists the values of the criteria for various alternatives. The use of the word "value" here is somewhat misleading since this technique allows for descriptive statements as the range of a criterion. For example, the range of a criterion might be "bad," "neutral," and "good." The DM is expected to select the most satisfactory alternative among those listed in the array. We can conjecture that the DM may use a sequential approach in evaluating the various possibilities. As in the cost-benefit technique mentioned earlier, this method assumes that a set of known feasible alternatives exists prior to the analysis.

Benayoun et al. (1970) use an interactive approach in which the algorithm (STEP Method--STEM) permits the decision maker to develop the search area at each stage by examining a table containing the value of each criterion function under alternate input conditions (policy vectors). Although the authors considered only linear criterion functions subject to linear constraints, they demonstrated that the power of the technique arises from its consideration of cases when the preference function is only implicitly known to the DM himself.

Geoffrion (1970) has also used this technique of man/machine interaction for multiple objective problems. He utilizes a question-answer format interspersed with optimizations based on the Frank-Wolfe (1956) algorithm to locate the DM's preferred solutions.

Partial Order Models. The essence of this group of models is an attempt to build a ranking relationship which delineates the part of the DM's preferences that can be accounted for by means of the available data. This is in contrast to a complete ordering of all his desires. The formulation of a ranking relationship among some alternative actions is an attempt to explicitly develop a partial ordering on the DM's preferences.

Uncertainty Reduction Models. Methods to reduce uncertainty and incomparability assume that the preference order is not sufficiently explicit to allow a decision as to the best choice. They consider the preference order a reflection of implicit quantities (i.e., marginal substitution rates) which are not well known. A reduction in the variation in some of those incomparable quantities allows the selection of a "best" choice in light of the remaining uncertainty and incomparability.

Maier-Rothe and Stankard (1970) have developed an approach to build a complete order on the set of alternative actions. Denote a particular n-tuple of scores on all criteria as

$$S_i = (s_1, \dots, s_n)$$

They then hypothesize that the set of n-tuples

$$T = \{S_1, \dots, S_m\}$$

is convex. By postulating the existence of an unknown utility function, they develop a set of bounds for the objective functions. This is accomplished by asking the DM a sequence of questions dealing with

his preferences. Preference relations are then constructed which reduce the area of uncertainty.

An alternate approach assumes that a complete ordering of the set of vector scores is defined by the mean of a classification rank assigned to each element in every vector. The method then examines the effect on the ranking caused by "unit" changes in the appropriate elements of the vector for some pairs of attributes.

Evaluation

With respect to the definition of the research problem in the last chapter, approaches 1, 3, and 4 of the previous classification seem inappropriate for various reasons. The aggregation models require a common denominator of measurement and we are dealing with preferences which can be based on unquantifiable or even undefinable attributes, such as beauty.

Partial order models develop comparisons between some pairs of alternatives and seem most appropriate for those situations in which the alternatives can be listed or grouped into sets. The method seems inappropriate when the decision variables and results can vary continuously, and there are nonlinear constraints present. There is also an implicit requirement that the set of feasible alternatives is known.

The uncertainty reduction models recognize that some aspects of the comparison process are not quantifiable, but they seem unable to handle constrained problems of the type we are considering.

The philosophy of the sequential models of group 2, however, seems to lend itself to our problem and we will explore this concept further to examine its appropriateness.

Discussion and Development

In striving for a "general" method of solution, we ask if individuals exhibit an overall similarity of preferences so that an "average" preference function can be constructed. In a gross sense, yes. Comfort is usually preferred to discomfort, love to hate, and so on. But the aspiration levels and the ordering of the goals are unique to each individual because of his stored experiences---the sum total of what he is. Tucker (1964, pp. 86-86) also observes that:

. . . the perception of situations and possibly the conception of the nature of laws of relations are also subject to individual differences. . . . Differences between individuals which exist in the perception and understanding of such systems (sets of relations) influence the perception and interpretation of information available to the person about the present state of the system, . . . and the relations between a system and its surroundings. Such judgments (perceptions) undoubtedly affect the optimality of subsequent situations.

The individual uniquely determines both the problem and its solution so the algorithm must in some manner reflect this uniqueness. The discussion in Chapter 2 indicated that it would be impossible to construct individual analytic preference functions, but the search models of group 2 have circumvented the difficulty because they actively include the DM in the search.

Although the man/machine interactive search has been used by Benayoun et al. (1970) and also by Geoffrion (1970) who described it as an "interactive method," the phrase "interactive programming" does not seem to have been proposed as a classification of this technique. We will define interactive programming as the inclusion of the DM himself in an algorithm so that he can direct a sequential search procedure. In this research the results from each cycle will be evaluated by the decision maker with possible aspiration level changes as a result. As he determines the trade-offs demanded by satisfying various goals he tends to bring what he wants more in line with what he can get.

But is the sequential approach reasonable? Can decision making take place when alternatives are explored in a serial manner? Koopmans (1964, p. 245) states very definitely that " . . . almost all choices in real life are sequential, 'piecemeal,' choices between alternative ways of narrowing down the presently existing opportunity rather than 'once-and-for-all' choices between specific programs visualized in full detail." The psychological reason for such serial processing is that man is finite. Simon's (1957) "bounded rationale" explains the limitations of memory and evaluative capacity under which decisions are made. Multiattributed alternatives are not considered "in toto."

Research indicates that individuals make decisions on only a few aspects of a multiattributed alternative although their impression may be that they have considered all of the variables affecting

it. Shepard (1964, p. 258) feels " . . . that the relative weights to be assigned to the component attributes are not always determinate and may, in fact, depend on the adoption of one of several incompatible but equally tenable systems of subjective goals." He continues to state (p. 264) that:

. . . results presented by DeSoto (1961) and by Osgood, Suci, and Tannenbaum (1957, pp. 119-116), for example, reveal a striking inability of subjects to take account of the independent way in which the objects vary along the different dimensions. Instead, there seems to be an overweening tendency to collapse all dimensions into a single "good versus bad" dimension with an attendant loss in detailed information about the configuration or pattern of attributes unique to any one object.

He also reports (p. 266) that experiments conducted by Hoffman (1960, pp. 126-127) and Pollack (1962) suggest that:

. . . although the weights actually controlling the subjects' responses are usually concentrated on only one or two attributes, the subjective weights reported by the subjects tended to be more evenly distributed over the whole set of attributes. Indeed, there is some indication in Pollack's findings that the announced subjective weights tended to err in the opposite direction of ascribing too much importance to the less important variables. Possibly our feeling that we can take account of a host of different factors comes about because, although we remember that at sometime or other we have attended to each of the different factors, we fail to notice that it is seldom more than one or two that we consider at any one time. In any case, the confidence that we have tended to invest in our rational ability to weight and combine many subjective factors appears to have been somewhat misplaced.

So a serial approach, rather than conflicting with established patterns of decision making, will tend to support them because of the information that is generated. Radner (1964, p. 211) comments that " . . . if information is not forgotten then the information on

which decisions are based is, in a certain sense, 'expanding through time.'" And so later choices can be guided by "new" preference functions which have changed due to the experiences of the individual up to the time of choice.

To summarize the discussion up to this point, then, an interactive algorithm in which serial choices are made does not seem to conflict with the "natural" decision-making procedure. If a suitable form of information presentation can be found, this method could greatly enhance sequential choice processes by helping delimit areas of search in a more objective manner. The key to multiobjective decision making seems to be both the amount of information and its timing. Too much causes confusion, too little results in indecision. We need information about our more important goals first. As the solution becomes more definite, we can assimilate additional information about the less important aspects of our alternatives. Since we are not actually optimizing the DM's preference function, it may be appropriate to view the algorithm we are attempting to develop as an information generating device which permits him to find a satisfactum, if not an optimum. With this in mind, we will proceed to formulate the algorithm.

Notation

The notation used in explaining the development of the algorithm is collected here as an aid to the reader.

Number of Decision Variables	N	
Number of Equality Constraints	L	
Number of Inequality Constraints	M	
Number of Goals or Objectives	T	
Number of Criterion Functions	T	
Policy Vector	$\underline{x} = (x_1, \dots, x_N)$	\underline{x} continuous
Constraint Set	$\underline{h} = \underline{H}(\underline{x}) = 0$	$\underline{H}(\underline{x})$ differentiable
	$\underline{g} = \underline{G}(\underline{x}) \geq 0$	$\underline{G}(\underline{x})$ differentiable
	$0 \leq \underline{b}_L \leq \underline{x} \leq \underline{b}_U < \infty$	
Criterion Functions	$\underline{z} = \underline{Z}(\underline{x})$	$\underline{Z}(\underline{x})$ differentiable
transformed	$\underline{y} = \underline{Y}(\underline{x})$	$y \in (0,1]$
Goal Levels	$\underline{GL} = (GL_1, \dots, GL_T)$	$\underline{GL} \in \Gamma[\underline{Z}(\underline{x})]$ (Γ = range)
Aspiration Levels	$\underline{AL} = (AL_1, \dots, AL_T)$	$\underline{AL} \in \Gamma[\underline{Z}(\underline{x})]$
transformed	$\underline{A} = (A_1, \dots, A_T)$	$\underline{A} \in (0,1]$
Goal Formulation	the manner in which z and AL are to be related for the t^{th} goal when achievement of that goal is added to the problem as a constraint. For example, if the goal is $z_t \geq AL_t$, then the goal formulation expressed as a constraint is	

$z_t - AL_t \geq 0$. We notate this concept as

$$GF_t(z_t : AL_t) \geq 0.$$

Dimensionless Indicator of Attainment

for the T goals $\underline{d} = \underline{DA}(\underline{x})$

Surrogate Objective Function $s = \sum_{t=1}^T d_t$

Preference Function $p = P(\underline{z})$

P undefined, p a subjective measure of worth

At the i^{th} cycle, with goals j, k, ... entered as constraints which must be satisfied at the current aspiration levels, the following definitions apply: (i = 0 prior to any search)

Policy Vector $\underline{x}_{i.jk...} = \{x_1^{i.jk...}, \dots, x_N^{i.jk...}\}$

Constraint Set $\underline{h} = \underline{H}(\underline{x}) = 0$
 $\underline{g} = \underline{G}(\underline{x}) \geq 0$
 $0 \leq \underline{b}_L \leq \underline{x} \leq \underline{b}_U < \infty$

plus $GF_j(z_j : AL_j^i) \geq 0$

$GF_k(z_k : AL_k^i) \geq 0$

etc.

Aspiration Levels Prior to the Cycle $\underline{AL}_i = \{AL_1^i, \dots, AL_T^i\}$

transformed AL $\underline{A}_i = \{A_1^i, \dots, A_T^i\}$

Values on All Criterion Functions $\underline{z}_{i.jk...} = \{z_1^{i.jk...}, \dots, z_T^{i.jk...}\}$

transformed z $\underline{y}_{i.jk...} = \{y_1^{i.jk...}, \dots, y_T^{i.jk...}\}$

Dimensionless Indicator of Attainment $\underline{d}_i = \{d_1^i, \dots, d_T^i\}$

Surrogate Objective Function $s_{i.jk...} = \sum_{t=1}^T d_t^i \quad t \neq j, k, \dots$

Preference Function $P_{i.jk...} = P_i[\underline{z}_{i.jk...}]$

Our philosophy with regard to the use of this notation will be to employ the minimum amount of subscripting that the context will permit. The explicit listing of the argument of a function will be omitted for clarity if possible. So we write Z_t instead of $Z_t(\underline{x})$. Similarly, if the text is concerned with the general characteristics of a function then the goal subscript will be omitted; i.e., criterion functions in general are referred to as Z .

The Algorithm

Transforms

Our development begins by transforming the original response surfaces z to y where y is defined on the interval $(0,1]$. In principle, then, for any goal:

$$y = Y(\underline{x}) = \frac{Z(\underline{x}) - z_{\min}}{z_{\max} - z_{\min}} + \epsilon(\underline{x}) \quad \text{where } \epsilon(\underline{x}) = \begin{cases} 0 & \text{if } Z(\underline{x}) > z_{\min} \\ \epsilon' & \text{if } Z(\underline{x}) = z_{\min} \end{cases}$$

with ϵ' defined as a constant equal to 10^{-300} . We may wish to divide by y and this procedure will prevent numerical difficulties on the computer. The value chosen for ϵ' reflects the range allowed for numeric values on a CDC 6400 computer.

However, z may be unbounded above and/or below. Even if the maximum and minimum values, z_{\max} and z_{\min} , exist, finding them may prove to be a formidable nonlinear programming problem in itself. And, in a physical problem, portions of the range of Z may be known to be unattainable. For these reasons, we choose to define a "relevant range" of Z for each goal as:

$$\Gamma[Z(\underline{x})] = [z_{\text{lower}}, z_{\text{upper}}]$$

Then on that range

$$y = Y(\underline{x}) = \frac{Z(\underline{x}) - z_{\text{lower}}}{z_{\text{upper}} - z_{\text{lower}}} + \epsilon(\underline{x}) \quad \text{where } \epsilon(\underline{x}) = \begin{cases} 0 & \text{if } Z(\underline{x}) > z_{\text{lower}} \\ \epsilon' & \text{if } Z(\underline{x}) = z_{\text{lower}} \end{cases}$$

so that y is in the interval $(0,1]$. (In examining the results of each iteration, it will be necessary to confirm that z has indeed remained in the specified interval.)

The AL may be transformed into A by substituting AL for Z in the previous transformation equation. Then A is in $(0,1]$.

Goal Formulations

Most practical goal formulations can be represented by one of the five types listed in Table 4.1. The concept of goal attainment refers to achievement of a particular aspiration level (AL) for that goal formulation rather than attainment of a goal level (GL). The GL will provide only an initial starting point for the algorithm. References made to a particular goal will be made to that goal formulation rather than the numeric value of the GL.

The dimensionless indicator of achievement, d , is defined for each type of goal by the corresponding DA equation in Table 4.1. The shape of d for each of the 5 types is shown in Figures 4.1-4.5. In each instance values of d greater than 1 imply that the goal is unsatisfied, and conversely. The critical fact in this construction is that the sign of any change in d can be interpreted the same for all goals. In our formulations, an increase in d always indicates that the corresponding goal is becoming less satisfied or more unsatisfied.

TABLE 4.1

Common Goal Formulations
(The t subscript has been omitted for clarity)

Type	Goal	Range of Acceptability	DA
I	"equals" $y=A$		$1/2(y/A + A/y)$
II	"at least" $y \geq A$		A/y
III	"at most" $y \leq A$		y/A
IV	"between" $A_1 \leq y \leq A_2$		$\frac{A_2}{(A_1 + A_2)} (A_1/y + y/A_2)$
V	"excluding" $y \geq A_2$ or $y \leq A_1$		$\frac{A_1 + A_2}{A_2} \frac{1}{(A_1/y + y/A_2)}$

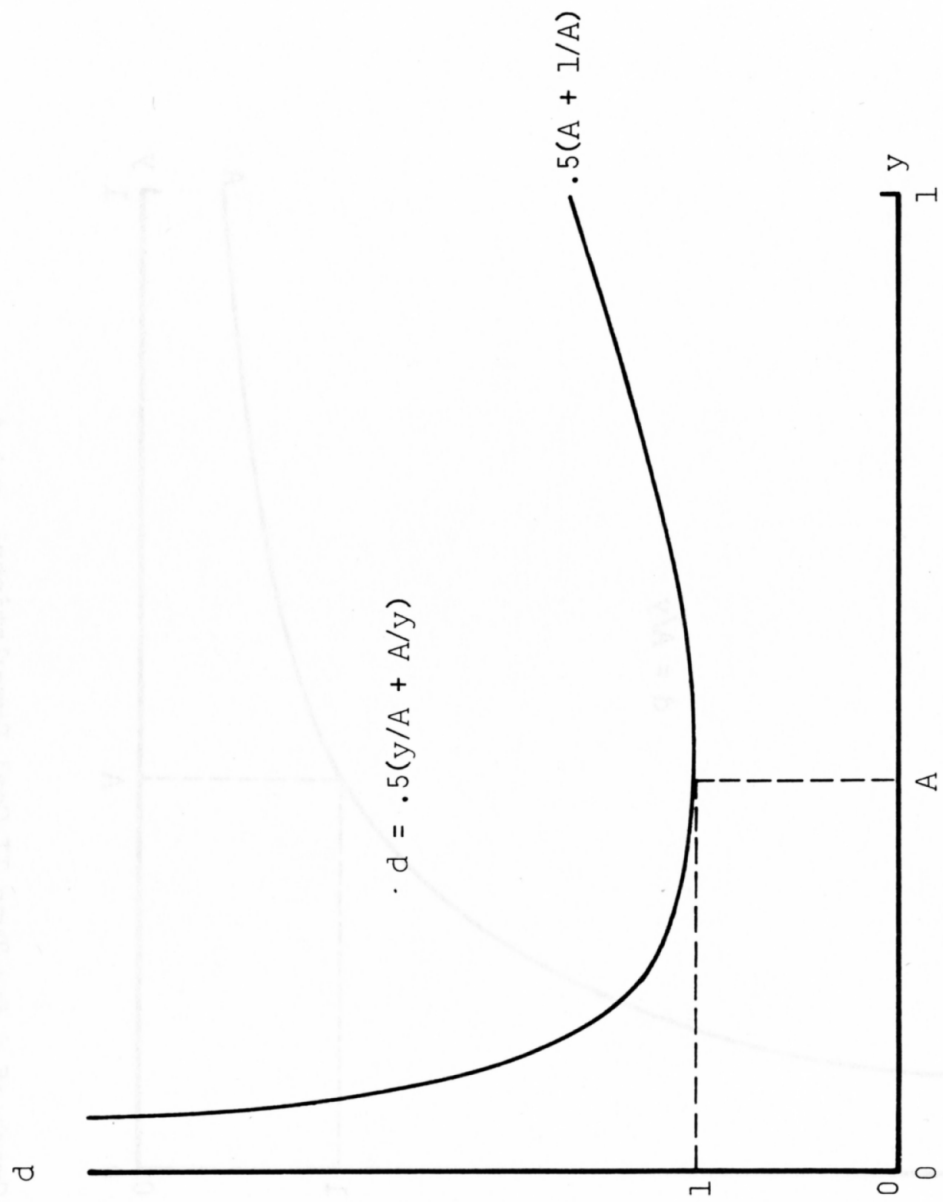


Figure 4.1 Graph of d for Type I Goal Formulations: $y = A$.

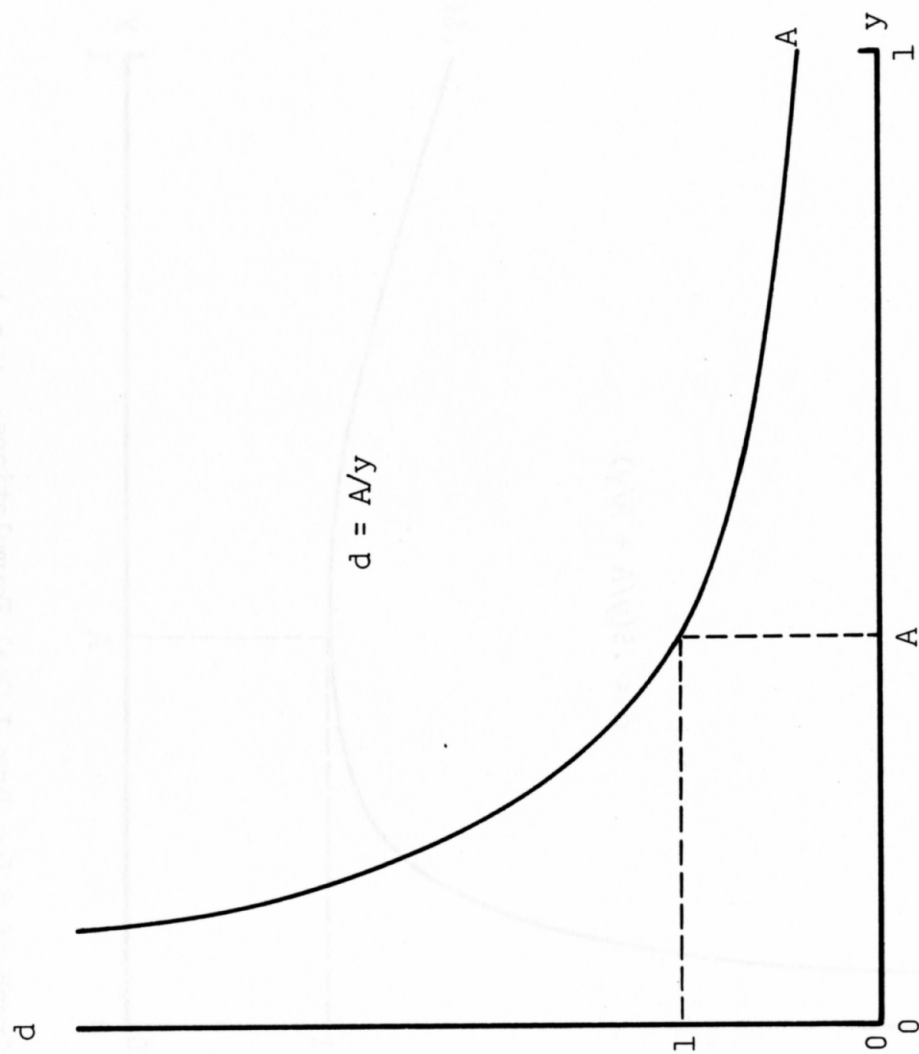


Figure 4.2 Graph of d for Type II Goal Formulations: $y \geq A$.

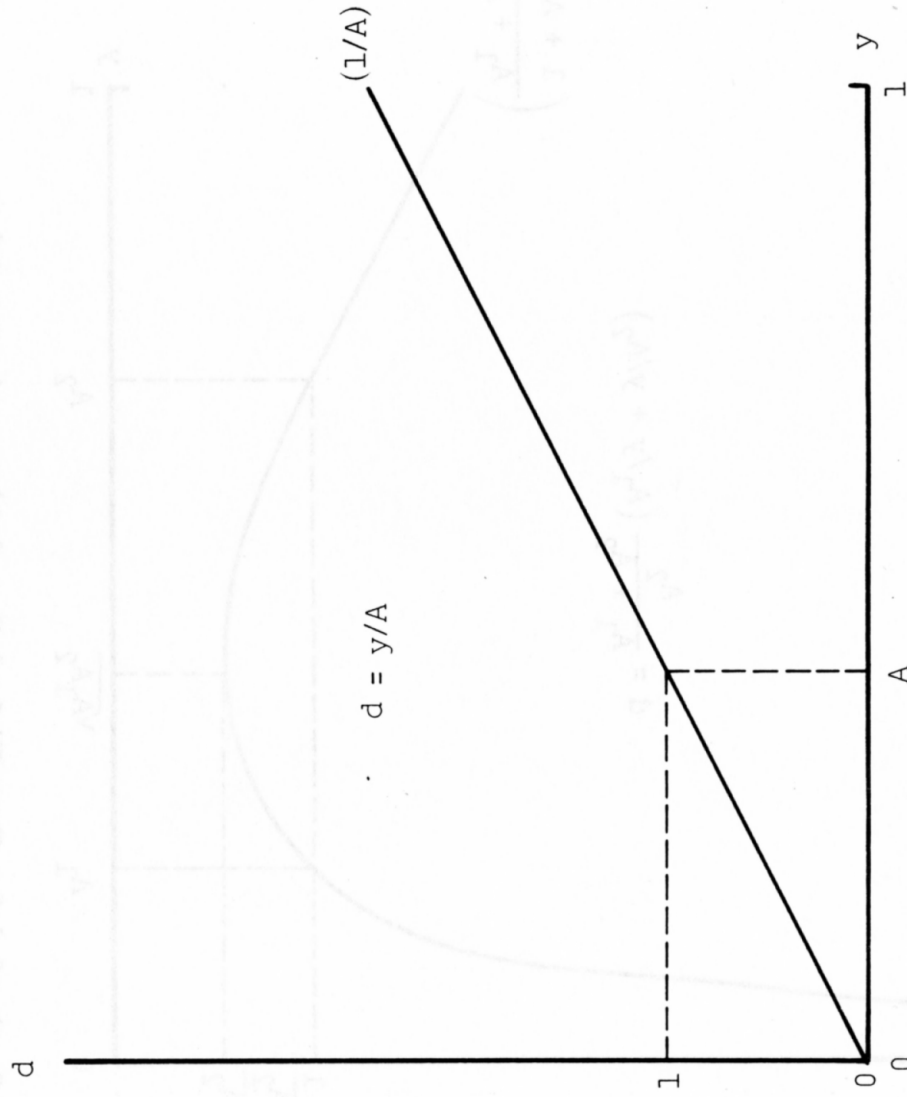


Figure 4.3 Graph of d for Type III Goal Formulations: $y \leq A$.

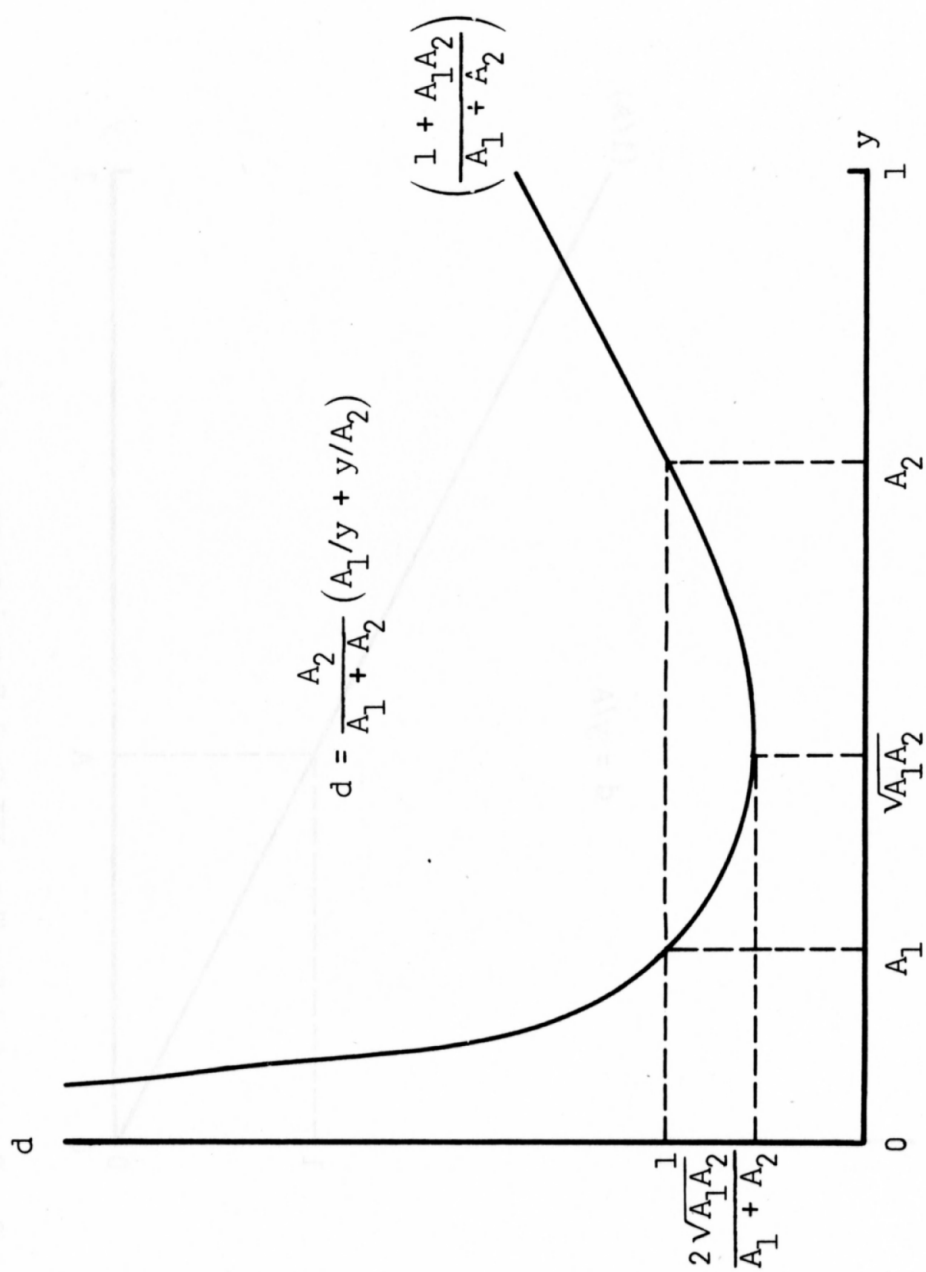


Figure 4.4 Graph of d for Type IV Goal Formulations: $A_1 \leq y \leq A_2$.

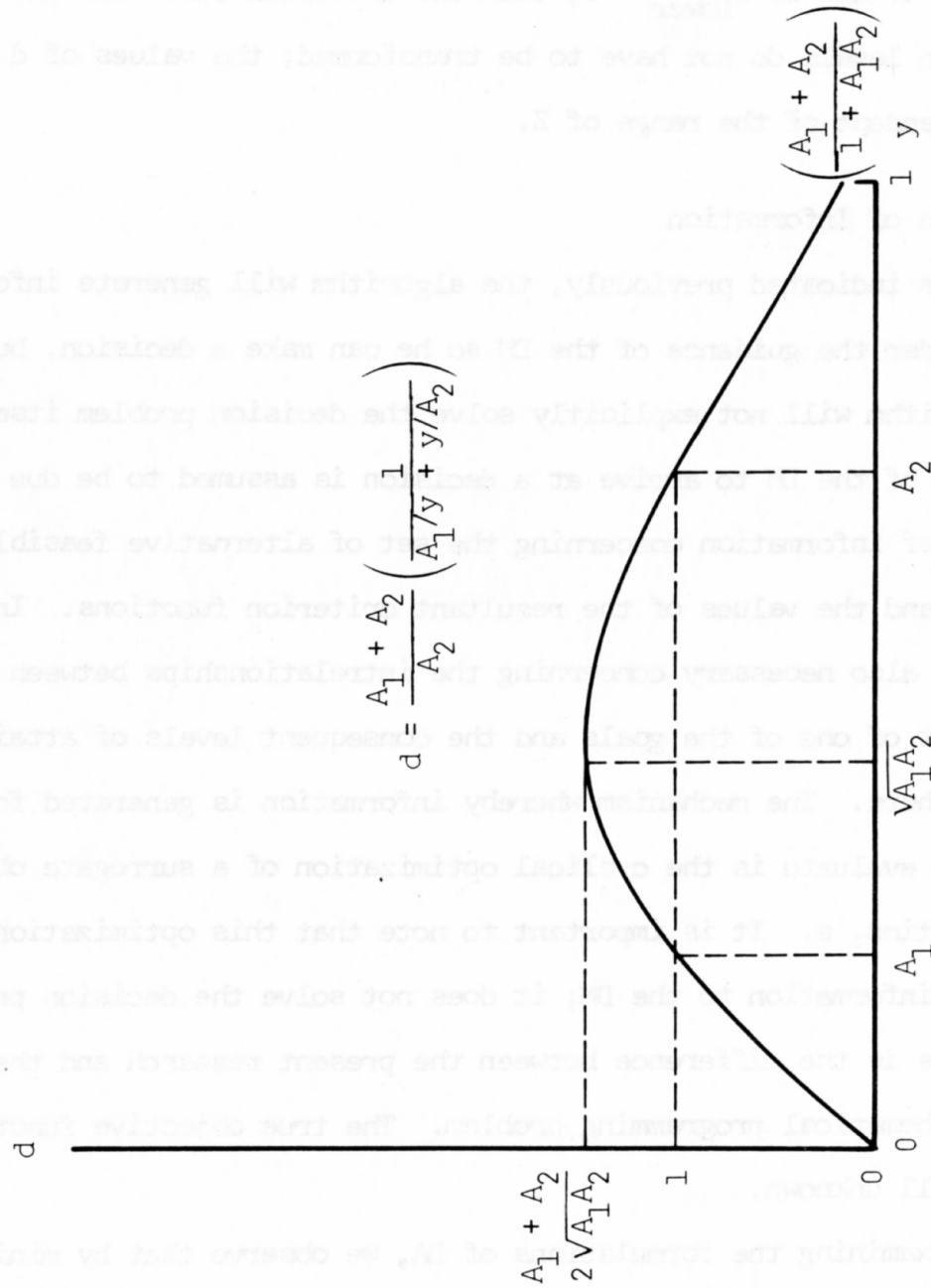


Figure 4.5 Graph of d for Type V Goal Formulations: $y \geq A_2$ or $y \leq A_1$.

Observe that if all original goal formulations are of types I, II, or IV and if $z_{\text{lower}} = 0$, then the criterion functions and aspiration levels do not have to be transformed; the values of d are independent of the range of Z .

Generation of Information

As indicated previously, the algorithm will generate information under the guidance of the DM so he can make a decision, but the algorithm will not explicitly solve the decision problem itself. Inability of the DM to arrive at a decision is assumed to be due to his lack of information concerning the set of alternative feasible policies and the values of the resultant criterion functions. Information is also necessary concerning the interrelationships between attainment of one of the goals and the consequent levels of attainment of the others. The mechanism whereby information is generated for the DM to evaluate is the cyclical optimization of a surrogate objective function, s . It is important to note that this optimization only provides information to the DM; it does not solve the decision problem. This is the difference between the present research and the usual mathematical programming problem. The true objective function, P , is still unknown.

Examining the formulations of DA, we observe that by minimizing:

$$s = \sum_t d_t$$

the value of each term in the solution will reflect whether or not

that goal has been satisfied with unsatisfied goals having values greater than 1. Of course, the nonlinearity of the DA prevents direct comparison among the values of d .

Before leaving the subject of the construction of DA and s we repeat that the minimization of s will provide information to the DM to help define the next cycle in the search for a satisfactum. Our problem is to find a satisfactum p^0 . The function s is not a substitute for P nor does optimization of s imply optimization of P in any sense. We have introduced this function as a tool to help the DM explore his preferences. If optimization of s would optimize P , either directly or indirectly, then this problem would be a "standard" nonlinear programming problem. The construction of DA and s has been completely arbitrary subject only to the directional property of d already discussed. It would be equally valid, for the purposes of this research, to attempt some alternate formulations of s and the DA, such as

$$s = \sum_{i=1}^T d_i,$$

but we feel that the additional complexity would not add to our work. We will return to this point in Chapter 7.

Iteration Scheme

Optimization Phase. Assume that the initial aspiration levels, \underline{AL}_0 are equal to the goal levels, \underline{GL} . Then the search for a satisfactum begins by solving $T+1$ optimization problems. The first is minimize:

$$s_1 = \sum_{t=1}^T d_t$$

subject to the original constraint set $\underline{h} = 0$ and $\underline{g} \geq 0$. T other problems are added by removing d_k from the surrogate objective function and adding the corresponding goal formulation as a constraint (a type IV or V formulation would involve 2 constraints). So minimize:

$$s_{1.k} = \sum_{\substack{t=1 \\ t \neq k}}^T d_t \quad k = 1, \dots, T$$

subject to the original constraints and $GF_k \geq 0$. The resultant optimal policy vectors are $\underline{x}_1, \underline{x}_{1.1}, \dots, \underline{x}_{1.T}$. Correspondingly, we calculate $\underline{z}_1, \underline{z}_{1.1}, \dots, \underline{z}_{1.T}$. There will now be up to $T+1$ feasible alternatives for the DM to evaluate. If some of the original goals cannot be attained even at the expense of the others, then there will be less than $T+1$ alternatives.

For example, let

$$\underline{AL}_0 = \underline{GL} = (300, 200, 400)$$

for 3 type II goals. We minimize s_1 and obtain

$$\underline{z}_1 = (200, 100, 300)$$

and so none of the goals have been attained at the current aspiration

level. Next add a constraint $z_1 \geq 300$ and delete d_1 from the objective function. Then minimize:

$$s_{1.1} = d_2 + d_3$$

and obtain:

$$\underline{z}_{1.1} = (300, 50, 200)$$

In a similar manner, obtain:

$$\underline{z}_{1.2} = (200, 200, 50)$$

and

$$\underline{z}_{1.3} = (25, 75, 400)$$

The DM can now examine these 4 feasible alternatives and if any are satisfactory the process terminates.

It may appear that none of the results in this example could be satisfactory because the aspiration levels, \underline{AL}_0 , were not attained for all goals at once. We must recall, however, that the aspiration levels have themselves changed as a consequence of this information. Knowing that he cannot attain \underline{AL}_0 , the DM may be content with one of the alternatives.

Evaluation Phase. The optimization phase provides information to estimate the effects of changing the aspiration level for any of the goals.

We can compare \underline{z}_1 and $\underline{z}_{1.1}$ visually in Figure 4.6. The shaded portion of each axis indicates goal attainment relative to \underline{AL}_0 .

Assuming the z_i are linearly related, dz_2/dz_1 and dz_3/dz_1 are constants. The DM uses this information to predict the approximate levels of z_2 and z_3 for a given aspiration level with z_1 entered as a constraint. Denote a desired level of attainment on goal 1 by \hat{AL}_1 . Then we can calculate the effect of this estimate entered as a constraint. We have:

$$\Delta \hat{z}_2 = (z_2^{1.1} - z_2^1) \cdot (\hat{AL}_1 - z_1^1) / (z_1^{1.1} - z_1^1)$$

and

$$\Delta \hat{z}_3 = (z_3^{1.1} - z_3^1) \cdot (\hat{AL}_1 - z_1^1) / (z_1^{1.1} - z_1^1)$$

Rather than ask the DM to solve these equations for trial values of \hat{AL}_1 , we observe that lines passing through m_1 and m_2 solve the equations graphically. (The points m_1 and m_2 are the points of intersection of z_1 and $z_{1.1}$ in Figure 4.6.) So the DM can use a ruler to "try out" different \hat{AL}_1 to see how they affect the other goals. This has been done for:

$$\hat{AL}_1 = 250$$

giving

$$\hat{z}_2 = 75$$

and

$$\hat{z}_3 = 250$$

This part of the example has implied that the first goal is the most important to the DM. We assume that he will examine the graphs for those goals which are the most vital to him, entering in trial aspiration levels as constraints in the manner indicated.

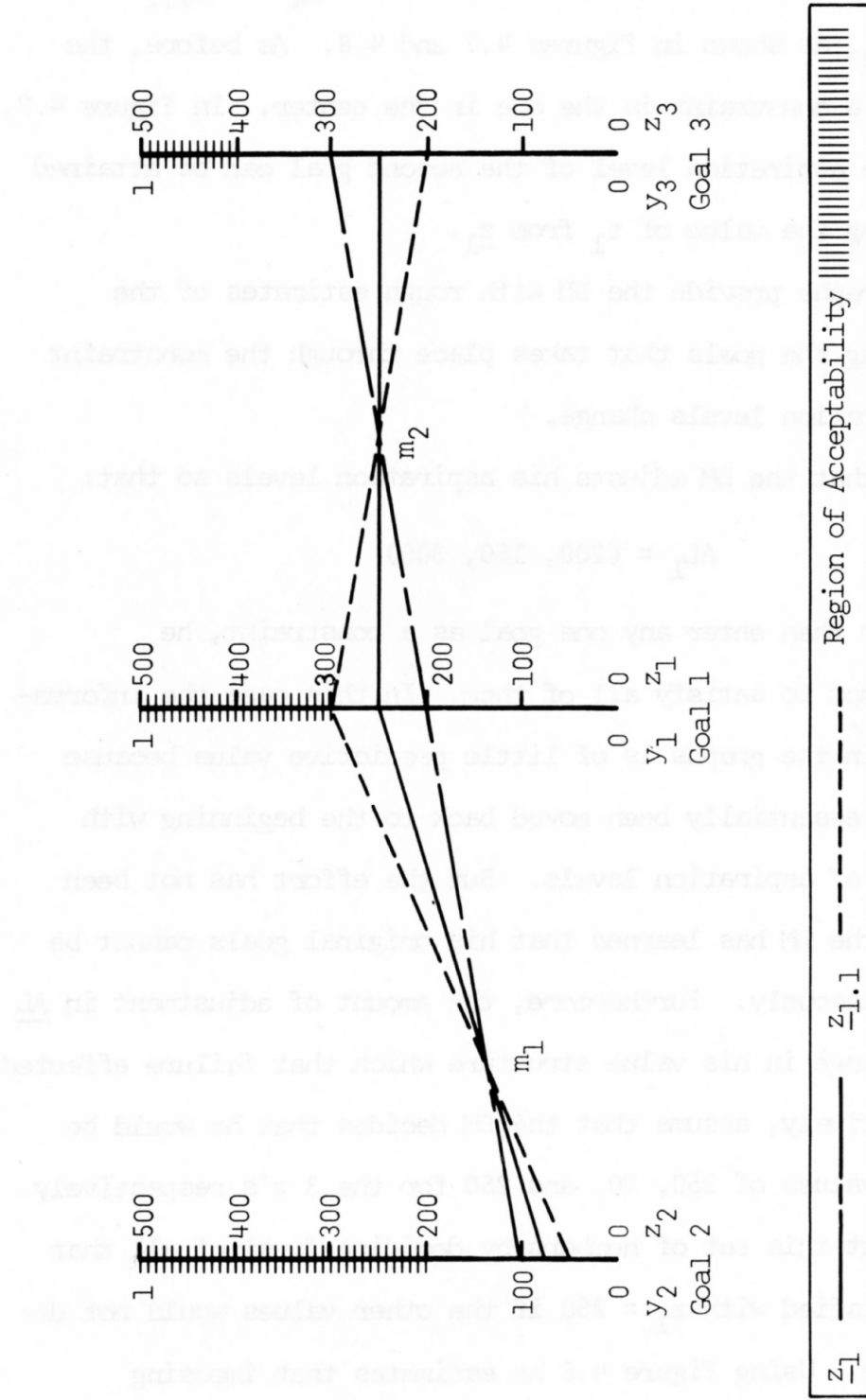


Figure 4.6 Goal 1 entered as a constraint.

We can also construct comparisons between \underline{z}_1 and $\underline{z}_{1.2}$, and also \underline{z}_1 and $\underline{z}_{1.3}$, as shown in Figures 4.7 and 4.8. As before, the goal entered as a constraint is the one in the center. In Figure 4.7, observe that the aspiration level of the second goal can be attained without affecting the value of \underline{z}_1 from \underline{z}_1 .

These graphs provide the DM with rough estimates of the interaction among the goals that takes place through the constraint set as his aspiration levels change.

Assume that the DM adjusts his aspiration levels so that:

$$AL_1 = (200, 150, 300)$$

and that, rather than enter any one goal as a constraint, he prefers to attempt to satisfy all of them. In this case the information contained in the graphs is of little predictive value because the problem has essentially been moved back to the beginning with a different set of aspiration levels. But the effort has not been wasted because the DM has learned that his original goals cannot be attained simultaneously. Furthermore, the amount of adjustment in \underline{AL} reflects the change in his value structure which that failure effected.

Alternatively, assume that the DM decides that he would be satisfied with values of 250, 70, and 250 for the 3 z 's respectively. He has arrived at this set of numbers by deciding first of all that he would be satisfied with $\underline{z}_1 = 250$ if the other values would not decrease too badly. Using Figure 4.6 he estimates that imposing $\underline{z}_1 \geq 250$ as a constraint would result in a \underline{z}_2 level of 70 and a \underline{z}_3

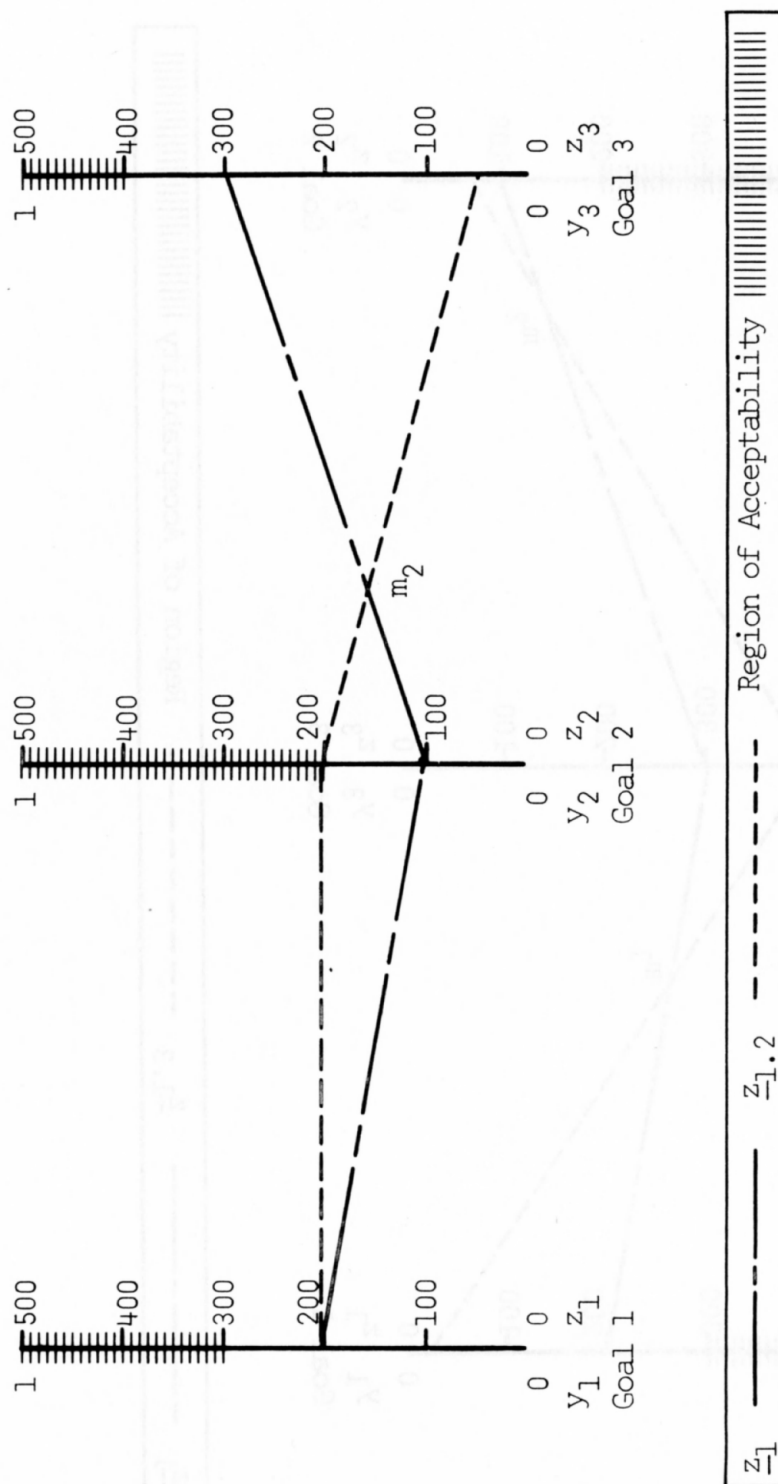


Figure 4.7 Goal 2 entered as a constraint.

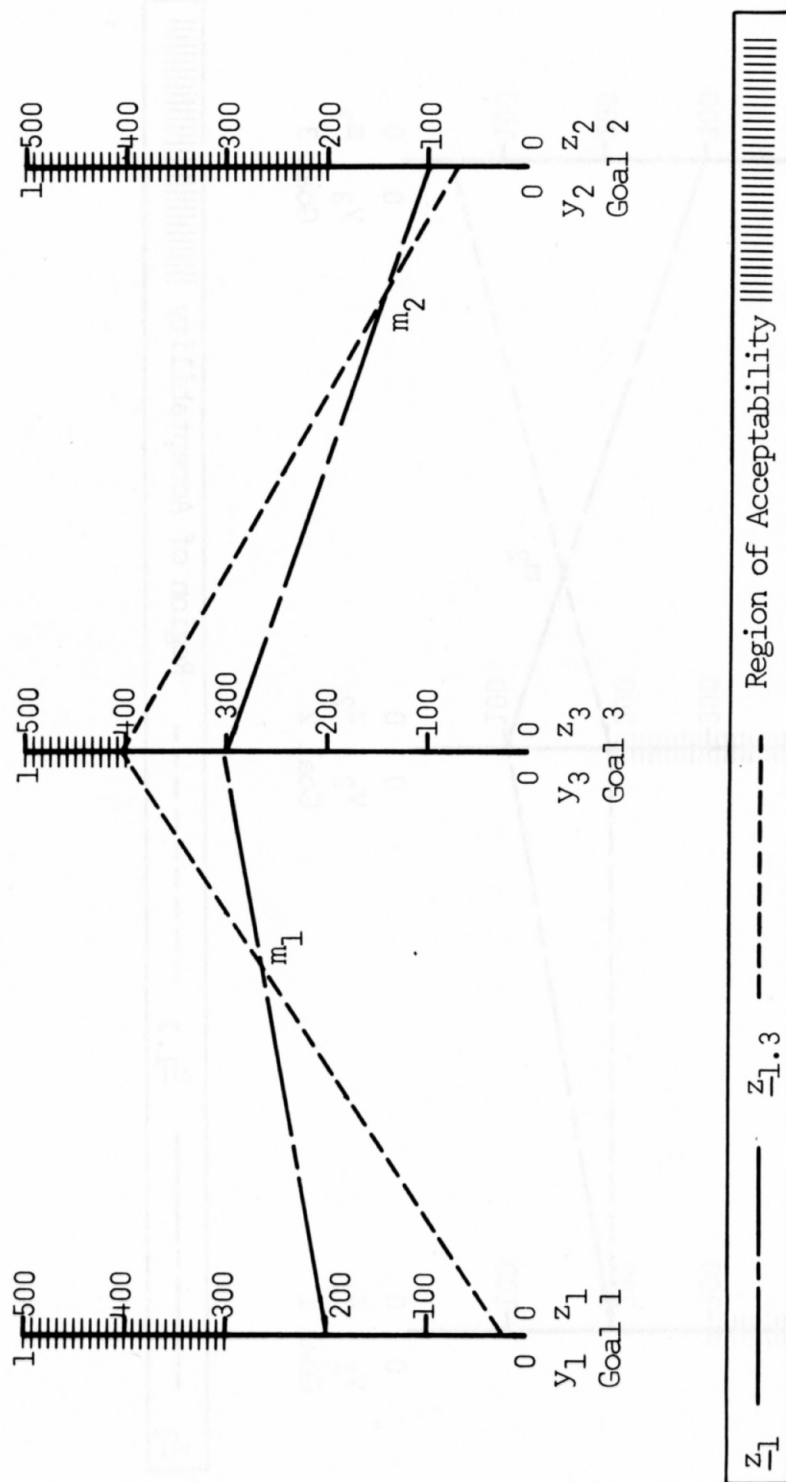


Figure 4.8 Goal 3 entered as a constraint.

level of 250 as linear approximations. This he can accept and so the new aspiration level vector is:

$$\underline{AL}_1 = (250, 200, 400) \quad 4.1$$

and the objective function is:

$$s = d_2 + d_3$$

with goal 1 entered as a constraint.

The reader may ask, why was the above aspiration vector used for \underline{AL}_1 instead of:

$$\underline{AL}_1 = (250, 70, 250) \quad 4.2$$

An argument can certainly be made that the DM, by his action of estimation, has in effect altered his aspiration levels for all of the goals rather than for only the first. There are several reasons for our choosing the former values for \underline{AL}_1 . First, we are supposing a serial process in which the aspiration levels are changed for only one goal at a time. Second, if the latter expression is used, the DM will not be able to determine from the next optimization phase if goals 2 and 3 could still be independently satisfied at the original levels. The goal formulations are entered as constraints with the current aspiration levels. Thus, on the auxiliary problem of the next cycle, the use of equation 4.1, would lead to the constraint for the second goal

$$g_1 = Z_2 - 200 \geq 0$$

being added to the constraint set, while equation 4.2 leads to

$$g_1 = Z_2 - 70 \geq 0$$

Satisfying the second constraint does not necessarily satisfy the first. We will give an example of this in Chapter 6 as part of the Bow River case study.

However, it is true that if the DM raises (in the sense of "making more difficult to achieve") his original goals at some cycle in the algorithm or raises (as above) a goal which has been lowered and entered as a constraint, then the algorithm will go through a cycle in which it is not known if the new levels can be achieved. The procedure is analogous to turning from one path to another--a new direction of travel is formed.

There is an additional psychological rationale for changing only 1 element of AL on each cycle. When the DM enters the new aspiration level as a constraint because the criterion function values are acceptable, he does so not on the basis of their individual values, but rather on the basis of their grouped impact as compared to the one goal he is examining (the Gestalt philosophy again). The values of the z_i which the DM "accepts" can be thought of as the current lower bounds upon his region of acceptability. He would, if necessary, accept values of $z_2 = 75$ and $z_3 = 250$ for $z_1 = 250$. However, he may wish to do some trading off between the second and third goals. Consequently, he now wishes to know if those goals can be individually satisfied for $z_1 = 250$. He still is serially processing the problem and will, in effect, form a new problem on the next cycle which consists of all the unsatisfied goals. The DM is "whittling

away" at the original problem a goal at a time. At the end of the process he has, hopefully, solved the original decision problem.

At this point the algorithm returns to the optimization phase and calculates $\underline{z}_{2.1}$, $\underline{z}_{2.12}$, and $\underline{z}_{2.13}$. These results are again evaluated in the same manner as previously. Observe that $\underline{z}_{2.1}$ is not equal to $\underline{z}_{1.1}$ because \underline{AL}_0 is not equal to \underline{AL}_1 .

So the i^{th} cycle consists of the calculation of $\underline{z}_{i.jk\dots}$ and $\underline{z}_{i.jk\dots m}$ where m ranges through all goals not entered as constraints on the principle problem of the i^{th} cycle. The number of secondary subscripts (j , k , etc.) is one less than the value of i . Recall that on the first cycle we find \underline{z}_1 and then $\underline{z}_{1.1}$ through $\underline{z}_{1.T}$. So we envision a process in which an additional goal is added to the constraint set on each cycle; this sequence corresponds to the importance of the goals.

However, the relative importance of each goal may shift as the flow of information progresses. In this case, the DM may start over at any point in the sequence. This could occur if a goal which initially was felt to be important was found to be relatively insensitive to changes in the aspiration levels and the other goals in the constraint set. In effect it maintains a stable z value. So its importance as part of the decision problem may be devalued by the DM.

At this time, the process returns to the optimization phase unless the DM desires to terminate the algorithm.

Termination. The search continues until a satisfactum is found. It is possible, of course, that no satisfactum can be found

due to the physical nature of the problem and a disinclination of the DM to modify his aspiration levels. Normally, however, the learning process that takes place as the search develops will provide the DM with alternatives consisting of feasible policy vectors, \underline{x} , and their corresponding results, \underline{z} , from which he can choose a policy which will be satisfactory.

Figure 4.9 summarizes the sequence of operations in this algorithm and the interactive nature of the technique.

Psychological Convergence

The termination of the algorithm implies convergence in two respects. First, the method of nonlinear optimization used within each iteration must converge to a solution for that particular cycle. Second, the DM must experience a "psychological convergence" such that he actually finds an acceptable alternative from the series of results generated by the search process. The next chapter is devoted to the development of the optimization program used in this research and questions of numerical convergence will be discussed there. In this section, we will discuss psychological aspects of termination.

Clearly, if the DM elects to continue restarting the search process there are an infinite number of locations possible because the variables are continuous. We argue that he will not do this for one of two reasons: (1) he finds a satisfactum in the manner that has been described; or (2) he insists that he cannot find a satisfactory alternative and terminates the procedure from sheer frustration.

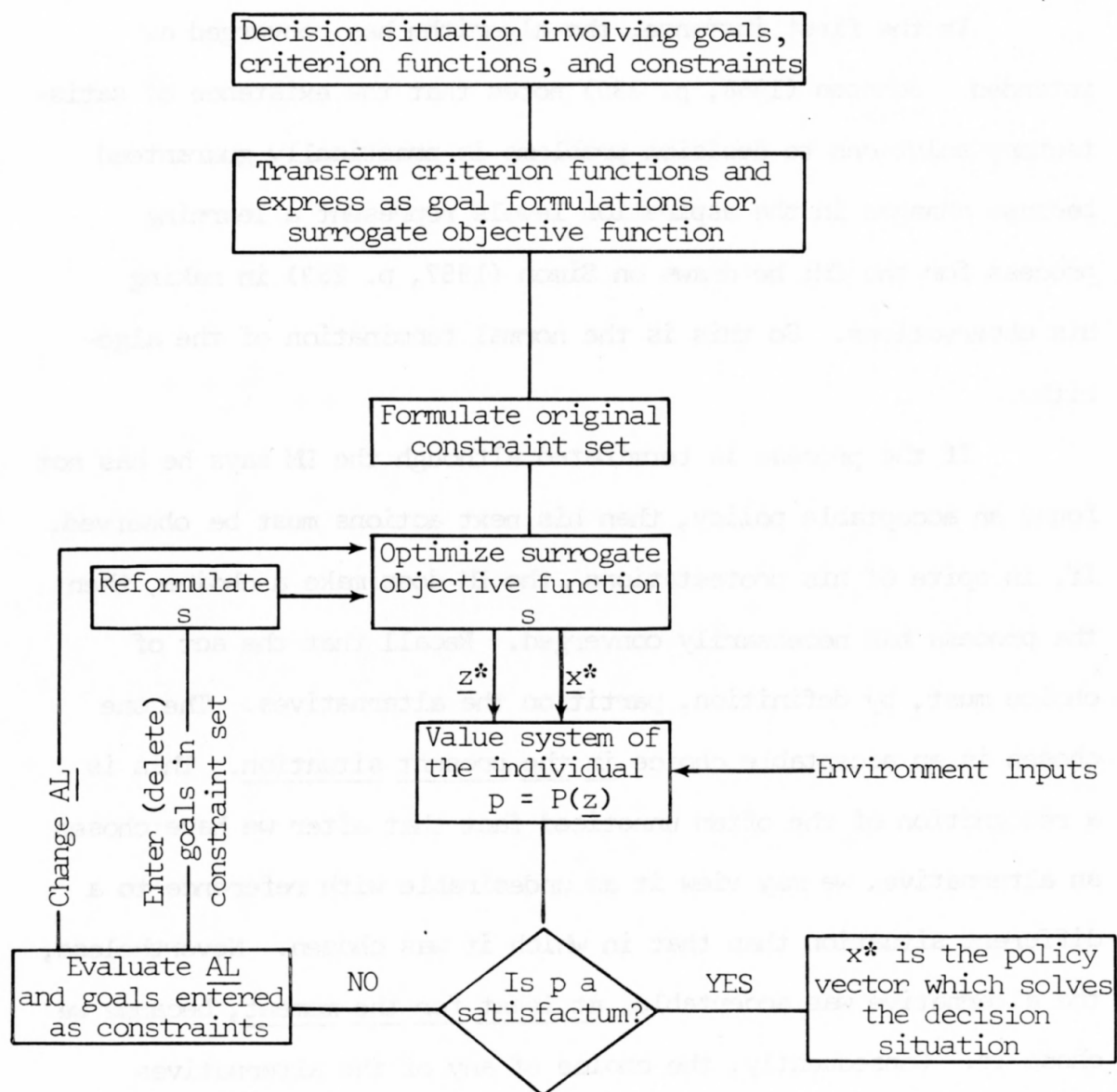


Figure 4.9 Simplified flowchart of the interactive algorithm.

In the first instance, the algorithm has converged as intended. Johnsen (1968, p. 339) notes that the existence of satisfactory solutions to decision problems is practically guaranteed because changes in the aspiration levels represent a learning process for the DM; he draws on Simon (1957, p. 253) in making his observations. So this is the normal termination of the algorithm.

If the process is terminated although the DM says he has not found an acceptable policy, then his next actions must be observed. If, in spite of his protestations, the DM does make a choice, then the process has necessarily converged. Recall that the act of choice must, by definition, partition the alternatives. The one chosen is an acceptable choice in the present situation. This is a recognition of the often unnoticed fact that after we have chosen an alternative, we may view it as undesirable with reference to a different situation than that in which it was chosen. Nevertheless, the alternative was acceptable, at least for the moment, because we chose it. Consequently, the choice of any of the alternatives presented by the search indicates a satisfactum.

But what if the DM just "throws in the towel?" He refuses to make a choice, insisting that none of the alternatives is acceptable. In this instance, then, the algorithm has failed to converge; no satisfactory solution to the decision problem has been located. Of course, in a broader sense, we can view the action of the DM as the substitution of a new problem for the old one, the new one being

"How can I stop all this?" Termination is actually a satisfactory alternative to that problem. And, even in this instance, we can conjecture that the information the DM has received may be of future positive value either in reformulating the problem, in working to modify the problem, or in leading to a gradual change in his value structure so that one of the alternatives will be acceptable.

In the next chapter, we will proceed with the development of an optimization technique which will be included in our overall algorithm.

CHAPTER 5

THE NONLINEAR OPTIMIZATION PROGRAM

Each cycle of the algorithm developed in the last chapter involves a number of nonlinear mathematical programming problems. These results are used by the DM in defining the problem for the next iteration. Consequently, we wish to choose a method which is both efficient, in terms of computer and set up time, and also accurate in that it actually converges to the true optimum in a finite amount of time. In particular, the technique chosen should be capable of detecting an inconsistent constraint set because mutually exclusive aspirations can be entered by the DM as constraints. The ease with which additional constraints can be inserted and deleted in the programming code will also affect our choice.

We will restate the optimization problem and then briefly review and evaluate some nonlinear optimization techniques to see which, if any, are applicable to the current research. The method actually used is then developed and the computer code for its implementation discussed.

Problem, Notation, Definitions

We have assumed that \underline{z} , \underline{h} , and \underline{g} are all differentiable (written \underline{z} is D) over the range of \underline{x} for the problem. However, the objective function which is actually optimized at each cycle is derived from the criterion functions, \underline{z} , and so we must examine the behavior of s .

The values of z are first normalized to y by a linear transformation, $L(z)$, so that if z is D, $L(z)$ is also. The values of y are then transformed to d so that $d \leq 1$ implies that the corresponding goal is satisfied. A general form of this transform is:

$$d = \left(\frac{1}{\alpha_1 y + \beta_1} \right) + \left(\alpha_2 y + \beta_2 \right)$$

(recall the five types of goals discussed in the last chapter).

Again $L(y)$ is D. The range of y is $(0,1]$ so that the fraction,

$$\frac{1}{\alpha_1 y + \beta_1}$$

is always defined. We restrict $\alpha_1 > 0$, $\beta_1 \geq 0$ so that for these conditions, the ratio is D.

Differentiation is a linear operation and so the sum of two differentiable functions is also differentiable; thus d is D , and s is D .

Then the problem to be solved in each iteration of the search process is of the form:

$$\min y = f(\underline{x}) \quad 5.1$$

subject to:

a set of L equality constraints

$$\underline{h} = \underline{H}(\underline{x}) \quad 5.2$$

a set of M inequality constraints

$$\underline{g} = \underline{G}(\underline{x})$$

a set of N bounds

$$0 \leq \underline{b}_L \leq \underline{x} \leq \underline{b}_U < \infty$$

with

y , \underline{h} , and \underline{g} all differentiable

The function $f(\underline{x})$ is referred to as the objective function and the constraints collectively as the constraint set, C . Different \underline{x} vectors are distinguished by a superscript, \underline{x}^i . A point \underline{x}^0 which satisfies the constraint set is a feasible point, and the set of such points,

$$F = \{ \underline{x} : \underline{x} \text{ satisfies } C \}$$

is the feasible region. If no feasible points exist (F empty), the constraint set is inconsistent.

The gradient of y is the vector of partial derivatives:

$$\nabla y = \nabla f(\underline{x}) = \begin{pmatrix} \partial y / \partial x_1 \\ \vdots \\ \partial y / \partial x_n \end{pmatrix}$$

When it is evaluated at a particular point \underline{x}^i , we write $\nabla y|_{\underline{x}^i}$.

The gradient points in a direction which maximizes the increase in y for a change in \underline{x} (see Beveridge and Schechter 1970, p. 410-412).

So $-\nabla y$ is the direction of greatest decrease.

A stationary point, \underline{x}^o , is one for which $\nabla y|_{\underline{x}^o} = \underline{0}$. If \underline{x}^o is in F and $f(\underline{x}^o + \underline{\Delta x}) \geq f(\underline{x}^o)$ for a small region of $\underline{\Delta x}$ around \underline{x} , then \underline{x}^o is a local minimum, \underline{x}^* . A local minimum is also a global minimum, \underline{x}^{**} , if $f(\underline{x}) \geq f(\underline{x}^*)$ for all $\underline{x} \in F$. The corresponding values of the function are y^o , y^* , and y^{**} .

If there are no equality constraints and if \underline{x}^* satisfies the inequalities so that $\underline{g} > 0$ and $\underline{b}_L < \underline{x}^* < \underline{b}_U$, then the constraints are "loose" and the solution is at an interior point of F . If any of the inequalities are satisfied as equalities, those constraints are "tight" and the solution is on the boundary of F . An interior solution indicates that the minimum value of the constrained function is equal to the minimum value of the unconstrained function.

If there are equality constraints present, then the feasible region becomes a hypersurface in E^n (Euclidean n -space where " n " is the dimensionality of \underline{x} , the number of variables in the problem) and all feasible points are necessarily on the boundary.

"Convergence" of an algorithm implies that a sequence of points $\underline{x}^1, \underline{x}^2, \dots$ resulting from applying an algorithm to a problem tends toward a solution (see Wolfe, 1970).

Review of Nonlinear Optimization Techniques

There are many nonlinear optimization techniques available, both analytical and numerical. These two groups are distinguishable in that analytical methods are essentially algebraic while numerical methods are geometric.

Analytical methods attempt to solve problems by deriving equations which define the solution. That is, given a problem consisting of a set of equations involving unknowns and parameters, these techniques develop expressions relating the unknown variables to the known parameters. However, the complexity of the analytic approach to nonlinear optimization limits its applicability to problems having few or no constraints and few (less than 5) variables. These considerations prompt us to rule out an analytic solution to the problem at hand.

For example, the method of Lagrange multipliers converts a problem in N unknowns and K constraints into one involving $N + K$ variables. But it requires the solution of $N + K$ simultaneous, potentially nonlinear equations.

Numerical methods use iterative processes which attempt to move from a point \underline{x}^i to a new point \underline{x}^{i+1} such that (for minimization) $f(\underline{x}^{i+1}) \leq f(\underline{x}^i)$. Many writers have discussed numerical nonlinear optimization in general. Wolfe (1963) and Box, Davies, and Swann

(1970) have written summaries of the numerical methods available. Dorn (1963) has also surveyed the field, while Beveridge and Schechter (1970) are more exhaustive in their treatment. Zoutendijk (1970a) provides some numerical examples for different techniques. Davies (1970) discusses some practical aspects of implementing different techniques. A very concise classification scheme with references is contained in Zoutendijk (1970b, pp. 546-54).

For our purposes, we can simply classify the methods as direct search (derivatives not required) and gradient (derivatives required).

Direct Search Methods

All of the direct search methods require evaluation of the objective functions at particular locations. The many variations differ in two ways: (1) the selection of the points at which the function is to be evaluated, and (2) the determination of the direction of search on the basis of those evaluations.

At one extreme, there is the univariate search method which changes one variable at a time, searching sequentially along an axis. The Sequential Simplex and the Complex Method (Constrained Simplex) of Box (1965) are more involved and use geometric polyhedrons to define the points at which the surface is to be evaluated. Powell (1964) and Rosenbrock (1960) have also developed sophisticated methods.

Gradient Methods

These methods move from \underline{x} to $(\underline{x} + a\underline{v})$ such that (for minimization) $f(\underline{x} + a\underline{v}) < f(\underline{x})$. The vector \underline{v} is parallel to the gradient of the function, $\nabla f(\underline{x})$, and "a" is a parameter controlling the amount of movement (referred to as "step size"). The mechanism by which "a" varies during the optimization and the means by which constraints are handled account for the differences between the methods.

Davidon (1959) has developed a very sophisticated method for determining unconstrained optima. However, it requires the calculation of second derivatives and developing the analytical expressions for them can be quite complex. For example, if the objective function is highly nonlinear and involves, say, 20 variables, the determination of all second derivatives would be formidable.

When constraints are added, the problem can be converted to an unconstrained objective function through the use of a penalty term which differs from zero when the constraints are not satisfied. Alternatively, a search for a feasible point can be made by minimizing a function of the unsatisfied constraints; then an unconstrained method can be employed. However, each point in the series of iterations must be checked for feasibility.

Reported Research

The many types of nonlinear optimization problems encountered make it difficult to compare different techniques. We will critically review some recent studies and then attempt to interpret them in view of our particular problem.

Wolfe (1963) made an examination of ten techniques but felt that insufficient evidence had been accumulated with regard to their usage to reach any conclusions.

In 1965, M. J. Box reported a study of direct search techniques with constrained problems. He studied the Complex algorithm (Box 1965), a modification of Rosenbrock's method (1960), and the original Rosenbrock method. In 1966 he compared those methods with that of Powell (1964) used in conjunction with transformations of the original variables to eliminate the constraints. He concludes that the use of transformations is a superior means of dealing with constrained problems. For example, if x is bounded by the interval $[a, b]$, that constraint can be removed by substituting

$$x = (b-a)|\sin(y)| + a$$

where y is unbounded. In 1970 he examined constrained optimization at some length and concluded that Carroll's (1961) method is quite effective when used with an algorithm for unconstrained optimization. (In all of this research Box considered problems involving up to 20 variables.)

Colville (1967) compared the results of applying 34 different algorithms to eight different problems. He finds that first, the coding of the algorithm for the computer greatly affects its performance; second, methods employing analytical derivatives are superior to those which use numerical evaluation of derivatives or no derivatives at all; third, the performance of many of the methods examined is problem dependent.

Research by Stocker (1969) compared five methods on 15 problems. Two of the methods are direct search, two are small step gradient techniques, and the fifth involves second derivatives. Four of the five methods had been previously evaluated by Colville (1967) while the last is based on research begun by DiBella (1963) and continued by Barnes (1967). Stocker finds that the direct search programs are undesirable because the computation time becomes excessive as the number of variables increases. He notes that one of the two direct search codes is most efficient with equality constraints while the other seems limited to inequality constraints. The gradient method involving numerical derivatives was inferior to all other methods. The two gradient techniques using analytical expressions were superior to direct search. The use of second derivatives is fraught with the possibility of human error so that methods employing that technique were not recommended. The code judged best overall was the one developed by DiBella and Barnes. It combines a gradient technique to locate feasible points with a linear programming approximation for moves within the feasible region.

Evaluation

Direct search methods for small problems can be made very efficient with regard to computer time. Box, Davies, and Swann (1970) regard them as preferable to gradient techniques that employ numerical derivatives regardless of the size of the problem. However, as the number of variables and constraints increases (i.e., more than 20 variables and constraints) the time required to find a solution becomes excessive. Since we anticipate the potential application of this method to large-scale problems, we do not view direct search as a practical method.

Gradient techniques are much more powerful (with regard to convergence time) but require more preparation for computer coding because derivatives must be taken. In general, gradient techniques for constrained problems employ either a penalty function or else attempt to locate a feasible point by minimizing a function (usually the sum or sum of squares) of the unsatisfied constraints. (See, for example, Wolfe 1963, pp. 69-70, Beveridge and Schechter 1970, pp. 443-449, or Davies 1970, pp. 98-99.) An unconstrained method is then used to find the optimum.

There are other gradient-based methods, however, which do not use this approach for constrained optimization. In these techniques, the gradient of the objective function defines the direction of search subject to constraints that have been linearized about a point. Some

examples are the cutting plane technique of Kelley (1960), the method of feasible directions due to Zoutendijk (1960), the gradient projection method of Rosen (1961), and the approaches of Hartley (1960) and Glass and Cooper (1965). The approach of Kelley (1960) as developed by Griffith and Stewart (1961) is attractive from a computational viewpoint and relatively easy to code for a computer. We have adopted their method and will present it in the next section.

A Linearization Technique

History of the Cutting-Plane Method

The term "cutting plane" was introduced by Gomory (1958) with regard to integer programming and adopted by Kelley (1960) to describe his method of nonlinear programming. Several variations of it have since been developed (Zangwill 1969, Ch. 14). Wolfe (1963, p. 82) describes this technique as:

. . . based on the idea that it (the constraint set) can be represented as the intersection of a sufficiently numerous set of half-spaces which contain it. . . . The main tool of the procedure is the representation of the constraints by first-order Taylor's series expansions.

Lest the reader be concerned that such approximations are "taboo," we hasten to note another, more recent statement by Wolfe (1970, p. 4):

The foundation of all our work in nonlinear programming is our ability to handle linear relationships, and linear approximations to nonlinear phenomena will be found at the bottom of every algorithm and every theorem in the field. It is thus not grossly overstating the case to say that the results we can get depend almost entirely on what we can determine about the relationships between a function f and the approximation to it given by the linearization

$$f(y) + u^T(x-y)$$

as a function of x , where y is some fixed point (and T denotes the transpose of u).

Collatz (1970, p. 286) echoes this comment:

For the general nonlinear problem no general theory exists and one is using for numerical purposes in most cases Newton's method and related methods.

Development of the Cutting-Plane Algorithm of Griffith and Stewart

In the optimization algorithm formulated by Griffith and Stewart (1961), a linear programming (LP) algorithm is iteratively applied to a nonlinear problem so that the solution of a linear problem converges to the solution of the nonlinear problem. Each LP iteration is the result of treating the gradient at a point as an objective function subject to linearized constraints (See Wolfe 1970, p. 26). The formulation which follows is that of Griffith and Stewart (1961) except as noted.

Returning to the problem defined by equations 5.1, 5.2, 5.3, and 5.4, we can linearize the functions by expanding them in a Taylor series and retaining only the first order terms. A prime denotes the transpose, vectors are normally column vectors, and $J_{\underline{x}, \underline{y}}$ is a Jacobian matrix

$$J_{\underline{x}, \underline{y}} = \begin{bmatrix} \partial x_1 / \partial y_1 & \dots & \partial x_1 / \partial y_m \\ \vdots & & \vdots \\ \partial x_n / \partial y_1 & \dots & \partial x_n / \partial y_m \end{bmatrix}$$

We have

$$f(\underline{x} + \underline{\Delta x}) = f(\underline{x}) + \nabla f(\underline{x})' \underline{\Delta x}$$

$$\underline{H}(\underline{x} + \underline{\Delta x}) = \underline{H}(\underline{x}) + \underline{J}_{\underline{H}, \underline{x}} \underline{\Delta x}$$

$$\underline{G}(\underline{x} + \underline{\Delta x}) = \underline{G}(\underline{x}) + \underline{J}_{\underline{G}, \underline{x}} \underline{\Delta x}$$

so

$$\Delta f(\underline{x}) = \nabla f(\underline{x})' \underline{\Delta x} \quad 5.5$$

$$\Delta \underline{H}(\underline{x}) = \underline{J}_{\underline{H}, \underline{x}} \underline{\Delta x} \quad 5.6$$

$$\Delta \underline{G}(\underline{x}) = \underline{J}_{\underline{G}, \underline{x}} \underline{\Delta x} \quad 5.7$$

If $h_i \neq 0$, then that constraint is violated by an amount $|h_i|$. Similarly, if $g_j < 0$, that constraint is violated by an amount $|g_j|$. From the linear approximation of the constraints evaluated at \underline{x} , we estimate that F can be entered if \underline{x} is changed an amount $\underline{\Delta x}$ such that $\Delta H_i(\underline{x}) = |h_i|$ and $\Delta G_j(\underline{x}) = |g_j|$ for $g_j < 0$. There will, in general, be an infinite number of change vectors, $\underline{\Delta x}$, that will satisfy the above conditions. However, the move should be made in such a manner as to also decrease $f(\underline{x})$. Or, if $f(\underline{x})$ is already at a minimum, then movement from that point to satisfy the constraints should be accomplished with a minimal increase in $f(\underline{x})$. In either instance, we wish to minimize $\nabla f(\underline{x})$.

The partial derivatives evaluated at a point are all scalars so that the system expressed by equations 5.6 and 5.7 is a set of linear equations. Similarly, $\nabla f(\underline{x}) \underline{\Delta x}$ is a linear function. So we

have a problem which can be solved by linear programming except that Δx can be negative. This difficulty can be circumvented by writing

$$\Delta x_i = \Delta x_i^+ - \Delta x_i^-$$

where

$$\Delta x_i^+ = \begin{cases} \Delta x_i & \Delta x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta x_i^- = \begin{cases} -\Delta x_i & \Delta x_i \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

The range of x_i is bounded so that

$$(b_{L_i} - x_i) \leq x_i \leq (b_{U_i} - x_i)$$

Then the change in x_i may be restricted by

$$\alpha_i \Delta x_i^+ + \beta_i \Delta x_i^- \leq a_i \quad 5.8$$

where

$$\alpha_i = \max \left(c_1, \frac{a_i}{b_{U_i} - x_i} \right)$$

$$\beta_i = \max \left(c_2, \frac{a_i}{x_i - b_{L_i}} \right)$$

a_i is the step size for variable i , and

c_1 and c_2 are a damping modification added by Barnes (1967) to reduce oscillations about ridges. If $\Delta x_i > 0$ on the previous move, $c_1 = 1$ and $c_2 = 2$. If $\Delta x_i = 0$ on the previous move (the starting condition),

$c_1 = c_2 = 1$. And if $\Delta x_i < 0$, $c_1 = 2$ and $c_2 = 1$. Griffith and Stewart use $c_1 = c_2 = 1$ for all circumstances. We added the condition for $\Delta x_i = 0$ to accelerate the progress from the initial point.

Equations 5.5, 5.6, and 5.7 can now be written:

$$\Delta f(\underline{x}) = \nabla f(\underline{x})' \underline{\Delta x}^+ - \nabla f(\underline{x})' \underline{\Delta x}^-$$

$$\Delta H(\underline{x}) = J_{H,\underline{x}} \underline{\Delta x}^+ - J_{H,\underline{x}} \underline{\Delta x}^-$$

$$\Delta G(\underline{x}) = J_{G,\underline{x}} \underline{\Delta x}^+ - J_{G,\underline{x}} \underline{\Delta x}^-$$

So the LP problems which are repetitively solved are of the form:

$$\text{minimize} \quad y = \nabla f(\underline{x})' \underline{\Delta x}^+ - \nabla f(\underline{x})' \underline{\Delta x}^-$$

subject to

$$J_{H,\underline{x}} \underline{\Delta x}^+ - J_{H,\underline{x}} \underline{\Delta x}^- = |h|$$

$$J_{G,\underline{x}} \underline{\Delta x}^+ - J_{G,\underline{x}} \underline{\Delta x}^- \geq |g|$$

$$\underline{\alpha} \underline{\Delta x}^+ + \underline{\beta} \underline{\Delta x}^- \leq \underline{a}$$

for $h_i \neq 0$, $g_j < 0$, and $\underline{\alpha}$ and $\underline{\beta}$ defined in equation 5.8. In the next section, we will discuss the simplex algorithm as implemented in the computer program.

The LP Algorithm

The basic theory of the Simplex solution of the LP is discussed by Hadley (1962, Chapters 3 and 4). We will assume the reader is familiar with the general algorithm and concentrate on some aspects which apply to our problem.

Hadley (1962, Section 4-5) discusses the use of an identity matrix composed of artificial and slack variables to obtain a starting basis. This idea is computationally convenient but, as he also states (1962, Chapter 5), the selection of the magnitude of the "cost" of the artificial variables can cause difficulties on a digital computer. These problems can be avoided, however, by recomputing for each tableau the criterion by which entering variables are chosen. This is the method we have chosen using a value of 10^{20} as the "cost" of the artificial variables.

Examples and Further Considerations

Examining the system in equation 5.9, we see that some of the constraints may be inconsistent because the linear approximation requires changes in the decision variables greater than the limits imposed by equation 5.8. Consequently, we expect that some of the artificial variables will remain in the solution. However, as we make successive moves the constraints will become satisfied.

We can exemplify this point and consider some additional issues through the use of a small example. For simplicity, we set $c_1 = c_2 = 1$ in equation 5.8. Our problem is:

$$\begin{array}{ll}
 \text{minimize} & y = x_1x_2 + x_1 + x_2 \\
 \\
 \text{subject to} & g_1 = x_1x_2 - 50 \geq 0 \\
 & 20 \geq x_1 \geq 0 \\
 & 20 \geq x_2 \geq 0
 \end{array}$$

The feasible region, F , is shown in Figure 5.1.

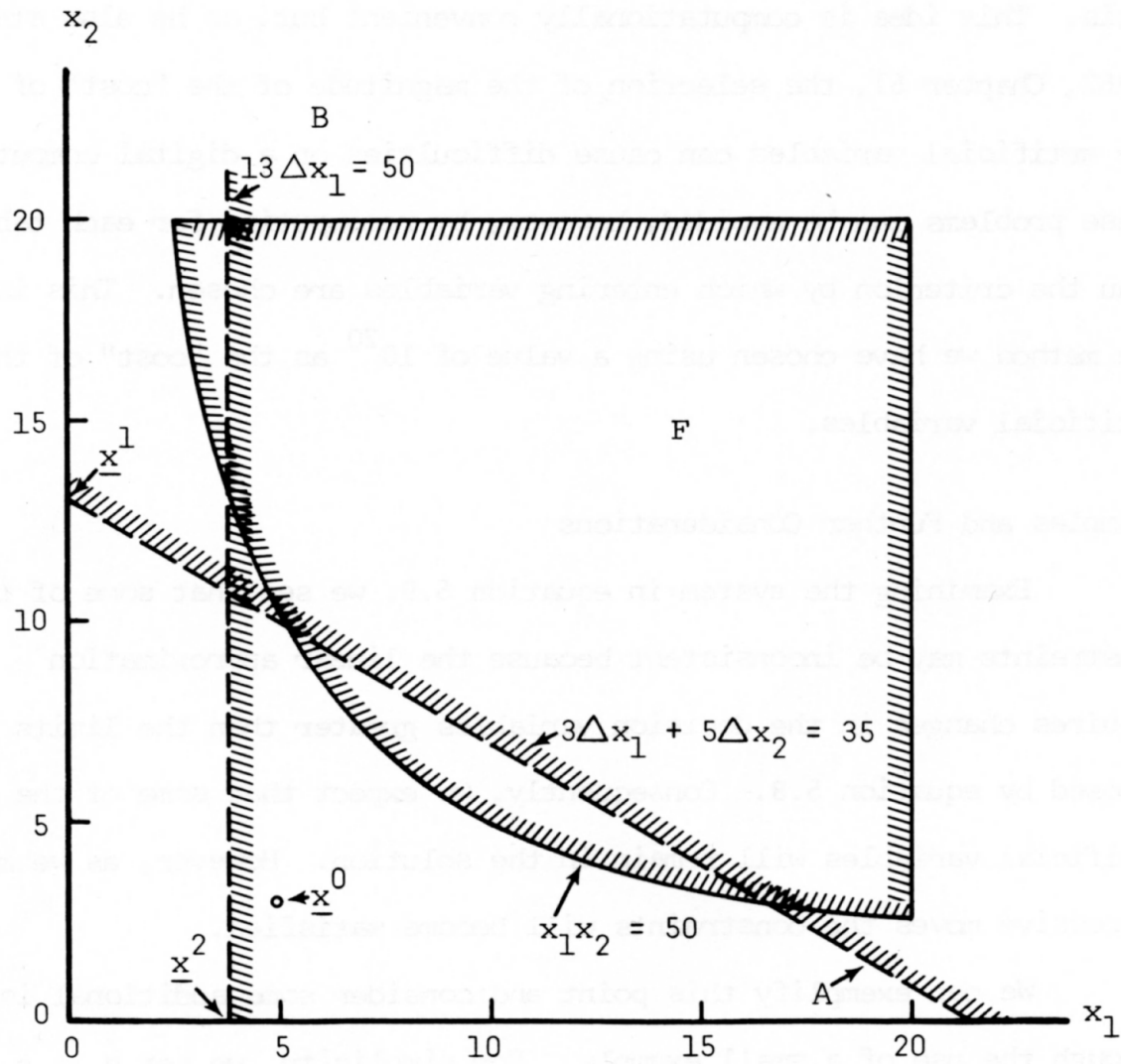


Figure 5.1 LP moves when stepsize is not limited

Initially, let the step size for each variable, a_i , be equal to the maximum of $(b_{U_i} - x_i, x_i - b_{L_i})$ so that the change in x_i is limited by the bounds on x_i . At a starting point $\underline{x}^0 = (5, 3)$, the value of g_1 is -35. The gradient of the constraint is:

$$\nabla g_1 = (x_2, x_1) = (3, 5)$$

so that the linearized constraint is:

$$\bar{g}_1 = 3\Delta x_1 + 5\Delta x_2 \geq 35$$

where the bar indicates the linearized version of the original constraint. The Δx_1 and Δx_2 axes have their origin at \underline{x}^0 so that we can graph the linearized constraint on the x_1 and x_2 axes. This is the line labelled "A" in Figure 5.1. Although Δx_1 and Δx_2 can be negative, they are constrained by equation 5.8 to keep x_1 and x_2 within their bounds. Graphically this is shown by the intersection of A with the x_1 and x_2 axes.

The gradient of y is:

$$\nabla y = (x_2 + 1, x_1 + 1) = (4, 5)$$

so that the LP problem becomes:

$$\begin{aligned} \text{minimize} \quad & 4\Delta x_1^+ - 4\Delta x_1^- + 6\Delta x_2^+ - 6\Delta x_2^- \\ \text{subject to} \quad & 3\Delta x_1^+ - 3\Delta x_1^- + 5\Delta x_2^+ - 5\Delta x_2^- \geq 35 \\ & 1\Delta x_1^+ + 3\Delta x_1^- \leq 15 \\ & \Delta x_2^+ + \frac{17}{3}\Delta x_2^- \leq 17 \end{aligned}$$

The solution to this problem is $\underline{x}^1 = (0, 13)$.

Repeating the process, we have a new LP problem:

$$\begin{array}{ll}
 \text{minimize} & 14 \Delta x_1^+ - 14 \Delta x_1^- + \Delta x_2^+ - \Delta x_2^- \\
 \text{subject to} & 13 \Delta x_1^+ - 13 \Delta x_1^- \geq 50 \\
 & \Delta x_1^+ + \alpha \Delta x_1^- \leq 20 \\
 & \frac{13}{7} (\Delta x_2^+) + \Delta x_2^- \leq 13
 \end{array}$$

where $\alpha \gg 0$ because of the division procedure in equation 5.8. The line B represents the linearized constraint for this problem. The solution to the LP occurs at $\underline{x}^2 = (50/13, 0)$.

It is apparent that wide oscillations can take place when there is no limitation on the step size for each move. We will now repeat the procedure setting $a_1 = a_2 = 1$. For convenience, the actual feasible region, F, is repeated in Figure 5.2. Beginning again at $\underline{x}^0 = (5, 3)$, we construct A representing the same constraint as before. But now additional constraints are imposed limiting Δx_i to 1. These are shown in Figure 5.2 by lines a, b, c, and d. It is apparent that the linearized constraint and the bounds on Δx_i are inconsistent. The LP problem is

$$\begin{array}{ll}
 \text{minimize} & 4 \Delta x_1^+ - 4 \Delta x_1^- + 6 \Delta x_2^+ - 6 \Delta x_2^- \\
 \text{subject to} & 3 \Delta x_1^+ - 3 \Delta x_1^- + 5 \Delta x_2^+ - \Delta x_2^- \geq 35 \\
 & 1 \Delta x_1^+ + 1 \Delta x_1^- \leq 1 \\
 & 1 \Delta x_2^+ + 1 \Delta x_2^- \leq 1
 \end{array}$$

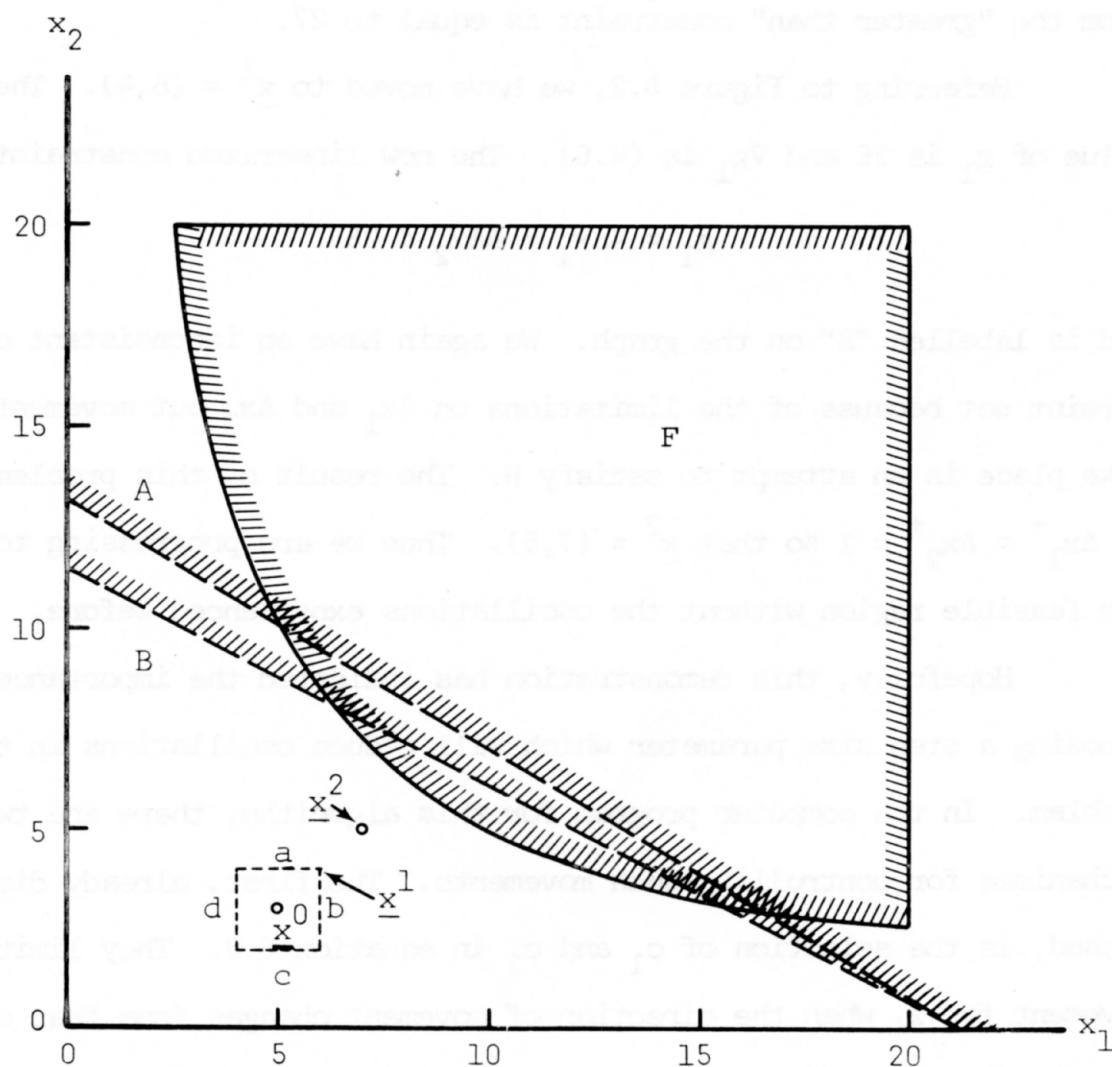


Figure 5.2 LP moves when stepsize is limited.

and although the problem is unsolvable, movement does take place within the region enclosed by lines a, b, c, and d. The final tableau from this problem shows $\Delta x_1^+ = \Delta x_2^+ = 1$ and the artificial variable from the "greater than" constraint is equal to 27.

Referring to Figure 5.2, we have moved to $\underline{x}^1 = (6,4)$. The value of g_1 is 26 and ∇g_1 is (4,6). The new linearized constraint is:

$$\bar{g}_1 = 4\Delta x_1 + 6\Delta x_2 \geq 26$$

and is labelled "B" on the graph. We again have an inconsistent constraint set because of the limitations on Δx_1 and Δx_2 but movement does take place in an attempt to satisfy B. The result of this problem is $\Delta x_1^+ = \Delta x_2^+ = 1$ so that $\underline{x}^2 = (7,5)$. Thus we are progressing toward the feasible region without the oscillations experienced before.

Hopefully, this demonstration has indicated the importance of choosing a step size parameter which will reduce oscillations in the problem. In the computer program for this algorithm, there are two mechanisms for controlling such movements. The first, already discussed, is the selection of c_1 and c_2 in equation 5.8. They limit the movement for x_i when the direction of movement changes from that of the previous LP. The second is an acceleration-deceleration scheme for varying the value of the step-size parameter. If four successive moves are made in the same direction, the step size is doubled; if a move is made in a direction opposite relative to the previous move, the step-size is halved. This procedure is taken from Barnes (1967). This allows us to locate "small" feasible regions and to converge to any desired degree of accuracy at a stationary point.

To continue the discussion, we will add the constraint

$$g_2 = x_2 + 2(x_1 - 10)^2 - 10 \geq 0$$

to our problem. Starting again at $\underline{x}^0 = (5, 3)$, we have $g_1 = -35$ and $g_2 = -2$. Initially, let $a_1 = a_2 = 1$. Then our first LP problem is:

$$\begin{aligned} &\text{minimize} && 4\Delta x_1^+ - 4\Delta x_1^- + 6\Delta x_2^+ - 6\Delta x_2^- \\ &\text{subject to} && 3\Delta x_1^+ - 3\Delta x_1^- + 5\Delta x_2^+ - 5\Delta x_2^- \geq 35 \\ &&& -2\Delta x_1^+ + 2\Delta x_1^- + \Delta x_2^+ - \Delta x_2^- \geq 2 \\ &&& \Delta x_1^+ + \Delta x_1^- \leq 1 \\ &&& \Delta x_2^+ + \Delta x_2^- \leq 1 \end{aligned}$$

In Figure 5.3, we have shown the feasible region for the nonlinear problem and the constraints formulated for this LP problem. The bounds on Δx_1 and Δx_2 will permit us to satisfy \bar{g}_2 , but the final tableau reveals that $\Delta x_1^+ = 1$ and $\Delta x_2^+ = 1$. Both artificial variables are still in the basis with values of 27 (down from 35) and 1 (down from 2).

Although \bar{g}_2 could have been satisfied, movement has been determined by the relative magnitudes of the partial derivatives. However, both \bar{g}_1 and \bar{g}_2 are closer to being satisfied. For comparison, movement of $\Delta x_1 = -1$ and $\Delta x_2 = 1$ would have satisfied \bar{g}_2 but then the artificial variable for \bar{g}_1 would be equal to 38.

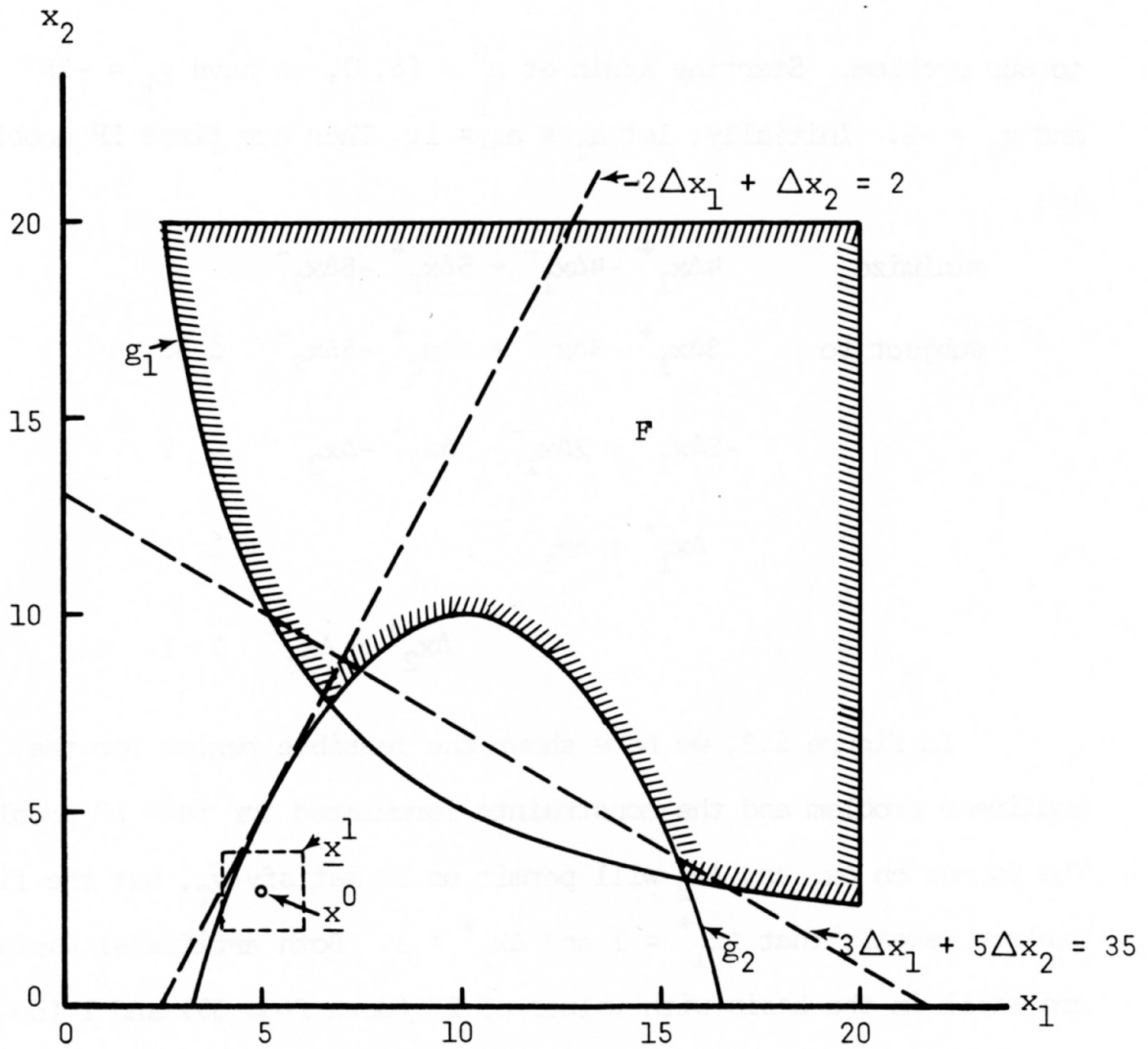


Figure 5.3 LP moves for two constraints.

This leads to an interesting question: Under what circumstances will there be no movement ($\Delta x = 0$) even though \underline{x} is outside the feasible region? Clearly, if the gradients of the objective function and all violated constraints are 0, no movement will take place because the LP tableau will appear as shown in Figure 5.4.

The step-size parameters are all assumed to be 1. There are m unsatisfied \geq constraints and no equality constraints in the original nonlinear problem. (Our remarks will be applicable also if equality constraints are present.) From this, we can make the more general observation that if $\nabla y = 0$ and

$$\sum_{i=1}^M \partial g_i / \partial x_j = 0 \quad \text{for } j = 1, \dots, n$$

then $\Delta x = 0$. So it is possible for the method to "stall" if it passes through such a point. In such a situation, the computer program terminates after printing an error message.

Now we pose another question: Can the algorithm cycle? That is, can a series of points $\underline{x}^k, \dots, \underline{x}^{n+k}$ be generated such that $\underline{x}^k = \underline{x}^{n+k}$ for $n > 1$? If n equals 1, then Δx equals 0; and we have previously discussed this case. Unfortunately, we cannot make a definitive statement in this case. The acceleration-deceleration mechanisms in the computer program appear to make such an occurrence unlikely but we are unable to construct a proof for such a statement. As a safeguard, we recommend the use of multiple starting points for questionable situations.

obj. fn.	$\Delta x_1^+ \Delta x_1^- \dots \Delta x_n^+ \Delta x_n^- \quad a_1 \dots a_m \quad s_1 \dots s_m \quad s_{m+1} \dots s_{m+n}$												Constraint Values	Costs of Variables in Initial Basis
	0	0	...	0	0	0	0	w	...	w	0	...	0	
\bar{g}_1	0	0	...	0	0	0	1	.	.	.	-1	.	b_1	w
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
\bar{g}_m	0	0	...	0	0	0	.	.	.	1	.	-1	b_m	w
Δx_1	1	1	.	1	0
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
Δx_n	1	1	1	0

Figure 5.4 LP Tableau for $\Delta x = 0$: Initial Basis

$a_1 \dots a_m$ artificial variables with cost $w \gg 0$
 $s_1 \dots s_{m+n}$ slack variables with cost 0
 $b_1 \dots b_m$ absolute value of the unsatisfied constraints, $|g_m|$

Although we cannot guarantee that the method will locate the feasible region for all problems, we have not in practice found any cases in which this region was not found. The computer program tries to anticipate these difficulties in the following ways:

1. Two starting points are used for each problem. The first is supplied by the user or is the final solution to the previous problem if multiple problems are being run. The second is based on the bounds for each variable and is formulated as

$$\underline{x}^0 = \underline{b}_L + 1/2 (\underline{b}_U - \underline{b}_L)$$

2. The termination criterion for each problem is based on three conditions: (a) $\Delta x \leq \text{EPIS}$ where EPIS is a convergence criterion supplied by the user; (b) $\Delta y \leq \text{EPIS}$; and (c) the sum of the squares of the violated constraints are satisfied to within a tolerance level CONVRG, a user supplied constant.

3. If conditions (a) and (b) from 2 are satisfied but (c) is violated, the program assumes that no feasible region exists and prints a message to that effect along with other pertinent information. If conditions (a) and (c) are satisfied but (b) is violated, the program again terminates with an appropriate message. This latter situation can occur when the objective function is very "peaked" in the neighborhood of a stationary point.

To demonstrate these checking procedures, we will modify our example so that the two constraints are inconsistent. The problem we

are going to work with is

$$\begin{aligned}
 &\text{minimize} && y = x_1 x_2 + x_1 + x_2 \\
 &\text{subject to} && g_1 = x_1 x_2 - 50 \geq 0 \\
 &&& g_2 = -x_2 - .2(x_1 - 10)^2 + 4 \geq 0 \\
 &&& 10^{200} \geq x_1 \geq -10^{200} \\
 &&& 10^{200} \geq x_2 \geq -10^{200}
 \end{aligned}$$

and, as Figure 5.5 reveals, the constraints g_1 and g_2 are inconsistent. The bounds on x_1 and x_2 have been increased so that equation 5.8 is effectively

$$c_1 \Delta x_i^+ + c_2 \Delta x_i^- \leq a_i$$

As before, the step size is initially set at 1.

At the point $\underline{x}^0 = (5, 5)$, we solve the following LP problem:

$$\begin{aligned}
 &\text{minimize} && 6\Delta x_1^+ - 6\Delta x_1^- + 6\Delta x_2^+ - 6\Delta x_2^- \\
 &\text{subject to} && 5\Delta x_1^+ - 5\Delta x_1^- + 5\Delta x_2^+ - 5\Delta x_2^- \geq 25 \\
 &&& 2\Delta x_1^+ - 2\Delta x_1^- - \Delta x_2^+ + \Delta x_2^- \geq 6 \\
 &&& \Delta x_1^+ + \Delta x_1^- \leq 1 \\
 &&& \Delta x_2^+ + \Delta x_2^- \leq 1
 \end{aligned}$$

The final tableau (which is not a solution) shows $\Delta x_1^+ = \Delta x_2^+ = 1$.

Moving to $\underline{x}^1 = (6, 6)$, we have

$$\begin{aligned}
 &\text{minimize} && 7\Delta x_1^+ - 7\Delta x_1^- + 7\Delta x_2^+ - 7\Delta x_2^-
 \end{aligned}$$

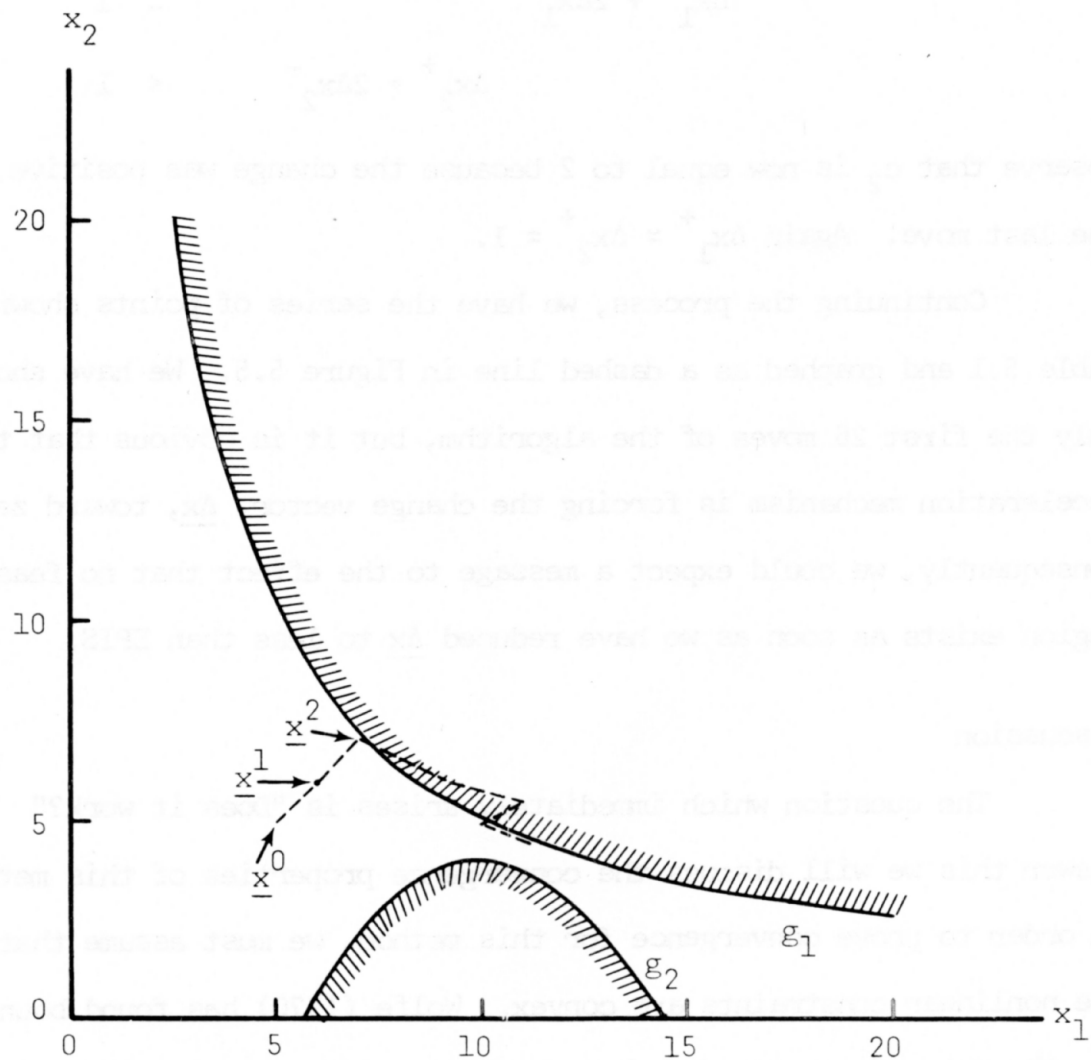


Figure 5.5 LP moves for an inconsistent constraint set.

$$\begin{aligned}
\text{subject to} \quad & 6\Delta x_1^+ - 6\Delta x_1^- + 6\Delta x_2^+ - 6\Delta x_2^- && \geq 14 \\
& 1.6\Delta x_1^+ - 1.6\Delta x_1^- - \Delta x_2^+ + \Delta x_2^- && \geq 5.2 \\
& \Delta x_1^+ + 2\Delta x_1^- && \leq 1 \\
& \Delta x_2^+ + 2\Delta x_2^- && < 1
\end{aligned}$$

Observe that c_2 is now equal to 2 because the change was positive on the last move. Again $\Delta x_1^+ = \Delta x_2^+ = 1$.

Continuing the process, we have the series of points shown in Table 5.1 and graphed as a dashed line in Figure 5.5. We have shown only the first 26 moves of the algorithm, but it is obvious that the deceleration mechanism is forcing the change vector, Δx , toward zero. Consequently, we could expect a message to the effect that no feasible region exists as soon as we have reduced Δx to less than EPIS.

Discussion

The question which immediately arises is "Does it work?" To answer this we will discuss the convergence properties of this method. In order to prove convergence for this method, we must assume that the nonlinear constraints are convex. Wolfe (1970) has found bounds for the rate of convergence when the constraints are linear but this hypothesis is too narrow for our use. We have not assumed that the constraint set is convex; what effect will this have on convergence? Wolfe (1970, p. 25-26) states that the algorithm's effectiveness:

. . . depends quite severely on the convexity assumption; in the absence of convexity it can cut away necessary portions of the problem.

TABLE 5.1
Sequence of Points for an Inconsistent
Constraint Set

Pt. #	x_1	x_2	y	g_1	g_2
0	5.00000	5.00000	35.00000	-25.00000	-6.00000
1	6.00000	6.00000	48.00000	-14.00000	-5.20000
2	7.00000	7.00000	63.00000	- 1.00000	-4.80000
3	8.00000	6.50000	66.50000	2.00000	-3.30000
4	9.00000	6.00000	69.00000	4.00000	-2.20000
5	11.00000	5.50000	77.00000	10.50000	-1.70000
6	10.09091	5.00000	65.545454	.45454	-1.00165
7	10.59091	4.70721	65.15172	- .14640	- .77704
8	11.09091	4.49880	65.48551	- .10420	- .73682
9	10.84091	4.60960	65.42281	- .02770	- .75103
10	10.96591	4.55901	65.51859	- .00632	- .74560
11	11.09091	4.50762	65.59210	- .00642	- .74563
12	11.02841	4.53360	65.56038	- .00162	- .74512
13	10.96591	4.55944	65.52373	- .00161	- .74603
14	10.99716	4.54659	65.54335	- .00040	- .74546
15	11.03841	4.53371	65.56172	- .00040	- .74523
16	11.02378	4.54017	65.55285	- .00010	- .74532
17	11.02060	4.53696	65.55753	- .00002	- .74528
18	11.02841	4.53374	65.56213	- .00003	- .74527
19	11.02450	4.53535	65.55985	- 6.28E-6	- .74527
20	11.02647	4.53455	65.56100	- 1.57E-6	- .74527
21	11.02841	4.53375	65.56215	- 1.57E-6	- .74527
22	11.02743	4.53415	65.56158	- 3.92E-7	- .74527
23	11.02792	4.53395	65.56187	- 9.80E-8	- .74527
24	11.02768	4.53405	65.56172	- 2.45E-8	- .74527
25	11.027921	4.53395	65.56187	- 6.13E-9	- .74527

However, Griffith and Stewart (1961, p. 379) are more optimistic:

In order to prove convergence to a unique solution, the problem to be solved via MAP (the authors' name for their cutting-plane algorithm) would have to meet the usual mathematical requirements of convex constraint space, concave objective function and continuous first partial derivatives. However, in actual practice, problems have been solved with MAP which do not fully satisfy all of these requirements; the type of problem which MAP can handle depends to a great extent on the ingenuity of the problem formulator. It has been our experience that the most important aspect of solving nonlinear programming problems is the degree of curvature in constraint space. In general, the objective function can always be linearized.

Furthermore, although this method requires convexity to guarantee convergence, alternatives applicable to the general problem we have posed (nonlinear objective function, nonlinear constraints) also require convexity of C to ensure convergence (Wolfe 1963, p. 69), so little can be gained in this regard by choosing an alternate method. We feel that the linearization algorithm seems to meet our needs. It has several advantages over the other methods which we have considered:

1. The function to be optimized is not a composite of the constraints as in the penalty functions approach.
2. LP programs exist which can easily handle large numbers of variables and constraints, and additional constraints can readily be entered into the LP tableau.

Although we cannot guarantee that the feasible region will be located if it exists, the use of multiple starting points will hopefully keep the algorithm from stalling or cycling. It is apparent

that, in the absence of those difficulties, the method will eventually locate the feasible region even if it consists of a single point (unless the coordinates of that point are irrational numbers; then we can only approximate it).

And even though this method may "cut-off" the global optimum, there is no assurance that other techniques could locate it for the type of problem we are considering. So in this regard there is little basis for choice.

There are also some disadvantages associated with this method:

1. The linear approximations may be quite slow in locating the feasible region and an optimum. The acceleration-deceleration scheme is an attempt to compensate for this but its success will depend upon the problem at hand.

2. The many LP programs that must be solved can dramatically increase the computation time required to find a solution. This will depend upon the location of \underline{x}^0 with respect to F and whether we have an interior or a boundary optimum. In the latter case the method is not at a particular disadvantage because most methods have trouble locating a boundary optimum.

The optimization process can be summarized by the simplified flow chart shown in Figure 5.6. The computer program is listed in Appendix B and various comments within it contain directions for the user. The subroutine USER is supplied for the problem under consideration. It contains the expressions for the objective function,

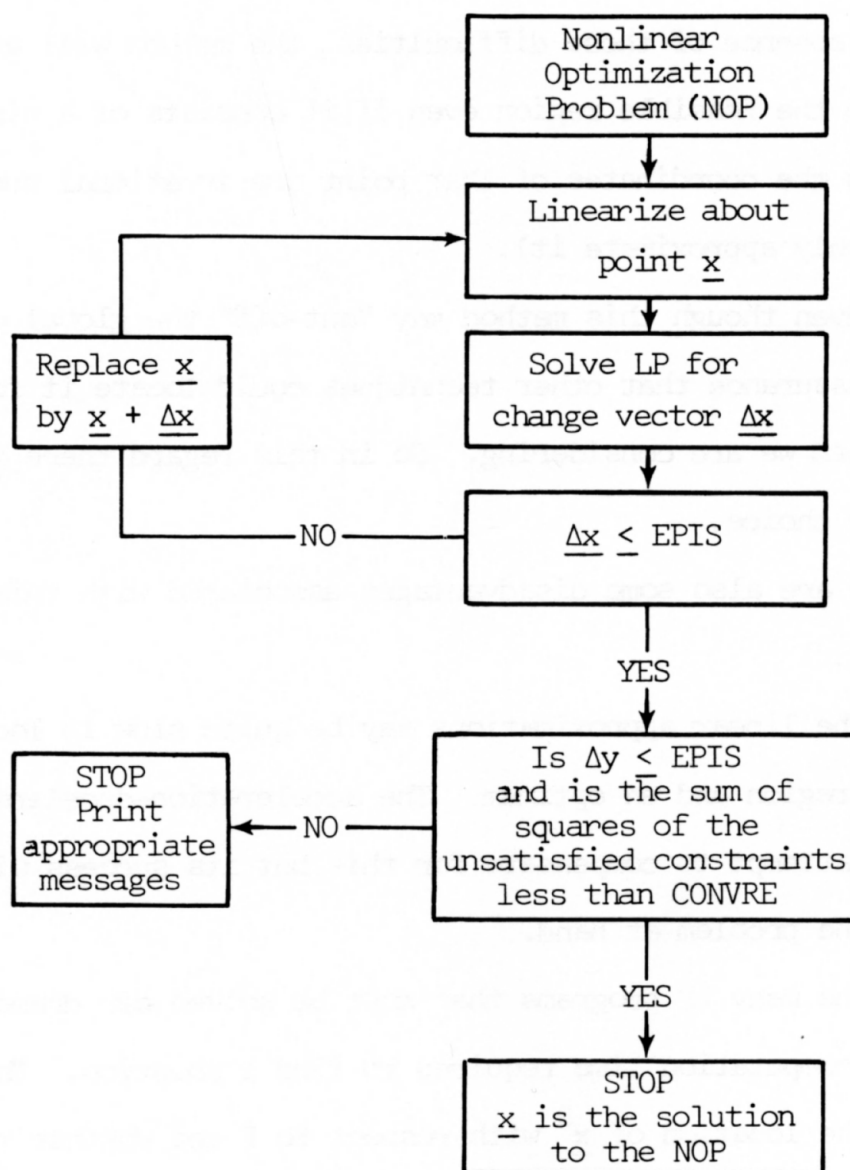


Figure 5.6 Flowchart for the nonlinear optimization algorithm.

the original constraint set, and the goal formulations for all objectives which are entered as constraints by the DM. Then the analytical expressions for the partial derivatives must be given. The subroutine shown in Appendix B is for the Bow River case study discussed in Chapter 6.

CHAPTER 6

A POLITICAL DECISION PROBLEM OF WATER POLLUTION CONTROL

This chapter illustrates an application of the algorithm which has been developed. We will begin with a small, completely artificial example to demonstrate the procedures involved in utilizing this method. Then a synthetic case study of a pollution control problem will be presented to show (hopefully) the usefulness of this technique.

Example Problem

Components

The example involves three goals, two bounded decision variables, and one nonlinear constraint. Their various formulations are expressed below.

Goals:

$$z_1 \geq AL_1 \qquad z_2 \geq AL_2 \qquad z_3 \leq AL_3$$

Criterion Functions:

$$z_1 = 10/x_1 \qquad z_2 = x_1/x_2 \qquad z_3 = 20/x_2$$

$$\Gamma(z_1) = [.1, 10] \qquad \Gamma(z_2) = [.1, 100] \qquad \Gamma(z_3) = [2, 20]$$

Constraints:

$$g_1 = 100 - x_1 - x_2^2 \geq 0$$

$$1 \leq x_1 \leq 1000$$

$$1 \leq x_2 \leq 1000$$

Goal Levels (initial aspiration levels):

$$GL_1 = 5 \quad GL_2 = 20 \quad GL_3 = 5$$

Setup Procedure

We begin by transforming the criterion functions:

$$y_1 = \frac{z_1 - .1}{10 - .1} + \epsilon_1(\underline{x}) \quad y_2 = \frac{z_2 - .1}{100 - .1} + \epsilon_2(\underline{x}) \quad y_3 = \frac{z_3 - 2}{20 - 2} + \epsilon_3(\underline{x})$$

where, as we have explained $\epsilon_i(\underline{x}) = \epsilon'$ when $z_i = z_{i\text{lower}}$. The initial aspiration levels are assumed equal to the goal levels,

$$\underline{AL}_0 = \underline{GL}$$

and so the values of \underline{A}_0 are

$$A_1^0 = \frac{5 - .1}{10 - .1} \quad A_2^0 = \frac{20 - .1}{100 - .1} \quad A_3^0 = \frac{5 - 2}{20 - 2}$$

(Note that ϵ is not needed in these transformations.)

The elements of \underline{d} are formulated from the structure of the goals:

$$d_1 = A_1/y_1 \quad d_2 = A_2/y_2 \quad d_3 = y_3/A_3$$

First Cycle

The principal problem to be solved on the first cycle is

$$\begin{aligned} &\text{minimize} && s_1 = d_1 + d_2 + d_3 \\ &\text{subject to} && g_1 = 101 - x_1 - x_2^2 \geq 0 \\ &&& 1 \leq x_1 \leq 1000 \\ &&& 1 \leq x_2 \leq 100 \end{aligned}$$

Expressing d_i in terms of AL_i and z_i , we have

$$\begin{aligned} s &= \left(\frac{4.9}{9.9} \right) / \left(\frac{z_1 - .1}{10 - .1} + \epsilon_1 \right) + \left(\frac{19.9}{99.9} \right) / \left(\frac{z_2 - .1}{100 - .1} + \epsilon_2 \right) + \left(\frac{z_3 - 2}{20 - 2} + \epsilon_3 \right) / \left(\frac{3}{18} \right) \\ &= 4.9 / (z_1 - .1 + 9.9\epsilon_1) + 19.1 / (z_2 - .1 + 99.9\epsilon_1) + (z_3 - 2 + 18\epsilon_3)/3 \end{aligned}$$

This can be rewritten as:

$$s = \frac{4.9}{z_1 - .1 + \epsilon_1} + \frac{19.9}{z_2 - .1 + \epsilon_2} + \frac{z_3 - 2 + \epsilon_3}{3}$$

Then in terms of the decision variables

$$s = \frac{4.9x_1}{10 - x_1(.1 - \epsilon_1)} + \frac{19.9x_2}{x_1 - x_2(.1 - \epsilon_2)} + 1/3 \left(\frac{20}{x_2} - 2 + \epsilon_3 \right)$$

An interior minimum was located at:

$$\underline{x}_1 = (7.488, 1.550)$$

with

$$s = 11.811$$

The values of the criterion functions are:

$$\underline{z}_1 = (1.34, 4.83, 12.90)$$

and the corresponding transformed values are:

$$\underline{y}_1 = (.125, .047, .606)$$

None of the goals has been attained as indicated by:

$$\underline{d}_1 = (3.96, 4.21, 3.63) > \underline{1}$$

The nonlinear nature of DA makes it generally impossible to compare the extent of nonattainment from these numbers, so it is not necessarily true that the smallest d signifies the "most satisfied" goal. Values of d for a specific goal may, however, be compared from one optimization to another (provided the aspiration level remains constant).

Also as part of the first cycle, we construct three auxiliary problems which attempt to satisfy each of the goals in turn. If $z_1 \geq 5$ is entered as a constraint, d_1 is deleted from the surrogate objective function giving:

$$\text{minimize} \quad s_{1.1} = d_2 + d_3$$

$$\text{subject to} \quad g_1 = 101 - x_1 - x_2^2 \geq 0$$

$$g_2 = 10/x_1 - 5 \geq 0$$

$$1 \leq x_1 \leq 1000$$

$$1 \leq x_2 \leq 1000$$

Similarly the second and third auxiliary problems are:

$$\begin{aligned}
 &\text{minimize} && s_{1.2} = d_1 + d_3 \\
 &\text{subject to} && g_1 = 101 - x_1 - x_2^2 \geq 0 \\
 &&& g_2 = x_1/x_2 - 20 \geq 0 \\
 &&& 1 \leq x_1 \leq 1000 \\
 &&& 1 \leq x_2 \leq 1000
 \end{aligned}$$

and

$$\begin{aligned}
 &\text{minimize} && s_{1.3} = d_1 + d_2 \\
 &\text{subject to} && g_1 = 101 - x_1 - x_2^2 \geq 0 \\
 &&& g_2 = 5 - 20/x_2 \geq 0 \\
 &&& 1 \leq x_1 \leq 1000 \\
 &&& 1 \leq x_2 \leq 1000
 \end{aligned}$$

The results from all four problems are tabulated in Table 6.1.

The information from these four problems is presented to the DM in the graphical form discussed in Chapter 4. Figures 6.1, 6.2, and 6.3 show the effects of imposing goal attainment at the current aspiration levels compared to solution of the principle problem. The DM can examine this display for gross interactions among the various goals caused by the constraint set.

To continue the development of this example, we will assume that the DM has modified his aspirations on goal three so that he

TABLE 6.1

Results of First Cycle on Example Problem

$$\underline{AL}_0 = \underline{GL} = (5, 20, 5)$$

Principal problem	$s_1 = d_1 + d_2 + d_3$			
$s_1 = 11.811$	$x_1 = (7.488, 1.550)$	$z_1 = (1.34, 4.83, 12.90)$	$y_1 = (.125, .047, .606)$	$d_1 = (3.96, 4.21, 3.63)$
Auxiliary problems	$s_{1.1} = d_2 + d_3$			
$s_{1.1} = 16.483$	$x_{1.1} = (2.000, 1.000)$	$z_{1.1} = (5.00, 2.00, 20.00)$	$y_{1.1} = (.495, .019, 1.000)$	$d_{1.1} = (1.00, 10.48, 6.00)$
	$s_{1.2} = d_1 + d_3$			
$s_{1.2} = 18.250$	$x_{1.2} = (20.000, 1.000)$	$z_{1.2} = (.500, 20.00, 20.00)$	$y_{1.2} = (.036, .199, .902)$	$d_{1.2} = (12.25, 1.00, 6.00)$
	$s_{1.3} = d_1 + d_2$			
$s_{1.3} = 13.543$	$x_{1.3} = (11.656, 4.000)$	$z_{1.3} = (.858, 2.91, 5.00)$	$y_{1.3} = (.077, .028, .107)$	$d_{1.3} = (6.47, 7.08, 1.00)$

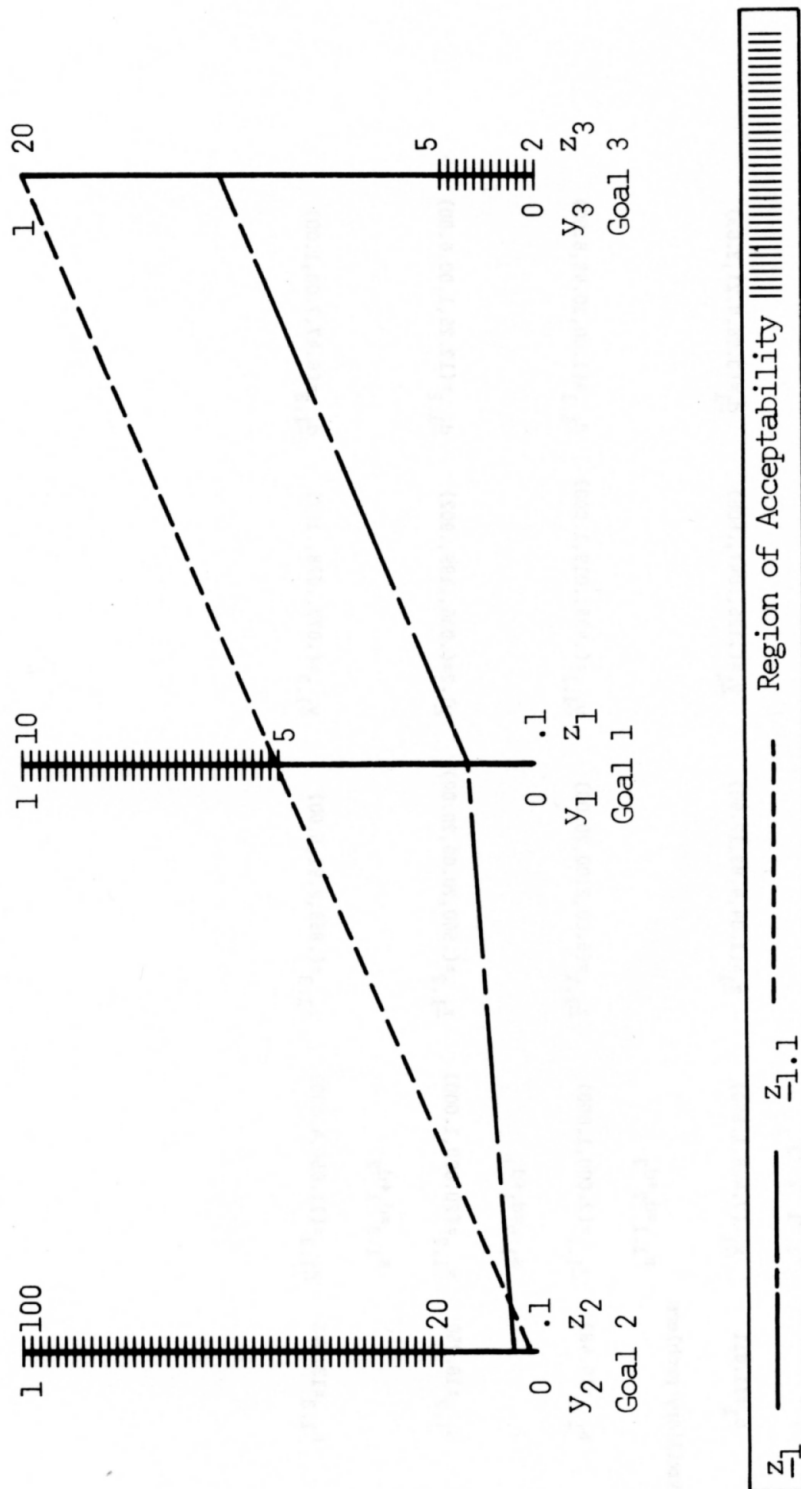


Figure 6.1 First Cycle: Goal 1 satisfied at $AL_0^1 = 5$.

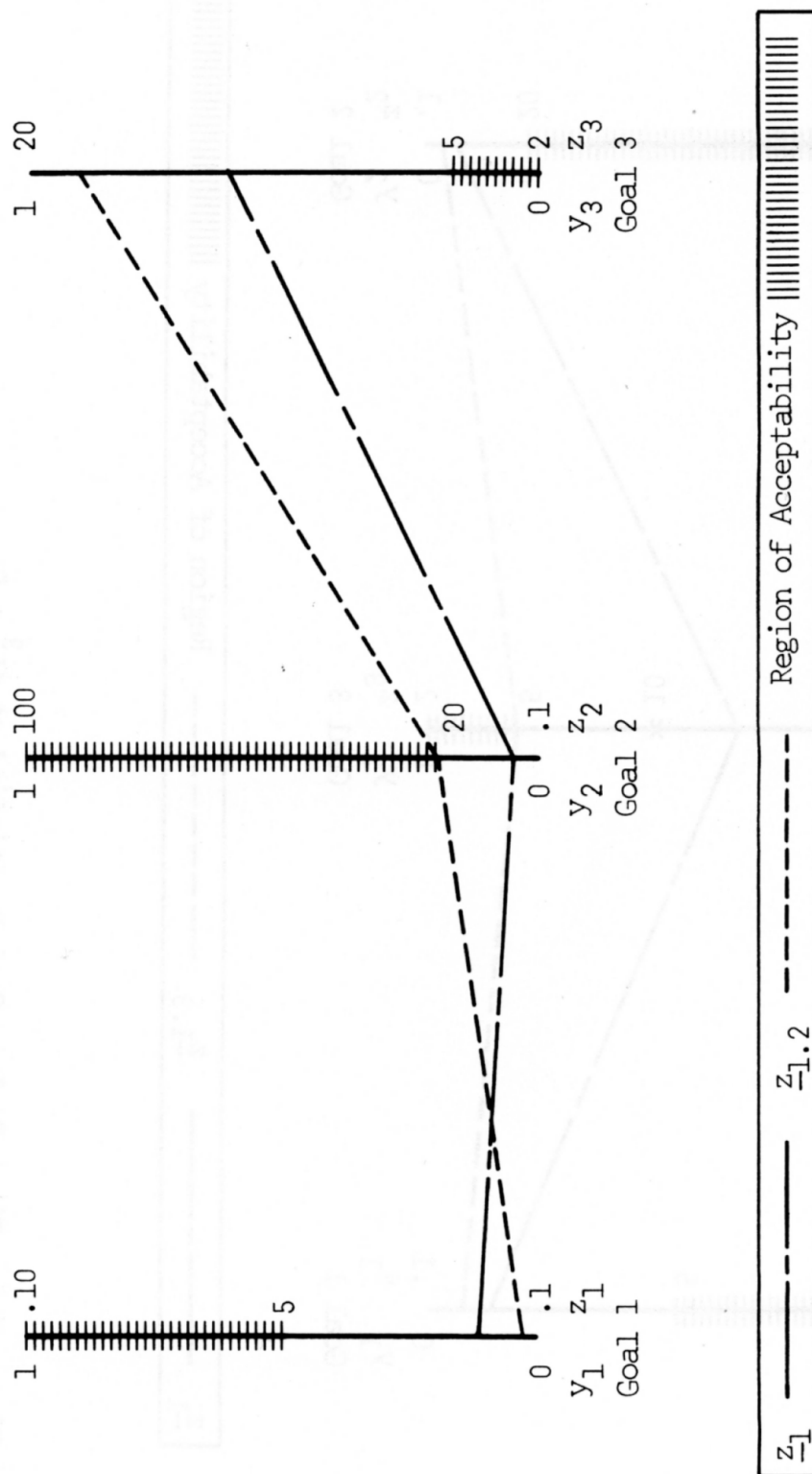


Figure 6.2 First Cycle: Goal 2 satisfied at $AL_0^2 = 20$.

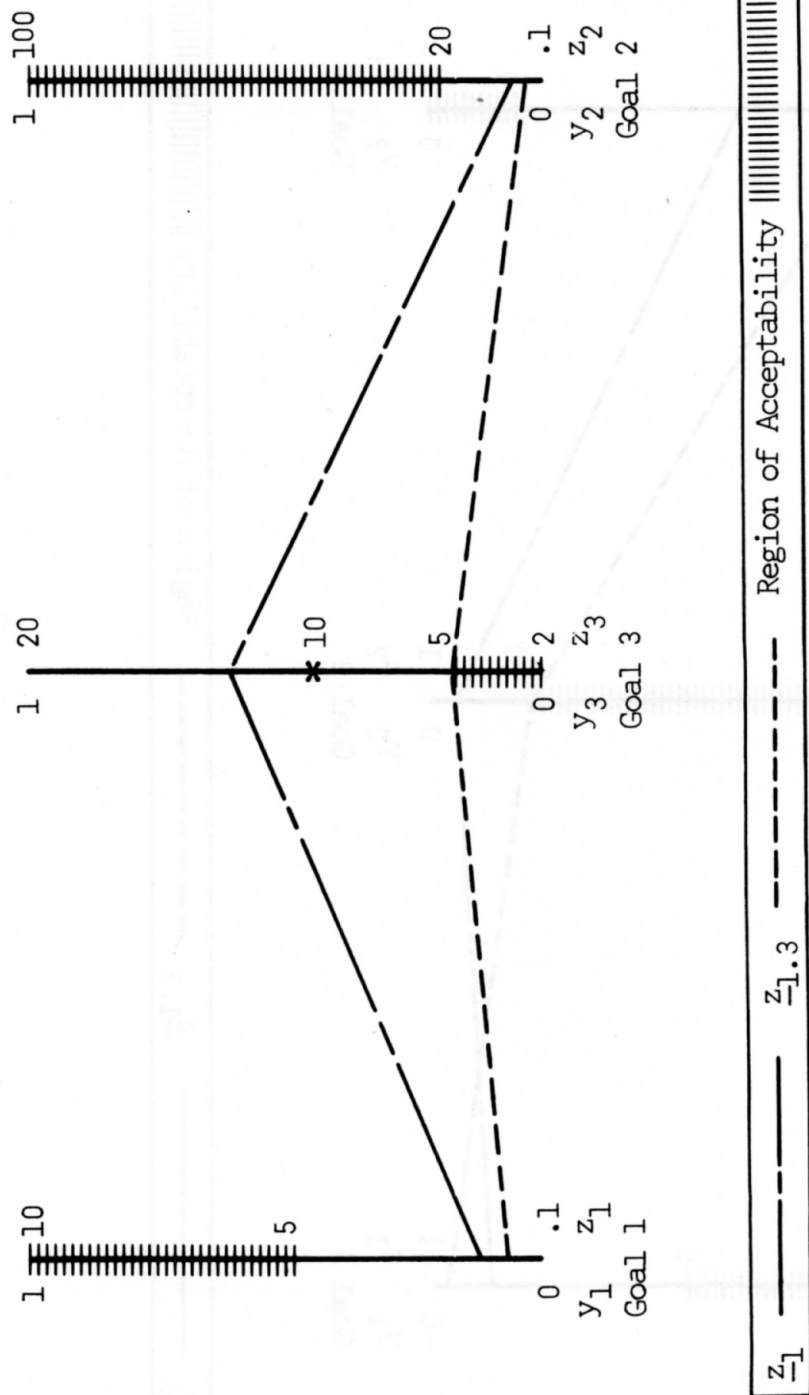


Figure 6.3 First Cycle: Goal 3 satisfied at $AL_0^3 = 5$.

will be content if $z_3 \leq 10$. He arrived at this decision by applying a ruler to Figure 6.3 (see Chapter 4) to predict the approximate levels of z_1 and z_2 for a given level of attainment on the third goal. The projected values of z_1 and z_2 for $z_3 = 10$ are $z_1 = 1.2$ and $z_2 = 4.1$.

Second Cycle

The aspiration levels are now $\underline{AL}_1 = (5, 20, 10)$ and achievement of goal three has been entered as a constraint. The principal problem to be solved on this cycle is:

$$\begin{aligned}
 &\text{minimize} && s_{2.3} = d_1 + d_2 \\
 &\text{subject to} && g_1 = 100 - x_1 - x_2^2 \geq 0 \\
 &&& g_2 = 10 - 20/x_2 \geq 0 \\
 &&& 1 \leq x_1 \leq 1000 \\
 &&& 1 \leq x_2 \leq 1000
 \end{aligned}$$

and the two auxiliary problems are:

$$\begin{aligned}
 &\text{minimize} && s_{1.31} = d_2 \\
 &\text{subject to} && g_1 = 100 - x_1 - x_2^2 \geq 0 \\
 &&& g_2 = 10 - 20/x_2 \geq 0 \\
 &&& g_3 = 10/x_1 - 5 \geq 0 \\
 &&& 1 \leq x_1 \leq 1000 \\
 &&& 1 \leq x_2 \leq 1000
 \end{aligned}$$

and

$$\begin{aligned}
 &\text{minimize} && s_{1.32} = d_1 \\
 &\text{subject to} && g_1 = 100 - x_1 - x_2^2 \geq 0 \\
 &&& g_2 = 10 - 20/x_2 \geq 0 \\
 &&& g_3 = x_1/x_2 = 20 \geq 0 \\
 &&& 1 \leq x_1 \leq 1000 \\
 &&& 1 \leq x_2 \leq 1000
 \end{aligned}$$

The results of these optimizations are collected in Table 6.2 and their effects on goals 1 and 2 displayed in Figures 6.4 and 6.5. Inspection of Table 6.2 reveals that the minimum of the primary problem is now on the boundary of F because \underline{x}_1 did not satisfy goal 3 at an aspiration level of 10. In comparing \underline{d}_1 and $\underline{d}_{2,3}$ we notice that both \underline{d}_1 and \underline{d}_2 have increased, indicating that goal 3 has been satisfied only at the expense of the other goals.

The results are again presented to the DM for his consideration. Assume that he will settle for $z_3 \leq 10$ and $z_1 \geq 3$. A ruler applied to Figure 6.4 gives an estimate of the result, which is $z_2 = 2.5$.

Third Cycle

The aspiration levels for this cycle are $\underline{AL}_2 = (3, 20, 10)$ and both goals 1 and 3 are entered as constraints. The primary problem is:

$$\begin{aligned}
 &\text{minimize} && s_{3.31} = d_2
 \end{aligned}$$

TABLE 6.2

Results of Second Cycle on Example Problem

$$\underline{AL}_1 = (5, 20, 10)$$

Additional Constraints

$$10 - 20/x_2 \geq 0$$

Principal problem	$s_{2.3} = d_1 + d_2$			
$s_{2.3} = 9.351$	$\underline{x}_{2.3} = (8.453, 2.000)$	$\underline{z}_{2.3} = (1.18, 4.23, 10.00)$	$\underline{y}_{2.3} = (.109, .041, .444)$	$\underline{d}_{2.3} = (4.52, 4.83, 1.00)$
Auxiliary problems	$s_{2.31} = d_2$			
$s_{2.31} = 22.131$	$\underline{x}_{2.32} = (2.000, 2.000)$	$\underline{z}_{2.32} = (5.00, 1.00, 10.00)$	$\underline{y}_{2.32} = (.495, .009, .444)$	$\underline{d}_{2.32} = (1.00, 22.13, 1.00)$
	$s_{2.32} = d_1$			
$s_{2.32} = 32.666$	$\underline{x}_{2.31} = (40.000, 2.000)$	$\underline{z}_{2.31} = (.25, 20.00, 10.00)$	$\underline{y}_{2.32} = (.015, .199, .444)$	$\underline{d}_{2.31} = (32.67, 1.00, 1.00)$

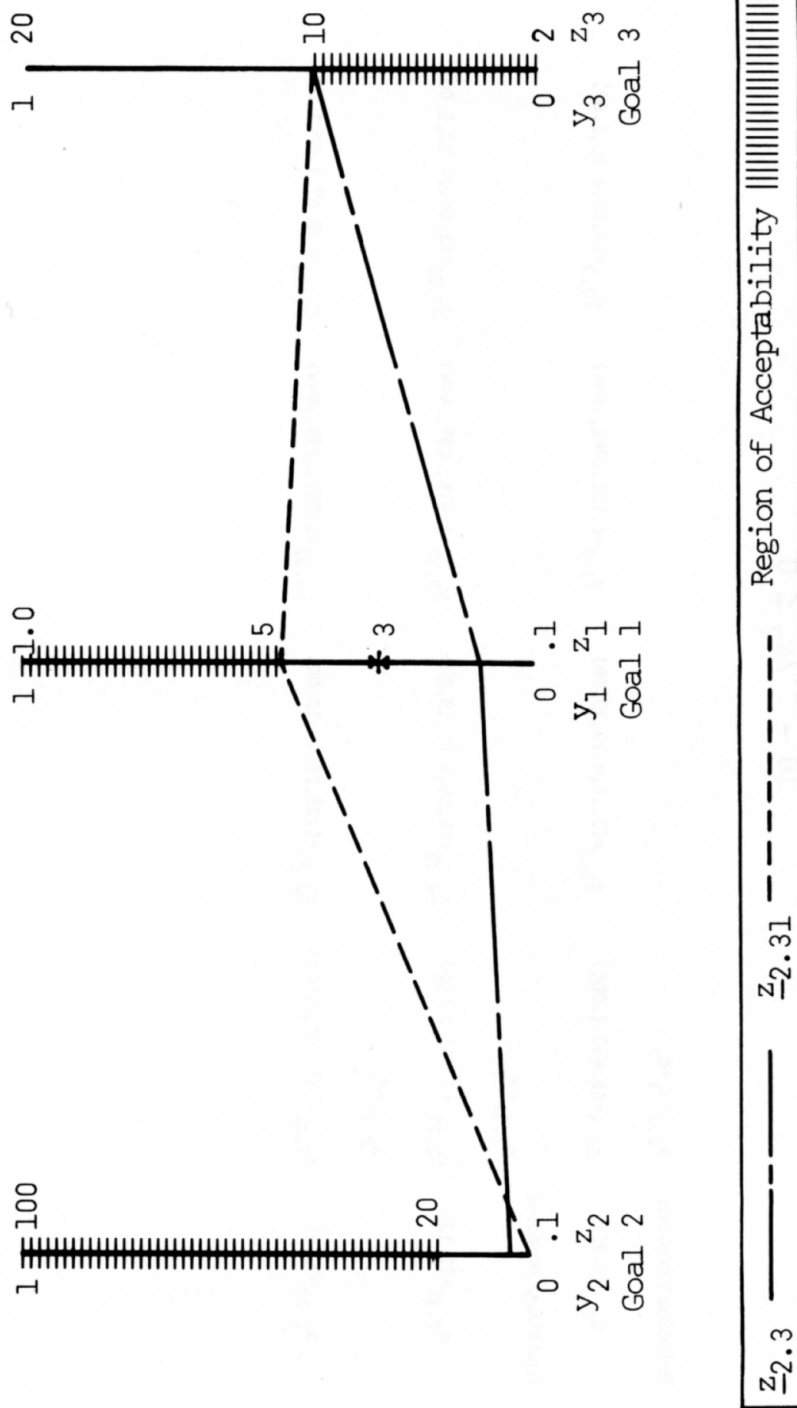


Figure 6.4 Second Cycle: Goal 1 satisfied at $AL_1^1 = 5$.

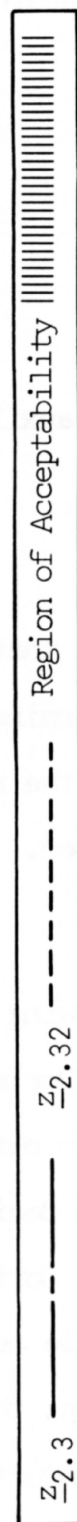
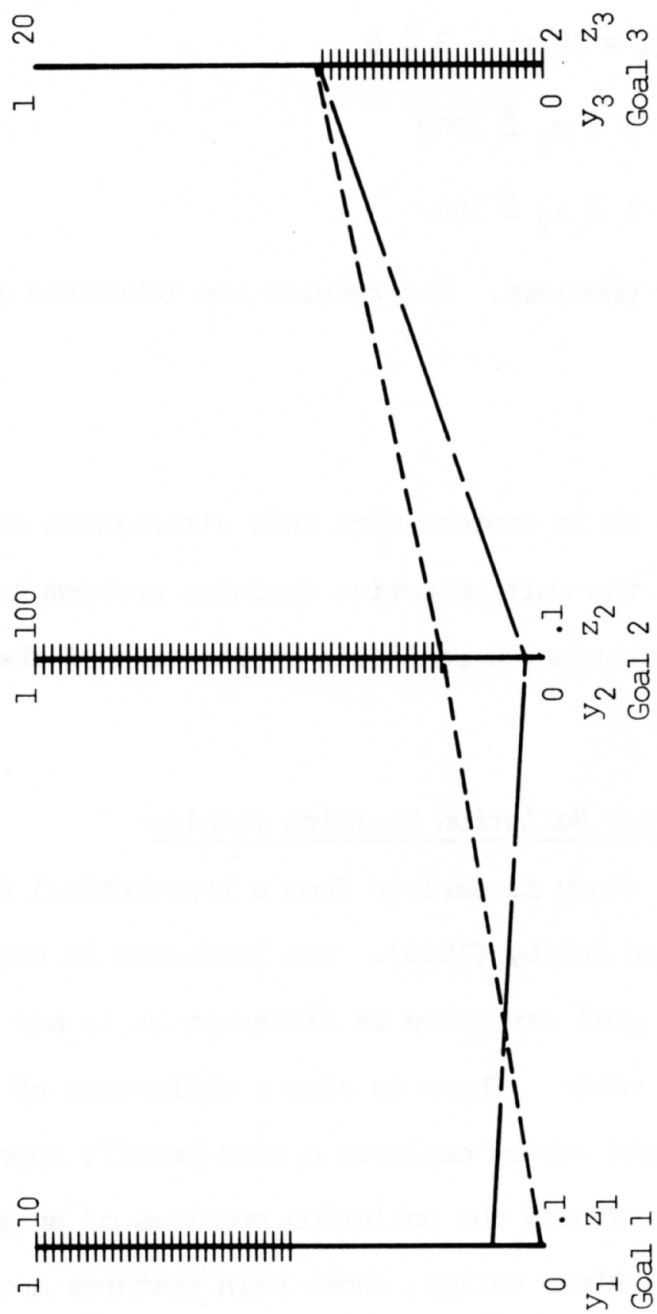


Figure 6.5 Second Cycle: Goal 2 satisfied at $AL_1^2 = 20$.

$$\begin{aligned}
\text{subject to } g_1 &= 100 - x_1 - x_2^2 \geq 0 \\
g_2 &= 10 - 20/x_2 \geq 0 \\
g_3 &= 10/x_1 - 3 \geq 0 \\
1 &\leq x_1 \leq 1000 \\
1 &\leq x_2 \leq 100
\end{aligned}$$

and there are no auxiliary problems. The results are tabulated in Table 6.3.

Termination

We assume that the DM is content with this alternative and so the process terminates; the multiobjective decision problem has been solved. In the next section, a realistic application of the method is developed.

A Water Pollution Decision Problem

The following case study is derived from a hypothetical example developed by Dorfman and Jacoby (1969). The framework in both studies is similar but our goal structure is different as is our conception of the decision maker. There is also a difference of technique in that Dorfman and Jacoby employed a cost-benefit approach.

The example centers around the pollution problems of an artificial river basin, the Bow River Valley, whose main features are shown in Figure 6.6. Industrial pollution is represented by the Pierce-Hall Cannery, located near the head of the valley. There are two sources of municipal waste, Bowville and Plympton. A state park is located

TABLE 6.3

Results of Third Cycle on Example Problem

$$\underline{AL}_2 = (3, 20, 10)$$

Additional Constraints

$$10 - 20/x_2 \geq 0$$

$$10/x_1 - 3 \geq 0$$

Principal problem $s_{3.31} = d_2$

$$s_{3.31} = 12.714 \quad \underline{x}_{3.31} = (3.333, 2.000) \quad \underline{z}_{3.31} = (3.00, 1.67, 10.00) \quad \underline{y}_{3.31} = (.293, .016, .444) \quad \underline{d}_{3.31} = (1.00, 12.71, 1.00)$$

(No Auxiliary Problems)

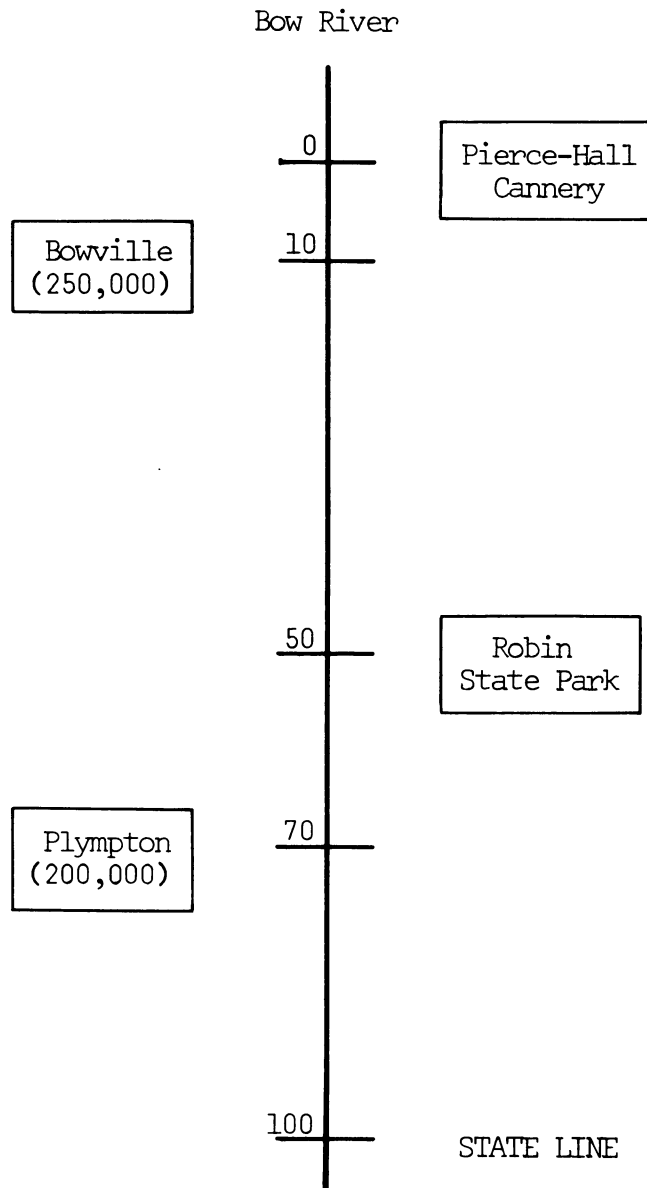


Figure 6.6 Main features of the Bow River Valley.

Numeric values indicate river miles.

between the cities. The lower end of the valley adjoins the state boundary line.

The Bow River Water Pollution Control Commission must set pollution standards for the entire valley. But it must act with an awareness of the effect of any additional effluent treatment costs on the economic health of the valley. Rather than model the interactive process of group decision-making in this political body, we will view it as "Big Brother," a term borrowed from Dupnick (1971) to denote a governmental decision-making body in the abstract. As Dupnick mentions, the term is actually borrowed from Orwell's Nineteen Eighty-Four (1949). Big Brother then is our decision maker.

The Bow River Valley Basin

The Bow River. The Bow River has a flow rate of 800 cubic feet per second (cfs) and a velocity of 0.5 feet per second (fps) during the summer drought months. No tributaries flow into the valley and we will assume no water losses due to evaporation, transpiration from trees, etc. The specification of water quality has been reduced to a single dimension, dissolved oxygen concentration (DO). We are ignoring floating solids, color, turbidity, coliform bacteria, taste and odor, temperature, pH, radioactivity, etc. Similarly, the waste content of the municipal and industrial effluents is assumed to be described by the number of pounds of biochemical oxygen demanding material (BOD) they carry. BOD is separated into carbonaceous and nitrogenous material. Again, we are ignoring many factors here.

Pollutants from upstream cities have diminished the DO concentration from a saturation level of 8.5 milligrams per liter (mg/l) to only 6.75 mg/l at the Pierce-Hall Cannery. The current summer DO levels at other points of interest are Bowville, 4.75 mg/l; Robin State Park, 2.0 mg/l; Plympton, 5.1 mg/l; and 1.0 mg/l at the state line.

Pierce-Hall Cannery. The Pierce-Hall Cannery produces slightly over seven million (M) equivalent cases (eqc) each year. One eqc represents 24 #303 cans. Primary waste treatment facilities in the form of screening and sedimentation equipment have already been installed but the waste stream still carries about one pound ultimate BOD for each case produced. The cannery discharges approximately 30 million gallons per day (mgd) with an ultimate carbonaceous demand (BOD_C) of 28,000 pounds per day (#/d) and an ultimate nitrogenous demand (BOD_N) of 19,000 #/d. (The gross or before treatment BOD loads are 40,000 #/d and 28,000 #/d respectively.)

In order to reduce the wastes further, additional treatment facilities would have to be installed. A consulting firm has developed the figures shown in Table 6.4 for various specific alternatives.

We assume that there is a continuous range of alternatives available so that the following relationship can be developed:

$$AAC = \frac{59}{1.09 - x^2} - 59$$

where AAC is the gross additional annual cost in thousands, and x is the proportionate reduction in gross BOD_C at the cannery. The influence of any additional cost is mitigated by the federal corporation tax so that the net additional cost is 60% of the annual cost.

TABLE 6.4
Cost of Additional Waste Treatment

<u>Type of Treatment</u> ¹	<u>Percent Gross BOD_c Removed</u>	<u>Gross Additional Annual Cost</u> ²		
		<u>Pierce-Hall</u>	<u>Bowville</u>	<u>Plympton</u>
Primary	30	\$ 0	\$ 0	\$ 0
Primary + A	80	72,000	650,000	550,000
Primary + B	90	151,000	1,368,000	1,157,000
Primary +B+C	95	256,000	2,305,000	1,950,000

1. "A" is low efficiency secondary treatment.

"B" is high efficiency secondary treatment.

"C" is tertiary treatment.

2. Primary treatment facilities are now in place so there is no additional cost.

The selling price of the product is \$3.50 per eqc yielding annual gross sales of \$25M. The firm's net operating revenues, after income taxes, are 1.5% of gross sales when only primary treatment is employed. This net profit of \$375,000 a year is a return of 7.5% on the stockholders' equity of \$5M. The firm is not a price leader and will not be able to raise its price appreciably even if a large increase in treatment costs is imposed. Nor is it aware of any changes in its methods of processing that would enable it to reduce its waste load at the current scale of operations. Therefore any increase in treatment costs would have to come out of net profits. For example, primary plus low efficiency secondary treatment would cost \$72,000. The net cost is \$43,200 ($.6 \times \$72,000$) so the new profit level is \$331,800 ($\$375,000 - \$43,200$). The return on equity would then decrease to 6.6%. The relationship between costs (AAC) and percent return on investment (r) is:

$$r = \frac{100}{5,000,000} (375,000 - .6AAC) \quad 6.1$$

The cannery will require additional financing within the next few years to replace worn out equipment and facilities; the likelihood of acquiring those funds is directly related to the level of net profits. On the other hand, the cannery's 800 employees come mainly from Bowville and make use of Robin State Park, so that improvement of the river will enhance the amenities available to them.

Bowville. Bowville is the major urban area of the basin with a population of 250,000. It discharges 51 mgd of effluent. Even after primary treatment removes 30% of both BOD_c and BOC_n , the

effluent contains 89,600 #/d of BOD_C (128,000 #/d gross) and 33,600 #/d of BOD_n (48,000 #/d gross). This load, together with the waste from the cannery, makes the river unsuitable for recreational use at Robin State Park.

The public works department of Bowville has prepared cost estimates for installation of additional treatment facilities similar to those considered by the cannery. These figures are shown in Table 6.4. The relationship between waste reduction and cost can be expressed as:

$$AAC = \frac{532}{1.09-x^2} - 532$$

where AAC is the gross additional annual cost in thousands, and x is the proportionate reduction in gross BOD_C at Bowville. However, the cost to Bowville of the additional treatment is reduced because the Federal Water Pollution Control Act provides a grant which covers 50% of the construction costs, which are about one-half of the total costs. So the city would pay only about 75% of the total cost.

Additional costs will affect the city's tax rate according to the relation:

$$\Delta t = 2.4 \times 10^{-3} AAC \quad 6.2$$

where Δt is the increase in the tax rate per thousand dollars assessed valuation. For example, additional costs of \$650,000 reduce to \$490,000 net costs so the tax rate increases by \$1.17 (i.e., $2.4 \times 10^{-3} \times .75 \times \650).

The current tax rate is already \$63.50 per thousand dollars assessed valuation, and the city comptroller believes that recent

increases in teachers' and firemens' salaries will raise the rate still higher. This fact is of considerable importance because Bowville's tax rates are already higher than Plympton's, which competes with Bowville for new industries.

Although Bowville's direct gain from improving the quality of the Bow River is small, cleaning up the river would attract more tourists and vacationers to the valley and permit the development of water based recreation at Robin State Park. The city's own park is so overcrowded that plans for expanding it have been considered. These changes would not be necessary if Robin State Park were usable.

Plympton. With a population of 200,000, Plympton is smaller than Bowville and somewhat less affluent. It has a primary treatment plant and after treatment the 43 mgd effluent of the city contains 67,000 #/d of ultimate BOD_c and 25,000 #/d of ultimate BOD_n . (The gross BOD loads are 95,700 #/d and 35,700 #/d respectively.) The costs of various levels of treatment at Plympton comparable to those for the cannery and Bowville are shown in Table 6.4. We can express the relationship between cost and amount of waste removal by:

$$AAC = \frac{450}{1.09-x^2} - 450$$

where AAC is the gross additional annual cost in thousands, and x is the proportionate reduction in gross BOD_c at Plympton. The cost of treatment at Plympton is less than for Bowville because Plympton is smaller. But the effect on the tax rate is reversed because Plympton is a poorer city with a lower value of taxable property per capita.

The relationship between cost and the tax rate is given by:

$$\Delta t = 3.33 \times 10^{-3} \text{ AAC} \quad 6.3$$

where Δt is the increase in the tax rate per thousand dollars assessed valuation, and AAC is the net additional annual cost to the city in thousands. An additional cost of \$550,000 reduces to a net cost of \$410,000 with an increase in the tax rate of \$1.37 (i.e., $3.33 \times 10^{-3} \times .75 \times \550).

Plympton has no recreational facilities of its own and is completely dependent upon the facilities of Robin State Park. Consequently, Plympton must bear its share of the cost of cleaning up the river. In addition, the city is more dependent upon tourism for revenues and for this reason would like to have Robin State Park improved. Finally, maintenance of an adequate DO level at the State Line is principally Plympton's responsibility.

Bow Valley Water Pollution Control Commission. The commission is made up of representatives from all three waste sources together with members of the state and federal government. Rather than explore the interactions of the group, we assume the existence of a composite individual referred to as "big brother," (BB). BB is a political figment who represents all of his constituents. The problem BB faces is to determine, first, a policy vector, \underline{x} , which satisfies the constraint that the DO level of the Bow River at the state line is greater than 3.5 mg/l. The components of \underline{x} , (x_1 , x_2 , x_3), are the proportionate reduction in carbonaceous waste load to be imposed upon Pierce-Hall, Bowville, and Plympton. We have assumed

that the proportionate reduction in BOD_n is a function of that for BOD_c so that only one set of values need be specified. The relationship used in this research is discussed in the next section. The values of all these variables are currently at 0.3. Having found a policy which satisfies the above constraint, BB must then locate any "better" policies. At this point, we define "better" as "producing a higher subjective value to BB." But BB has been assumed to reflect the social group he represents; consequently, we may suppose that higher valuation by BB implies higher valuation by the group. If we wish to view "valuation" as "utility," then BB seeks to increase the collective utility of the group. The framework in which our analysis takes place allows BB to alter his value structure, corresponding to changes in social values (i.e., the relatively recent ecological movement).

We are not trying to define and discuss collective utility formally; this would require far more space than we have here. Rather, we are seeking to relate the assumptions made in this case to others which have been accepted elsewhere and hence make ours more palatable. The interested reader is referred to Dupnick (1971) and Dupnick and Duckstein (1971). Lesourne (1964) is one of the principal contributors in this area.

BB is looking at six indicators of the worth of any decision (6 goals): the DO levels at Bowville, Robin State Park, and Plympton; the percent return on investment at the Pierce-Hall cannery, and the addition to the tax rate for Bowville and Plympton. Initially, BB

would like to raise the DO level at all the above locations to at least 6.0 mg/l. But BB also wants the percent return on investment at Pierce Hall to remain above 6.5% and the addition to the tax rate at both Bowville and Plympton to remain below \$1.50 per thousand assessed valuation.

So BB must impose regulations upon his constituents. But they are political and economic bodies whose individual interests do not coincide and whose interests are not entirely separate from those of BB. We would be very hesitant about attempting to model such a complex situation if we were required to express BB's preferences analytically. Fortunately, the algorithm which we have developed allows BB to evaluate the results of each alternative policy subjectively and decide which is "best."

Streeter-Phelps Model of DO Concentration in Streams

The decomposition of organic waste in a stream reduces the DO level at a rate proportionate to the concentration of waste in the stream. Streeter and Phelps (1925) have proposed a model of this process. They assume that 1 gram of BOD absorbs 1 gram of DO. If the DO level is already zero, anaerobic decomposition takes place. We will assume that the current raw waste treatment requirements will ensure that the DO level is above zero so that this aspect can be omitted from the model. As the DO level falls below the saturation level, g_s , additional oxygen is absorbed into the water from the atmosphere. The magnitude of g_s is influenced by temperature and flow

characteristics of the river; the average value for the Bow River during the summer months is 8.5 mg/l. These two opposing processes, deoxygenation and reoxygenation, determine the final DO level.

The proportion of carbonaceous versus nitrogenous substances affects the decomposition process because the oxidation of the nitrogenous material begins some time after that of the carbonaceous. The effects of the two wastes are additive so that the impact of the nitrogenous component can be approximated as a dummy waste source located downstream. The distance used in this example is 20 miles downstream.

The effects of the three different waste sources are also additive so that changes in the DO level for the river are the sum of changes caused by variations in waste reduction at each source.

The proportionate reductions in gross BOD_n and BOD_c after treatment are related by the expression:

$$w = \frac{.39}{1.39 - x^2} \quad 6.4$$

where w is the proportionate amount of gross BOD_n removed, and x is the proportionate amount of gross BOD_c removed. For example, if a method of treatment is selected which removes 90% of the BOD_c , then according to the above equation, 67% of the BOD_n is removed. We will assume that the equation is valid for all of the different types of treatment considered and, for simplification, that it applies to the cannery, Bowville, and Plympton.

In the Streeter-Phelps model, the impact of a change in waste load at point i as measured at point j downstream is given by:

$$d_{ij} = -\alpha \frac{k_1}{F(k_2 - k_1)} \left(10^{\frac{-k_1 m}{v}} - 10^{\frac{-k_2 m}{v}} \right)$$

where d_{ij} is the transfer coefficient in units of (mg/l)/(#/d)

α is a dimensionality constant equal to 10^6 mg/l

F is the flow rate of the stream in #/d

m is the distance in miles between points i and j

v is the velocity of the stream in miles/d

k_1 and k_2 are constants characteristic of the stream
measured in units of 1/d

$k_1 = .2/\text{day}$ for BOD_c , $k_1 = .3/\text{day}$ for BOD_n , $k_2 = .4/\text{day}$ (for both)

For a derivation, the reader is referred to Chapter 4 of Fair, Geyer and Okun (1968). The transfer coefficients for the points of interest are displayed in Table 6.5.

On the basis of our additivity assumptions, then, the water quality at point j , g_j , is calculated from the equation:

$$g_j = - \sum_i (d_{ij}^c L_i^c (x_i - .3) + d_{ij}^n L_i^n (w_i - .3)) + \bar{g}_j \quad 6.5$$

where

d_{ij}^c is the carbonaceous transfer coefficient between points i and j

d_{ij}^n is the corresponding nitrogenous transfer coefficient

L_i^c is the gross BOD_c load for source i

L_i^n is the corresponding gross BOD_n load

x_i is the proportionate reduction in L_i^c

TABLE 6.5

Carbonaceous and Nitrogenous Transfer Coefficients

Cell values above the diagonal are carbonaceous coefficients.
 Cell values below the diagonal are nitrogenous coefficients.

All values are times 10^{-5} ; i.e., carbonaceous coefficient for

Pierce-Hall to Bowville is 5.68×10^{-5} (mg/l) / (#/d)

		TO			
		Bowville	Robin State Park	Plympton	State Line
FROM	Pierce-Hall	-5.68 —	-1.31 -3.15	-.442 -.771	-.083 -.073
	Bowville	0	-2.18 -5.53	-.764 -1.60	-.145 -.162
	Plympton	—	—	0 0	-3.49 -7.33

w_i is the proportionate reduction in L_i^n ; as explained previously,

$$w_i = f(x_i)$$

\bar{g}_j is the current DO level at point j

$$i = \begin{cases} 1 & \text{Pierce-Hall Cannery} \\ 2 & \text{Bowville} \\ 3 & \text{Plympton} \end{cases}$$

$$j = \begin{cases} 1 & \text{Bowville} \\ 2 & \text{Robin State Park} \\ 3 & \text{Plympton} \\ 4 & \text{State Line} \end{cases}$$

Formulation

BB must set values on three decision variables: x_1 , x_2 , and x_3 . They are the proportion of BOD_c that the cannery, Bowville, and Plympton must remove from their waste streams before discharging them in the Bow River. From a knowledge of \underline{x} , we can calculate the corresponding reduction in BOD_n that will take place. Then the DO levels at Bowville, Robin State Park, and Plympton can be calculated. The costs of such treatment can be determined and along with their subsequent effects on profits and taxes.

The criterion functions for goals 1, 2, and 3 and the constraint regarding DO level at the state line are based on equations 6.4 and 6.5. The last three goals are based on equations 6.1, 6.2, and 6.3 respectively.

The decision variables are initially set at .3 and the starting goal levels are:

$$\underline{GL} = (6, 6, 6, 6.5, 1.5, 1.5)$$

As the reader recalls, we will set $\underline{AL}_0 = \underline{GL}$.

Goals and Criterion Functions.

Goal 1: DO level at Bowville = $z_1 \geq \underline{AL}_1$

$$z_1 = 6.5 + 5.68 \cdot 10^{-5} \cdot 4.0 \cdot 10^4 \cdot (x_1 - .3)$$

$$\Gamma(z_1) = (0, 8.5]$$

Goal 2: DO level at Robin State Park = $z_2 \geq \underline{AL}_2$

$$\begin{aligned} z_2 = & 3.7 + (1.31 \times 10^{-5} \cdot 4.0 \times 10^4 \cdot (x_1 - .3) + 3.15 \cdot 10^{-5} \cdot 2.8 \cdot 10^4 \cdot (w_1 - .3)) \\ & + (2.18 \cdot 10^{-5} \cdot 1.28 \times 10^5 \cdot (x_2 - .3) + 5.53 \cdot 10^{-5} \cdot 4.8 \cdot 10^4 \cdot (w_2 - .3)) \end{aligned}$$

$$\Gamma(z_2) = (0, 8.5]$$

Goal 3: DO level at Plympton = $z_3 \geq \underline{AL}_3$

$$\begin{aligned} z_3 = & 5.2 + (4.42 \cdot 10^{-6} \cdot 4.0 \cdot 10^4 \cdot (x_1 - .3) + 7.71 \cdot 10^{-6} \cdot 2.8 \cdot 10^4 \cdot (w_1 - .3)) \\ & + (7.64 \cdot 10^{-6} \cdot 1.28 \cdot 10^5 \cdot (x_2 - .3) + 1.60 \cdot 10^{-5} \cdot 4.8 \cdot 10^4 \cdot (w_2 - .3)) \end{aligned}$$

$$\Gamma(z_3) = (0, 8.5]$$

Goal 4: Percent return on equity at Pierce-Hall Cannery =

$$z_4 \geq \underline{AL}_4$$

$$z_4 = \frac{10^2}{5 \cdot 10^6} \left(3.75 \cdot 10^5 - .6 \cdot \left(\frac{59}{1.09 - x_1^2} - 59 \right) \right)$$

$$\Gamma(z_4) = (0, 7.5]$$

Goal 5: Addition to the tax rate at Bowville = $z_5 \geq AL_5$

$$z_5 = 2.4 \times 10^{-3} \cdot \left(\frac{532}{1.09 - x_2^2} - 532 \right)$$

$$\Gamma(z_5) = (0, 10]$$

Goal 6: Addition to the tax rate at Plympton = $z_6 \geq AL_6$

$$z_6 = 3.33 \cdot 10^{-3} \cdot \left(\frac{450}{1.09 - x^2} - 450 \right)$$

$$\Gamma(z_6) = (0, 12]$$

Constraints and Bounds.

Constraints: DO level at the State Line = $g_1 \geq 0$

$$\begin{aligned} g_1 = & 1.0 + (8.3 \cdot 10^{-7} \cdot 4.0 \cdot 10^4 \cdot (x_1 - .3) + 7.3 \cdot 10^{-7} \cdot 2.8 \cdot 10^4 (w_1 - .3)) \\ & + (1.45 \cdot 10^{-6} \cdot 1.28 \cdot 10^5 \cdot (x_2 - .3) + 1.62 \cdot 10^{-6} \cdot 4.8 \cdot 10^4 (w_2 - .3)) \\ & + (3.49 \cdot 10^{-5} \cdot 9.57 \cdot 10^4 \cdot (x_3 - .3) + 7.33 \cdot 10^{-3} \cdot 3.57 \cdot 10^4 (w_3 - .3)) \\ & - 3.5 \end{aligned}$$

Bounds: Proportionate reduction in gross BOD_c

$$\underline{.3} \leq \underline{x} \leq \underline{1.0}$$

Transformed Equations.

$$y_i = \frac{z_i - 0}{8.5 - 0} + \epsilon \quad i = 1, 2, 3$$

$$y_4 = \frac{z_4 - 0}{7.5 - 0} + \epsilon \quad y_5 = \frac{z_5 - 0}{12 - 0} + \epsilon \quad y_6 = \frac{z_6 - 0}{15 - 0} + \epsilon$$

The A_i are determined similarly.

Then

$$d_i = A_i/y_i \quad i = 1, \dots, 4$$

and

$$d_i = y_i/A_i \quad k = 5, 6$$

So

$$s = \sum_{i=1}^6 d_i$$

Solution

In the following discussion, only the principal problem will be expressed for each cycle. The auxiliary problems are constructed in the same manner as in the example earlier in the chapter. For ease of notation, we will write the constraint for the D0 level at the state line as $g_1 \geq 0$, recognizing that it is formulated in the previous sections. Graphical information will be presented at each cycle for only the goal that we assume BB adjusts and enters as a constraint. However, the complete numerical results from each cycle are tabulated.

The pattern of adjustments that BB makes are of course, merely an example. An alternate view of BB's preference would imply an alternative ranking of the goals and would lead to a different satisfactum.

First Cycle. The principal problem to be solved initially is

$$\text{minimize} \quad s_1 = \sum_{i=1}^6 d_i$$

$$\begin{aligned} \text{subject to} \quad & g_1 \geq 0 \\ & .3 \leq \underline{x} \leq 1.0 \end{aligned}$$

The solutions to this problem and the auxiliary problems are shown in Table 6.6. Examining these results, it is apparent that the change in the tax rate at Plympton (goal 6) is relatively independent of attainment or nonattainment of the other goals because the reduction in BOD_C at Plympton is heavily influenced by the DO constraint at the state line. It seems reasonable to choose an aspiration level for that goal and enter it as a constraint. From Figure 6.7, BB assesses the impact of such an action and sets a new aspiration level of 1.55 for goal 6. BB argues that such an increase is not really different from the original goal of 1.50.

Second Cycle. The principal problem for this cycle is

$$\begin{aligned} \text{minimize} \quad & s_{2.6} = \sum_{i=1}^5 d_i \\ \text{subject to} \quad & g_1 \geq 0 \\ & g_2 = 1.55 - z_6 \geq 0 \\ & .3 \leq \underline{x} \leq 1.0 \end{aligned}$$

Table 6.7 contains the results of this cycle.

BB notes that goal 2 is closely related to goals 1 and 3, and enters it as a constraint on this cycle. Figure 6.8 shows the effects of satisfying this goal. It is apparent that the addition to the tax rate at Bowville is going to be drastically affected if a DO level of

TABLE 6.6

Results of the First Cycle for the Bow River Case Study

$$\underline{AL}_0 = (6.0, 6.0, 6.0, 6.5, 1.5, 1.5)$$

Principal problem $s_1 = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 6.099$		
$\underline{x}_{1,1} = (.8332, .7188, .8221)$	$\underline{z}_1 = (5.960, 4.066, 5.772, 6.421, .712, 1.591)$	$\underline{y}_1 = (.701, .478, .679, .856, .071, .132)$
Auxiliary problems $s_{1,1} = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 5.071$		
$\underline{x}_{1,1} = (.8502, .7180, .8219)$	$\underline{z}_{1,1} = (6.000, 4.092, 5.780, 6.280, .709, 1.589)$	$\underline{d}_{1,1} = (1.000, 1.466, 1.038, 1.035, .473, 1.060)$
$\underline{x}_{1,2} = (.9517, .9564, .8094)$	$\underline{z}_{1,2} = (6.231, 6.000, 6.366, 4.364, 4.506, 1.462)$	$\underline{d}_{1,2} = (.963, 1.000, .942, 1.490, 3.004, .975)$
$\underline{x}_{1,3} = (.8953, .8236, .8169)$	$\underline{z}_{1,3} = (6.080, 4.792, 6.000, 5.896, 1.467, 1.537)$	$\underline{d}_{1,3} = (.987, 1.252, 1.000, 1.103, .372, 1.025)$
$\underline{x}_{1,4} = (.8219, .7188, .8222)$	$\underline{z}_{1,4} = (5.936, 4.048, 5.768, 6.500, .713, 1.592)$	$\underline{d}_{1,4} = (1.011, 1.482, 1.040, 1.000, .475, 1.061)$
$\underline{x}_{1,5} = (.8164, .8369, .8175)$	$\underline{z}_{1,5} = (5.923, 4.684, 5.971, 6.536, 1.500, 1.542)$	$\underline{d}_{1,5} = (1.013, 1.281, 1.005, .994, 1.000, 1.022)$
$\underline{x}_{1,6} = (.9102, .9027, .8133)$	$\underline{z}_{1,6} = (6.136, 5.353, 6.173, 5.502, 2.523, 1.500)$	$\underline{d}_{1,6} = (.978, 1.121, .972, 1.181, 1.582, 1.000)$

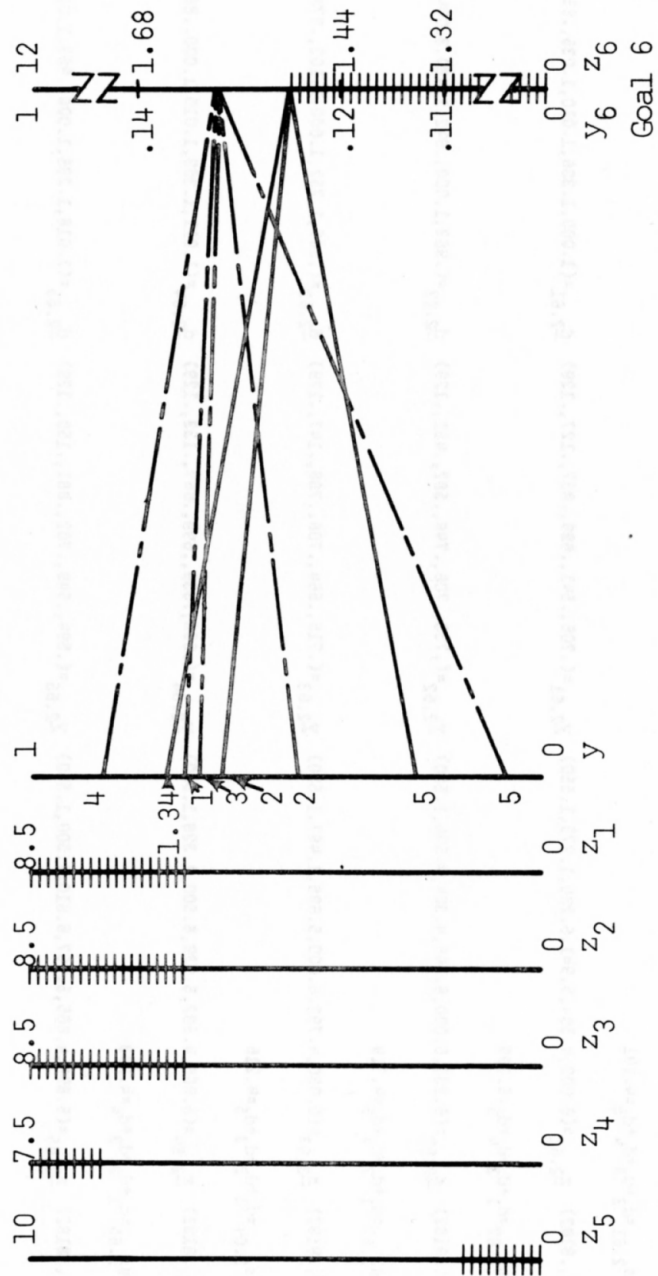


Figure 6.7 Bow River Case Study: Cycle 1, Goal 6 satisfied at $AL_6 = 1.5$.

TABLE 6.7

Results of the Second Cycle for the Bow River Case Study

$$\underline{AL}_1 = (6.0, 6.0, 6.0, 6.5, 1.5, 1.55)$$

Principal problem $s_{2.6} = d_1 + d_2 + d_3 + d_4 + d_5 = 5.198$

$$\underline{x}_{2.6} = (.8633, .8100, .8182) \quad \underline{z}_{2.6} = (6.029, 4.601, 5.942, 6.154, 1.250, 1.550) \quad \underline{y}_{2.6} = (.709, .541, .699, .821, .125, .129) \quad \underline{d}_{2.6} = (.995, 1.304, 1.010, 1.056, .833, 1.000)$$

Auxiliary problems $s_{2.61} = d_2 + d_3 + d_4 + d_5 = 4.201$

$$\underline{x}_{2.61} = (.8502, .8130, .8182) \quad \underline{z}_{2.61} = (6.000, 4.594, 5.941, 6.280, 1.275, 1.550) \quad \underline{y}_{2.61} = (.706, .541, .699, .837, .127, .129) \quad \underline{d}_{2.61} = (1.000, 1.306, 1.010, 1.035, .850, 1.000)$$

$$s_{2.62} = d_1 + d_3 + d_4 + d_5 = 6.399$$

$$\underline{x}_{2.62} = (.9517, .9564, .8182) \quad \underline{z}_{2.62} = (6.231, 6.000, 6.356, 4.364, 4.506, 1.550) \quad \underline{y}_{2.62} = (.733, .706, .749, .582, .451, .129) \quad \underline{d}_{2.62} = (.963, 1.000, .942, 1.490, 3.004, 1.000)$$

$$s_{2.63} = d_1 + d_2 + d_4 + d_5 = 4.319$$

$$\underline{x}_{2.63} = (.8853, .8336, .8182) \quad \underline{z}_{2.63} = (6.080, 4.792, 6.000, 5.896, 1.467, 1.550) \quad \underline{y}_{2.63} = (.715, .554, .706, .786, .147, .129) \quad \underline{d}_{2.63} = (.987, 1.252, 1.000, 1.103, .978, 1.000)$$

$$s_{2.64} = d_1 + d_2 + d_3 + d_5 = 4.216$$

$$\underline{x}_{2.64} = (.8219, .8182, .8182) \quad \underline{z}_{2.64} = (5.966, 4.582, 5.939, 6.500, 1.328, 1.550) \quad \underline{y}_{2.64} = (.698, .539, .699, .867, .133, .129) \quad \underline{d}_{2.64} = (1.011, 1.309, 1.010, 1.000, .886, 1.000)$$

$$s_{2.65} = d_1 + d_2 + d_3 + d_4 = 4.293$$

$$\underline{x}_{2.65} = (.8042, .8369, .8182) \quad \underline{z}_{2.65} = (5.896, 4.665, 5.967, 6.610, 1.500, 1.550) \quad \underline{y}_{2.65} = (.694, .549, .702, .881, .150, .129) \quad \underline{d}_{2.65} = (1.018, 1.286, 1.006, .983, 1.000, 1.000)$$

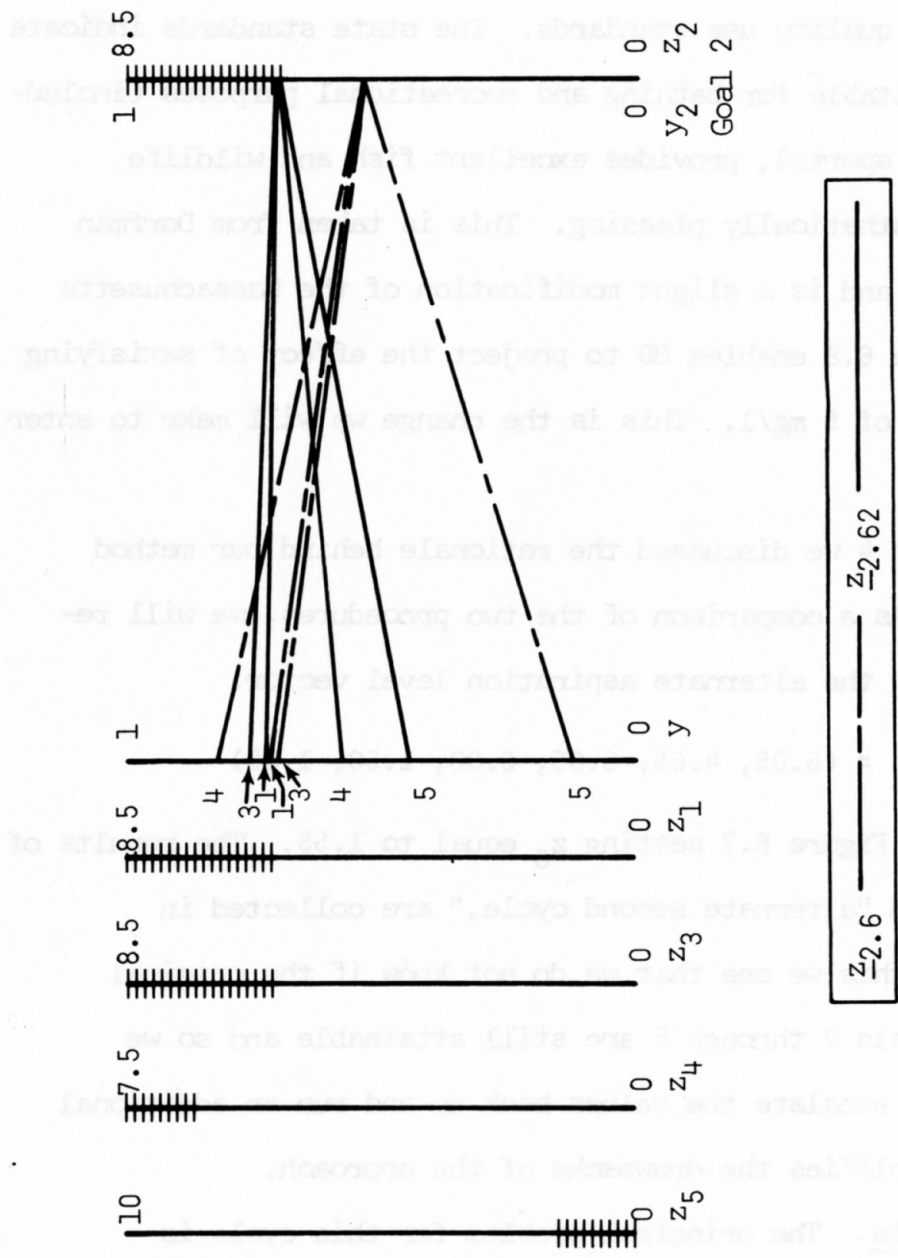


Figure 6.8 Bow River Case Study: Cycle 2, Goal 2 satisfied at $AL_2 = 5.0$.

Goal 6 is not included because it is satisfied under all conditions at $AL_6 = 1.55$.

6 mg/l at Robin State Park is to be achieved. At this point BB discovers that the goal of 6 mg/l was set rather arbitrarily without regard to accepted water quality use standards. The state standards indicate that 5 mg/l is suitable for bathing and recreational purposes (including water contact sports), provides excellent fish and wildlife habitat, and is esthetically pleasing. This is taken from Dorfman and Jacoby (1969) and is a slight modification of the Massachusetts standards. Figure 6.8 enables BB to project the effect of satisfying goal 2 at a level of 5 mg/l. This is the change we will make to enter the third cycle.

In Chapter 4 we discussed the rationale behind our method of altering AL. As a comparison of the two procedures, we will repeat cycle 2 using the alternate aspiration level vector:

$$AL_1 = (6.05, 4.65, 5.95, 6.00, 1.50, 1.55)$$

which is based on Figure 6.7 setting z_6 equal to 1.55. The results of that cycle, termed "alternate second cycle," are collected in Table 6.8. From this we see that we do not know if the original objectives for goals 2 through 5 are still attainable and so we would have to now escalate the values back up and run an additional cycle. This exemplifies the drawbacks of the approach.

Third Cycle. The principal problem for this cycle is

$$\text{minimize} \quad s_{3.62} = d_1 + d_3 + d_4 + d_5$$

TABLE 6.8

Results of the Alternate Second Cycle for the Bow River Case Study

$$AL_1 = (6.05, 4.65, 5.95, 6.00, 1.50, 1.55)$$

Principal problem $s_{2.6} = d_1 + d_2 + d_3 + d_4 + d_5 = 4.823$

$$x_{2.6} = (.8633, .8100, .8182) \quad z_{2.6} = (6.030, 4.601, 5.942, 6.154, 1.250, 1.550) \quad y_{2.6} = (.709, .541, .699, .821, .125, .129) \quad d_{2.6} = (1.003, 1.011, 1.001, .975, .833, 1.000)$$

Auxiliary problems

$$s_{2.61} = d_1 + d_2 + d_3 + d_4 + d_5 = 3.822$$

$$x_{2.61} = (.8722, .8078, .8182) \quad z_{2.61} = (6.050, 4.606, 5.943, 6.058, 1.232, 1.550) \quad y_{2.61} = (.712, .542, .699, .808, .123, .129) \quad d_{2.61} = (1.000, 1.009, 1.001, .990, .821, 1.000)$$

$$s_{2.62} = d_1 + d_2 + d_3 + d_4 + d_5 = 3.849$$

$$x_{2.62} = (.8789, .8128, .8182) \quad z_{2.62} = (6.065, 4.650, 5.956, 5.978, 1.273, 1.550) \quad y_{2.62} = (.714, .547, .700, .797, .127, .129) \quad d_{2.62} = (.997, 1.000, .999, 1.004, .849, 1.000)$$

$$s_{2.63} = d_1 + d_2 + d_3 + d_4 + d_5 = 3.837$$

$$x_{2.63} = (.8789, .8098, .8182) \quad z_{2.63} = (6.065, 4.632, 5.950, 5.978, 1.248, 1.550) \quad y_{2.63} = (.714, .545, .700, .797, .125, .129) \quad d_{2.63} = (.997, 1.004, 1.000, 1.004, .832, 1.000)$$

$$s_{2.64} = d_1 + d_2 + d_3 + d_4 + d_5 = 3.822$$

$$x_{2.64} = (.8771, .8066, .8182) \quad z_{2.64} = (6.061, 4.609, 5.942, 6.000, 1.222, 1.550) \quad y_{2.64} = (.713, .542, .699, .800, .122, .129) \quad d_{2.64} = (.998, 1.009, 1.001, 1.000, .814, 1.000)$$

$$s_{2.65} = d_1 + d_2 + d_3 + d_4 + d_5 = 3.928$$

$$x_{2.65} = (.8007, .8369, .8182) \quad z_{2.65} = (5.888, 4.660, 5.965, 6.630, 1.500, 1.550) \quad y_{2.65} = (.712, .548, .700, .800, .150, .129) \quad d_{2.65} = (1.028, .998, .997, .904, 1.000, 1.000)$$

$$\begin{aligned}
&\text{subject to } g_1 \geq 0 \\
&g_2 = 1.55 - z_6 \geq 0 \\
&g_3 = z_2 - 5.0 \geq 0 \\
&.3 \leq \underline{x} \leq 1.0 \\
&\text{with } \underline{AL}_2 = (6.0, 5.0, 6.0, 6.5, 1.5, 1.55)
\end{aligned}$$

Table 6.9 contains the results of this cycle.

The continued economic existence of the Pierce-Hall cannery is important to the welfare of the whole valley, but particularly to Bowville. Almost the entire work force of 800 people live at Bowville; and if the cannery goes out of business, they will create a burden on the city because there are no other immediate sources of employment for them. Consequently, it is to Bowville's advantage to keep the cannery operating if such an action does not penalize the city too much. BB recognizes this situation and decides that a firm bound on the return on investment for the cannery must be entered as a constraint. This is goal 4 and examination of Figure 6.9 enables BB to adopt 6.0% as an acceptable level of return. This constraint is entered into the problem as input into the next cycle.

Fourth Cycle. The principal problem to be solved in this cycle is

$$\begin{aligned}
&\text{minimize } s_{4.624} = d_1 + d_3 + d_5 \\
&\text{subject to } g_1 \geq 0 \\
&g_2 = 1.55 - z_6 \geq 0
\end{aligned}$$

TABLE 6.9

Results of the Third Cycle for the Bow River Case Study

$$\underline{AL}_2 = (6.0, 5.0, 6.0, 6.5, 1.5, 1.55)$$

Principal problem	$s_{3.62}=d_1+d_2+d_4+d_5=4.299$			
$x_{3.62}=(.8945,.8617,.8182)$	$\underline{z}_{3.62}=(6.101,5.000,6.065,5.765,1.798,1.550)$	$y_{3.62}=(.718,.588,.713,.769,.180,.129)$	$\underline{d}_{3.62}=(.983,1.000,.889,1.127,1.199,1.000)$	
Auxiliary problems	$s_{3.621}=d_3+d_4+d_5=3.317$			
$x_{3.621}=(.8867,.8642,.8182)$	$\underline{z}_{3.621}=(6.083,5.000,6.065,5.877,1.833,1.550)$	$y_{3.621}=(.716,.588,.714,.784,.183,.129)$	$\underline{d}_{3.621}=(.986,1.000,.989,1.106,1.222,1.000)$	
	$s_{3.623}=d_1+d_4+d_5=3.310$			
$x_{3.623}=(.9063,.8579,.8182)$	$\underline{z}_{3.623}=(6.127,5.000,6.063,5.573,1.746,1.550)$	$y_{3.623}=(.721,.588,.713,.743,.175,.129)$	$\underline{d}_{3.623}=(.979,1.000,.990,1.166,1.164,1.000)$	
	$s_{3.624}=d_1+d_3+d_4=3.393$			
$x_{3.624}=(.8219,.8609,.8182)$	$\underline{z}_{3.624}=(5.936,5.000,6.071,6.500,2.090,1.550)$	$y_{3.624}=(.698,.588,.714,.867,.209,.129)$	$\underline{d}_{3.624}=(1.011,1.000,.988,1.000,1.393,1.000)$	
	$s_{3.625}=d_1+d_3+d_4=3.484$			
$x_{3.625}=(.9547,.8369,.8182)$	$\underline{z}_{3.625}=(6.237,5.000,6.056,4.244,1.500,1.550)$	$y_{3.625}=(.734,.588,.712,.566,.150,.129)$	$\underline{d}_{3.625}=(.962,1.000,.991,1.532,1.000,1.000)$	

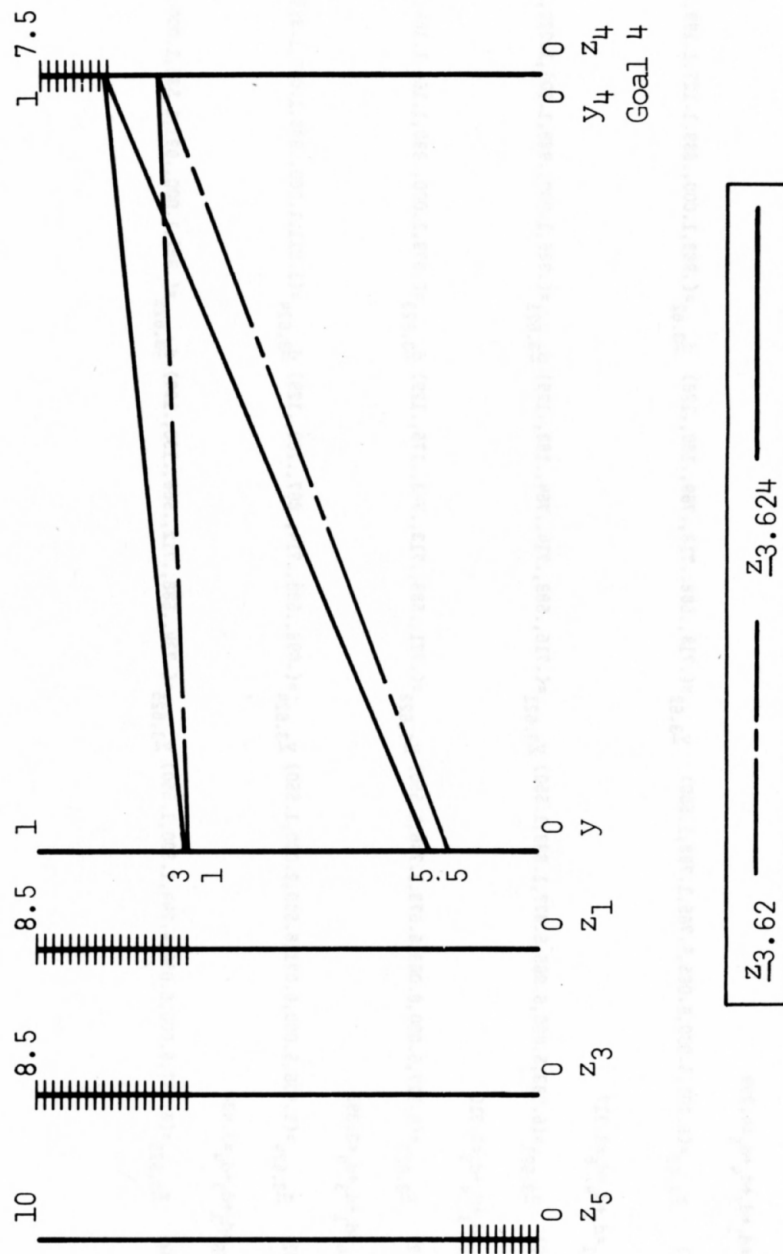


Figure 6.9 Bow River Case Study: Cycle 3, Goal 4 satisfied at $AL_4 = 6.0$. Goals 2 and 6 are excluded because they are satisfied under all alternatives.

$$g_3 = z_2 - 5.0 \geq 0$$

$$g_4 = z_4 - 6.0 \geq 0$$

$$.3 \leq \underline{x} \leq 1.0$$

The results are tabulated in Table 6.10.

In attempting to impose attainment of goal 5 as an additional constraint, BB is informed by the computer program that he has apparently formed an inconsistent constraint set (see Chapter 5). After several attempts with alternate starting points to find a feasible point, he concludes that AL_5 will indeed have to be modified or else he will have to reevaluate the previous aspirations he has developed and satisfied. Fortunately, raising AL_5 is not logically at odds with the development up to this point. BB rationalizes such an action by noting that Bowville is already at an advantage with respect to Plympton in the realm of tax rates. Consequently, in BB's desire to provide equitable treatment to all his constituents and to alleviate that above mentioned disparity, he modifies his aspirations regarding goal 5.

BB observes in Table 6.10 that the value z_5 is hovering around 1.87 as the various other goals are satisfied, so he has to decide if such an aspiration level is compatible with his other desires. In particular, it is not overly burdensome to Bowville. We will assume that BB does allow AL_5 to be raised to 1.90 and enter into the next cycle.

TABLE 6.10

Results of the Fourth Cycle for the Bow River Case Study

$$\underline{AL}_3 = (6.0, 5.0, 6.0, 6.0, 1.5, 1.55)$$

Principal problem $s_{4.624} = d_1 + d_3 + d_5 = 3.228$

$$\underline{x} = (.8771, .8670, .8182) \quad \underline{z} = (6.061, 5.000, 6.066, 5.000, 1.874, 1.55) \quad \underline{y} = (.713, .588, .714, .800, .187, .129) \quad \underline{d} = (.990, 1.000, .989, 1.000, 1.249, 1.000) \\ \underline{u}_{4.624}$$

Auxiliary problems $s_{4.6241} = d_3 + d_5 = 2.238$

$$\underline{x} = (.8771, .8670, .8182) \quad \underline{z} = (6.061, 5.000, 6.066, 6.000, 1.874, 1.550) \quad \underline{y} = (.713, .588, .714, .800, .187, .129) \quad \underline{d} = (.990, 1.000, .989, 1.000, 1.249, 1.000) \\ \underline{u}_{4.6241}$$

$$s_{4.6243} = d_1 + d_5 = 2.238$$

$$\underline{x} = (.8771, .8670, .8182) \quad \underline{z} = (6.061, 5.000, 6.066, 6.000, 1.874, 1.550) \quad \underline{y} = (.713, .488, .714, .800, .187, .129) \quad \underline{d} = (.990, 1.000, .989, 1.000, 1.249, 1.000) \\ \underline{u}_{4.6243}$$

$$s_{4.6245} = d_1 + d_3$$

(APPARENT INCONSISTENT CONSTRAINT SET)

Fifth Cycle and Termination. The principal problem to be solved on this cycle is

$$\begin{aligned}
 &\text{minimize} && s_{5.6245} = d_1 + d_3 \\
 &\text{subject to} && g_1 \geq 0 \\
 & && g_2 = 1.55 - z_6 \geq 0 \\
 & && g_3 = z_2 - 5.0 \geq 0 \\
 & && g_4 = z_4 - 6.0 \geq 0 \\
 & && .3 \leq \underline{x} \leq 1.0
 \end{aligned}$$

The results of this cycle are shown in Table 6.11. From this information BB decides that he has reached a satisfactum. BB's policy decision is to impose waste reduction requirements of 88% on the cannery, 87% on Bowville, and 82% on Plympton. The results have been rounded off because it is difficult to maintain fine control on waste treatment processes.

Discussion

This example has attempted to demonstrate the applicability of this algorithm to realistic problems. It shows that the solution or satisfactum which is finally reached is a result of the individual value structure of the DM — BB in this case. We can imagine that a change in BB's constituency would alter the choice of an acceptable policy. In addition, if some members of the commission are able to exert extraordinary influence, this too would be reflected in BB's

TABLE 6.11

Results of the Fifth Cycle for the Bow River Case Study

$$\underline{AL}_4 = (6.0, 5.0, 6.0, 6.0, 1.90, 1.55)$$

Principal problem	$S_{.6245} = d_1 + d_3 = .990$				
$x = (.8771, .8688, .8182)$ $S_{.6245}$	$z = (6.061, 5.013, 6.070, 6.000, 1.900, 1.550)$ $S_{.6245}$	$y = (.713, .590, .714, .800, .190, .129)$ $S_{.6245}$	$d = (.990, .997, .998, 1.000, 1.000, 1.000)$ $S_{.6245}$		
Auxiliary problems	$S_{.62451} = d_3 = .988$				
$x = (.8771, .8688, .8182)$ $S_{.62451}$	$z = (6.061, 5.013, 6.070, 6.000, 1.900, 1.550)$ $S_{.62451}$	$y = (.713, .590, .714, .800, .190, .129)$ $S_{.62451}$	$d = (.990, .997, .998, 1.000, 1.000, 1.000)$ $S_{.62451}$		
	$S_{.62453} = d_1 = .990$				
$x = (.8771, .8688, .8182)$ $S_{.62453}$	$z = (6.061, 5.013, 6.070, 6.000, 1.900, 1.550)$ $S_{.62453}$	$y = (.713, .590, .714, .800, .190, .129)$ $S_{.62453}$	$d = (.990, .997, .998, 1.000, 1.000, 1.000)$ $S_{.62453}$		

choices. Of course, in the example, the pattern of choices reflected our own biases.

The results obtained by Dorfman and Jacoby are not comparable to ours because we have changed most of the numerical values in their original formulation. However, we can comment that their methodology seems less satisfactory than ours for two reasons:

- 1) They use an explicit goal weighting scheme which assumes that the weights are constants. This does not seem to provide the DM an opportunity to change his mind as he receives information concerning the set of available alternatives.

- 2) The weighting scheme also requires that the DM be able to develop a complete ordering on the goals before receiving any information concerning available alternatives. In our method, we ask only that the DM be able to select the most important unsatisfied goal at each cycle of the algorithm. This allows him to build, or perhaps discover, a complete ordering as the information is generated.

CHAPTER 7

DISCUSSION, CONCLUSIONS

Hopefully, the preceding examples have clarified the mechanics of the algorithm we have developed so that we can examine its limitations and progress to some conclusions regarding the technique. Our work has been principally exploratory, so we will be limited in the number of definite conclusions we can propose. In discussing the limitations of the research, we will make suggestions for future areas of exploration.

We can look back to determine whether the objectives of this research have been accomplished, in particular, whether we solved the problem stated in Chapter 3. Our answer to this is a cautious "yes." It is a cautious reply because the algorithm has not been tested with an actual decision-making individual or group in a real decision situation.

Discussion

The many limitations of this research can be divided into those associated with the definition and scope of the research problem and those associated with the algorithm developed during the project.

Scope and Definition of the Research Problem

1. The objectivity requirements regarding goal attainment limit the class of decision situations to those whose criterion functions can be expressed analytically. However, it is entirely possible that many decision problems exist in which the criterion function is an implicit evaluation in itself. As an example, the level of employee morale may, in the final analysis, best be measured by an intuitive feeling on the part of a skilled manager rather than an objective response surface. To some extent, this is a reflection of our understanding of the forces at work, and the future may reveal objective measurements which will make such problems amenable to solution.

2. As indicated previously, it is unlikely that the DM can comprehend more than 7 goals at one time. There are $7!$ possible rankings of this set of goals, so it is an implicit requirement that the DM be able to partition them into subsets that can be ranked. This assures that at each iteration he will be able to decide for which goal(s) to fix the aspiration levels. We have assumed that the pattern in which the aspirations are assigned reflects the DM's implicit ranking of the goals. It is not necessary that this be true, but the process could continually cycle if the DM is in an oscillatory frame of mind. Note that a "re-ranking" of these goals, a realignment of the value structure does not cause any difficulties unless the DM oscillates between alternate value structures and the corresponding aspiration levels remain unchanged. At this

point we are again in the area of psychological convergence to a satisfactum and our earlier comments in Chapter 4 still apply.

Future work with this type of algorithm (Interactive Programming) might be directed toward an investigation of the behavior of a DM for different types of decision situations. In particular, two questions might be investigated. First, do DM's in general tend to develop a single unmodified ranking of the goals when solving such problems interactively? Or do they "change their minds" frequently and restart the solution procedure with different rankings and goals based on their previous results? Second, is there a relationship between the number of goals and the number of times that alternate rankings are developed?

3. The assumption that the preference function, although implicit, possesses completeness has been questioned by Aumann (1964, p. 222). He argues that the decision maker may not have "well-defined preferences" between two alternatives. However, the alternatives the decision maker is faced with in the search procedure are, we believe, comparable because they differ only within the framework of the overall problem. That is, the elements in the vector of values from the set of criterion functions vary in magnitude, but not in nature. The decision maker may be called upon to compare the vectors (pollution of 10ppm, 10% return on investment) vs. (pollution of 20ppm, 15% return on investment). This is, we maintain, not the same problem as deciding between (pollution of 10ppm, 10% return on investment) vs. (wearing gray suit to office, dinner afterwards).

4. The subject of group decision-making has been purposely avoided in this research because it is so complex. However, the pervasiveness of this form of problem-solving (the ever-present committee) requires us to address ourselves to the subject and comment upon the applicability of this algorithm to such decision-making problems. We have sidestepped the issue of the Bow River Valley Water Commission by inserting a synthetic individual, BB, into the situation as a DM. Now we wish to ask: Can this method be used by a group of people who must interact and arrive at a decision?

We note that this process (decision-making) is more complex for groups because it involves synthesizing the individual preferences of the members. More exactly, it requires a subtle flow of influence and persuasions so the individuals' aspirations are altered sufficiently to allow some measure of accord with respect to the various alternatives presented.

As the decision-making process begins, ". . . we have the possibility that each judge has his own unique bias for judging an optimum. When this happens, any average would be inappropriate; a different optimum would have to be determined for this judge." (Gulliksen 1964, p. 73) Obviously, if each member of the group refused to modify his aspirations, then each would be pursuing his own optimum and no decisions could be made. However, since decisions are actually made in the real world, people must be willing to make concessions to the group.

One possible explanation for this willingness to shift is that, although the individual's preference structure is well-defined on important (to him) issues, it is far less rigid on matters which are the normal subject of debate. As an illustration, a committee discussion concerning whether or not to legalize euthanasia would generate much heated debate because the subject touches directly upon fundamental values which guide the individual's life.

Thus the usefulness of our algorithm in such a situation is based on its ability to enable the various DM's to explore not only their own alternatives but those of the other members of the group. We envision an application in which each DM uses the algorithm independently to find his satisfactum. After this is accomplished, pooling of results will hopefully put the individuals in a position to make additional cycles while modifying their aspiration levels until at least a majority are satisfied with an alternative. An added benefit of this approach is that, subject to our previous assumptions, it will show inconsistencies in the aspirations of the various members and allow them to work out a satisfactory solution.

Characteristics of the Algorithm

1. The transformations on the goal formulations, Table 4.1, are not linear and vary among the different types of goals. Some are more sensitive than others to changes in the transformed criterion functions and (indirectly) to changes in the policy vector. So the goals are not treated identically in the surrogate objective

function and this prevents direct comparison of the values for the dimensionless indicator of attainment.

We have been content to find a set of transformations that have the property of giving a value greater than "one" if the goal is not satisfied with respect to the current aspiration level for that goal. But in accomplishing this, we have acquired the difficulties listed above. We are unable to find a set of transformations which are an improvement, in that regard, over the current set but research in this area would perhaps produce them. This would be highly desirable because this situation is likely the most severe mathematical limitation on this research.

2. The development of the surrogate objective function has been predicated on the requirement that the criterion functions be continuous so that the indicator of attainment, d , varies continuously. In an alternate problem formulation involving integer solutions, a different optimization mechanism would have to be imbedded in the algorithm along with a means of adding and deleting constraints. The formulation of d can remain the same, but it will not be continuous. In a more general vein, a mixed integer nonlinear programming problem would require a different mechanism. The expansion of this technique into such areas represents a potential subject for further research.

3. We have indicated that the selection of values denoting the relevant range for each criterion function is arbitrary provided the actual variation is contained in the specified interval. This

ensures that the transformed criterion function, y , is in the interval $(0,1]$. It is tempting to automatically set the range for all functions at

$$\Gamma = [a, b]$$

where $a \rightarrow -\infty$ and $b \rightarrow +\infty$. However, if this is done, then

$$y = \frac{z - z_{\text{lower}}}{z_{\text{upper}} - z_{\text{lower}}} \rightarrow .5$$

for all values of z . Consequently, the variation in y is very small and the surrogate objective function, s , is essentially a constant rather than a function of the policy vector, \underline{x} .

But trying to tightly bracket the variation in z also contains a pitfall. If z should, during the course of the optimization phase, pass outside the specified range, Γ , then the requirements of the algorithm are not satisfied and invalid (from our point of view) results can occur. This is particularly true in the instance where z becomes less than z_{lower} because then y is negative, the corresponding d term is negative, and so s , the function we are minimizing, will contain a negative term.

Consequently, the range specified for each function should be the minimum and maximum values of the function evaluated at the bounds of the decision variables. This is based on the fact that, if the optimization phase starts at a point within those bounds, then it is impossible for those bounds to be exceeded due to the construction of the algorithm.

4. The programming requirements for the optimization phase present a more pragmatic limitation. The complexity and cost of nonlinear programming will probably restrict application of this method to "significant" decision problems. The setting up of the problem may require a substantial time delay implying that the decision situation does not require immediate resolution. We have tried to make the program as general as possible, but the formulation of the surrogate objective function and the derivation of the expressions for the partial derivatives may be quite tedious.

5. The response surfaces have been assumed deterministic. A stochastic criterion function could be used by setting the response surface equal to the mean values of the function at each point. This may, however, prove quite unsatisfactory if the distribution is skewed or possesses a large variance. Our inquiries into the literature have revealed that there is not, apparently, a satisfactory algorithm for stochastic nonlinear programming; consequently, this topic may be difficult to explore. Parenthetically, we note that the difficulties attendant in developing a universally accepted deterministic nonlinear programming algorithm foretell a long delay in the development of a similar stochastic technique.

Comparison with Alternate Approaches

Any assessment of the algorithm should include a comparison with other methods for solving the same or similar problems. The literature in the area is still somewhat sparse but there are a few alternative techniques which we might consider.

Initially, we can return to the classification of Roy (1970) for a source of alternatives. We have already discussed his various categories in deciding upon the philosophy of solution that we felt best suited to our needs. Looking back, we can compare our work with that of others he cited. The method of Benayoun et al. (1970) is limited to linear frameworks and so is more restrictive than our method; the work by Geoffrion (1970) is more closely related to ours. That method has been applied (Geoffrion, Dyer, and Feinberg [1971]) to the operation of an academic department. Geoffrion supposes, as do we, the existence of an ordinal preference function (structure) but he also requires that it be concave, increasing and differentiable. We note that Geoffrion et al. (1971) also regard 6-8 goals as a reasonable number for the DM to consider at one time. The DM supplies explicit information regarding substitution rates between commodities at any point in the constraint space. This effectively specifies a direction of movement. The DM must also select the amount of movement to be allowed. Our opinion is that this procedure

represents a fairly formidable task to set before a DM, requiring him to verbalize trade-off rates that he may be unwilling to specify.

The second part of each iteration in that method, selection of amount of movement, is done in a graphical manner analogous to that employed in this paper. As a final note, it is unclear how different types of goals, such as an interval type goal, can be represented in this system.

Johnsen (1968, Chapter 9) discusses many multiple goal models, most of which require a weighting scheme in one guise or another. However, he also examines the concept of satisfying, rather than optimizing models. In particular he notes that classical optimization methods do not even apply in many problem situations. Extending the psychological implications of this thought, he considers learning and adaptation models as valid tools for the solution of multiple objective problems. So in his regard for the role of information in decision making and in his interpretation of the psychological aspects of it, we would appear to be echoing similar thoughts. But his solution to the problem is via simulation of a system designed to have certain specific properties. It seems that such an approach would be highly desirable for problems involving, say organizational design, where the decision problem involves the choice of a system or model. Our algorithm would not apply in such a case unless all of the criterion functions could be specified. However, simulation seems inappropriate in determining answers for a situation in which the operating system is fixed for the duration of the decision

situation. That is, in the Bow River example, the simulation approach would be useful in attempting to formulate a "better" system with regard to location of the cities, allowed levels of population, etc. (Although "better" would have to be defined.) However, it would be more cumbersome in trying to locate a satisfactum for the specific problem we examined.

Conclusions

In reflecting upon these alternative methods, we conclude that our research is particularly useful because the restrictions it places upon the constraint set, response surfaces, and preference function are minimal when compared to other methods. It seems capable of attacking realistic mathematical formulations without any modification. Specifically we have found the following:

(1) The method of converting the various types of goals into a surrogate objective function explicitly recognizes that there are different types of goal formulations. In particular, it removes from consideration the concept of maximize (or minimize) as a goal in itself.

(2) The method of presenting the results graphically increases the feasibility of actually implementing this algorithm because the method does not require unusual analytical skills on the part of the DM.

(3) Although the computer program requires that a subroutine be written for it for each problem, the size of the problem that can be solved is approximated by the storage required for the LP phase

$$(L+M+1)N + (L+M+N+1)(L+2M+3N) \leq \text{core size of computer}$$

where L is the number of equality constraints including goals expressed as equalities, M is the number of inequality constraints including goals expressed as inequalities, and N is the number of decision variables. For example, in the Bow River problem, M was 7 and N was 3, so the program required approximately 24 plus 253 storage locations for the LP phase. Thus the program can be implemented on almost any size computer.

The time required to solve the problem is, of course, a function of the problem itself. We cannot give any formulas to compute that time. But we can note that the total central processor time required to solve the Bow River example was 18 seconds on a CDC 6400. This included using 2 starting points for each problem.

(4) The psychological framework upon which the algorithm has been built seems to be a reasonable description of real-world decision making. Consequently the algorithm is expected to produce results more reflective of the perceptions and biases of the DM who uses it. This is, we argue, a necessary ingredient for a useful decision-making tool in areas where personal judgment plays an all important part in the evaluation of the results.

(5) The interactive feature allows the DM to develop a ranking of the goals and to acquire a feel for their interrelationships with the constraint set. This learning process is, itself, an important aspect of the algorithm.

(6) The investigation of the realistic synthetic example of the Bow River has shown how a DM can locate at an acceptable alternative.

To summarize, although there are limitations upon the algorithm developed and the class of decision situations to which it can be applied, the method is nevertheless a viable framework within which to formulate decision problems and to generate feasible acceptable alternatives.

APPENDIX A

MULTIPLE GOAL FORMULATION REFERENCES

The following list provides a sample of authors who have indicated that a multiple goal formulation is normally a part of every decision-making problem.

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APPENDIX B

MULTIPLE OBJECTIVE PROBLEM SOLVING PROGRAM (MOPS)

```

C      PROGRAM MOPS(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C      SUBROUTINE USER IS WRITTEN FOR EACH PROBLEM
C      CONSTRAINTS ARE ENTERED, EQUALITY CONSTRAINTS FIRST
C      INCLUDE ALL POSSIBLE GOAL FORMULATIONS. THEN THE
C      CORRESPONDING FORMULAS FOR THE FIRST PARTIAL
C      DERIVATIVES ARE ENTERED.
C
C
C      INPUT DATA
C      NX      NO. OF INDEPENDENT VARIABLES
C      NE      NO. OF EQUALITY CONSTRAINTS
C      NN      NO. OF INEQUALITY CONSTRAINTS
C      CONVRG   TERMINATION CRITERION
C      EPIS     TERMINATION CRITERION
C      NGOALS   NO. OF GOALS
C      DX      INITIAL STEP SIZE VECTOR
C      X        INITIAL VARIABLE VECTOR
C      XMIN AND XMAX  VECTORS OF BOUNDS ON X
C
C      FOR EACH CYCLE, INPUT DATA
C      ITITLE   80 ALPHANUMERIC CHARACTERS
C      IEX      VECTOR OF GOALS IN SURROGATE OBJECTIVE FUNCTION
C              (1=GOAL IN, 0=GOAL OUT)
C      ALEVEL   ASPIRATION LEVEL VECTOR
C      THE USER IS ADVISED TO EXAMINE SUBROUTINE USER TO
C      DETERMINE THE METHOD FOR EXPRESSING THE VARIOUS
C      GOALS AND CONSTRAINTS.
C
C      COMMON /1/ X(10),DEL(10),NX,NN,NC,INQ,VALQ,KK
C      COMMON /2/ OBJFN,CONSTR(10)
C      COMMON /3/ R(15,25)
C      COMMON /6/ IEXIT(10),ALEVEL(10)
C      DIMENSION B(25), OBJ(25), MOVE(10), DX(10)
C      DIMENSION IEX(10), XMAX(10), XMIN(10), XMOVE(10)
C      1, RJ(15), IEXIT(10), ITITLE(8), XORIG(10), DXORIG(10)
C      EQUIVALENCE (NX,N)
C      READ 650, NX,NE,NN,CONVRG,EPIS,NGOALS

```

```

XLRG=1.E20
NC=NE+NN
NM1=NE
NM2=NE+1
NM3=NC
NM4=NC+1
NM5=NX+NC
NM6=NM5+1
NM7=NM5+NX
NM8=NM7+NX
READ 660, (DX(I),I=1,N)
READ 660, (X(I),I=1,N)
READ 660, (XMIN(I),I=1,NX)
READ 660, (XMAX(I),I=1,NX)
DO 110 I=1,N
DXORIG(I)=DX(I)
110 XORIG(I)=XMIN(I)+.5*(XMAX(I)-XMIN(I))
120 PRINT 670
READ 680, ITITLE
IF (ITITLE(1).EQ.10HSTOP) GO TO 640
READ 690, (IEX(I),I=1,NGOALS)
READ 700, (ALEVEL(I),I=1,NGOALS)
ILAST=3HNO
IFIRST=3HYES
130 CONTINUE
IDAVID=0
IF (ILAST.EQ.3HYES) GO TO 120
CALL PERMUTE (IFIRST,ILAST,IEX,NGOALS)
PRINT 710, NX,NE,NN,CONVRG,EPIS,NGOALS
PRINT 720, ((DX(I),XMIN(I),X(I),XMAX(I))),I=1,N)
140 CONTINUE
YOLD=0
YNEW=1.E10
ITRIGER=0
IPOITT=0
DO 150 I=1,N
DX(I)=DXORIG(I)
150 CONTINUE
DO 160 I=1,N
XMOVE(I)=0
160 MOVE(I)=1
PRINT 730, ITITLE
II=0
DO 180 I=1,NGOALS
IF (IEXIT(I)) 170,180,170
170 II=II+1
IIEXIT(II)=I
180 CONTINUE

```

```

      IF (II.GT.0) GO TO 190
      PRINT 740
      GO TO 200
190   PRINT 750, (IIEXIT(J),J=1,II)
200   PRINT 760, (ALEVEL(I),I=1,NGOALS)
      CALL WRITEX (1,-1)
      CALL SECOND (TIME)
      PRINT 770, TIME
210   INDEX=0
      SUMSQ=0.0
      IF (IPOITT.LT.15000) GO TO 220
      CALL WRITEX (3,IPOITT)
      GO TO 130
220   CONTINUE
      IF (NE) 250,250,230
230   DO 240 I=1,NE
      KK=1
      INQ=I+1
      CALL USER
      RJ(I)=VALQ
240   SUMSQ=SUMSQ+RJ(I)*RJ(I)
250   IF (NC-NM2) 290,260,260
260   DO 280 I=NM2,NC
      KK=1
      INQ=I+1
      CALL USER
      RJ(I)=VALQ
270   SUMSQ=SUMSQ+RJ(I)*RJ(I)
280   CONTINUE
290   INDEX=INDEX+1
C     COMPUTE MATRIX NM5XNM5
      DO 310 I=1,NM5
      DO 300 J=1,NM5
300   R(J,I)=0.
310   R(I,I)=1.
      YDELTA=ABS(YNEW-YOLD)
      YOLD=YNEW
C     CHECK TO SEE IF TERMINATION ALLOWED
      IF (ITRIGER.NE.2) GO TO 330
      ICK=3
      IF (SUMSQ.GT.CONVRG) ICK=4
      IF (YDELTA.GT.EPIS) ICK=5+ICK-3
C     ICK=3  NORMAL TERMINATION
C     ICK=4  DELX LT EPIS  DELY LT EPIS  SUMSQ GT CONVRG
C     INCONSISTENT CONSTRAINT SET
C     ICK=5  DELX LT EPIS  DELY GT EPIS  SUMSQ LT CONVRG
C     KEEP GOING

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```

C      ICK =6  DELX LT EPIS  DELY BT EPIS  SUMSQ GT CONVRG
      IF (ICK.EQ.5) GO TO 330
      IPOITT=IPOITT-1
      CALL WRITEX (ICK,IPOITT)
      IF (IDAVID.EQ.1) GO TO 130
      IDAVID=1
      DO 320 I=1,N
320    X(I)=XORIG(I)
      GO TO 140
330    CONTINUE
      KK=2
      INQ=1
      CALL USER
C      COMPUTE MATRIX NM5 X NM8
      DO 420 I=1,NX
      DO 340 J=NM4,NM5
340    R(J,I+NM5)=0.
      R(J,I+NM7)=0.
      RKW1=RKW2=1
      IF (XMOVE(I)) 350,370,360
350    RKW1=2.
      RKW2=1.
      GO TO 370
360    RKW1=1.
      RKW2=2.
C      COMPUTE VALUES FOR FIRST ROW OF MATRIX AFTER
C      UNIT MATRIX
370    IF ((XMAX(I)-X(I)).EQ.0.) GO TO 380
      R(I+NM3,I+NM5)=AMAX1(RKW1,(DX(I)/(XMAX(I)-X(I))))
      GO TO 390
380    R(I+NM3,I+NM5)=AMAX1(RKW1,(DX(I)/1.E-100))
390    IF ((X(I)-XMIN(I)).EQ.0.) GO TO 400
      R(I+NM3,I+NM7)=AMAX1(RKW2,(DX(I)/(X(I)-XMIN(I))))
      GO TO 410
400    R(I+NM3,I+NM7)=AMAX1(RKW2,(DX(I)/1.E-100))
410    B(I+NM3)=DX(I)
      OBJ(NM5+I)=DEL(I)
      OBJ(NM7+I)=-DEL(I)
420    CONTINUE
      DO 430 J=1,NC
      KK=2
      INQ=J+1
      CALL USER
      DO 430 I=1,N
      R(J,I+NM7)=DEL(I)
      R(J,I+NM5)=-DEL(I)
430    CONTINUE
      DO 440 I=1,NM3

```

```

440  OBJ(I)=XLRG
      DO 450 I=NM4,NM5
450  OBJ(I)=0.
      IF (NE) 490,490,460
460  DO 490 I=1,NE
      B(I)=ABS(RJ(I))
      IF (RJ(I)) 470,490,490
470  DO 480 J=NM6,NM8
480  R(I,J)=-R(I,J)
490  CONTINUE
      KSIZE=NM8
      IF (NN) 560,560,560
C    CHOOSE ABSOLUTE VALUE OF INEQUALITY CONSTRAINTS
500  DO 550 I=NM2,NM3
      B(I)=ABS(RJ(I))
      IF (RJ(I)) 520,520,510
510  OBJ(I)=0.
      GO TO 550
520  KSIZE=KSIZE+1
      DO 530 J=1,NM5
530  R(J,KSIZE)=0.
      R(I,KSIZE)=-1.
      OBJ(KSIZE)=0.
      DO 540 J=NM6,NM8
540  R(I,J)=-R(I,J)
550  CONTINUE
560  CONTINUE
      CALL LP1 (NM5,KSIZE,OBJ,B)
      KONVRG=1
      DO 630 I=1,NX
      DXMOVE=AMAX1(0.,OBJ(I+NM5))-AMAX1(0.,OBJ(I+NM7))
      IF (DXMOVE*XMOVE(I)) 570,610,580
570  DX(I)=.5*DX(I)
      MOVE(I)=1
      GO TO 610
580  MKTI=MOVE(I)
      GO TO (590,590,590,600), MKTI
590  MOVE(I)=MOVE(I)+1
      GO TO 610
600  DX(I)=2.*DX(I)
      MOVE(I)=1
610  XMOVE(I)=DXMOVE
      X(I)=X(I)+DXMOVE
      IF (ABS(DXMOVE)-EPIS) 630,630,620
620  KONVRG=2
630  CONTINUE
      ITRIGER=ITRIGER+1
      IF (KONVRG.EQ.2) ITRIGER=0

```

```

      CALL WRITEX (2,IPOITT)
      YNEW=OBJFN
      GO TO 210
640  CONTINUE
650  FORMAT (3I5,E12.0,17XE12.0,12XI5)
660  FORMAT (6E12.0)
670  FORMAT (1H1)
680  FORMAT (8A10)
690  FORMAT (7I1)
700  FORMAT (7F10.0)
710  FORMAT (1H1*NX*I3,5X*NE*I3,5X*NN*I3,5X*CONVRG*E12.4,
15X*EPIS*E12.5,5X*NGOALS*I4)
720  FORMAT (1H010X*DX*8X*XMIN*9X*X*9X*XMAX*(/1H05X4E12.5))
730  FORMAT (1H1/1H 80(1H*)/1H08A10)
740  FORMAT (1H0*THERE ARE NO GOALS IN THE OBJECTIVE*
1* FUNCTION. PROGRAM IS LOOKING FOR A FEASIBLE POINT*
2 TO THE CONSTRAINT SET*)
750  FORMAT (1H0,*GOALS IN THE OBJECTIVE FUNCTION*
1/1H ,10(I2,2X))
760  FORMAT (1H0,*ASPIRATION LEVELS*/1H ,10E12.4)
770  FORMAT (35H0THE COMPUTATION TIME IN SECONDS IS/
1(5X,F10.5))
      END

```

```

      SUBROUTINE LP1 (M,N,OBJ,B)
      COMMON /3/ A(15,25)
      DIMENSION OBJ(25), SAVE(25), COST(25), ICOST(25),
1X(25), NDX(25), B(25)
      ITNO=0
      DO 10 I=1,N
10  SAVE(I)=OBJ(I)
      DO 20 I=1,M
      COST(I)=SAVE(I)
      ICOST(I)=I
      NDX(I)=I
20  CONTINUE
30  ITNO=ITNO+1
      IF (ITNO.GT.1000) GO TO 180
      Z=0
      DO 60 J=1,N
      S=-OBJ(J)
      DO 40 I=1,M
40  S=S+A(I,J)*COST(I)
      IF (S-Z) 60,60,50
50  K=J
      Z=S
60  CONTINUE
      C K IS THE ENTERING VARIABLE

```



```

      IF (Z) 140,140,70
70    DO 80 I=1,M
80    X(I)=A(I,K)
      L=0
      Q=1.E200
      DO 100 I=1,M
      IF (X(I)) 100,100,90
90    Q1=B(I)/X(I)
      IF (Q1.GE.Q) GO TO 100
      Q=Q1
      L=I
100   CONTINUE
C     L IS THE EXITING VARIABLE.
      IF (L.LE.0) GO TO 170
      K1=NDX(L)
      NDX(L)=K
      E=X(L)
      X(L)=0
      DO 110 J=1,N
110   A(L,J)=A(L,J)/E
      B(L)=B(L)/E
      DO 130 I=1,M
      DO 120 J=1,N
120   A(I,J)=A(I,J)-A(L,J)*X(I)
130   B(I)=B(I)-B(L)*X(I)
      ICOST(L)=K1
      COST(L)=SAVE(K)
      GO TO 30
140   DO 150 I=1,N
150   OBJ(I)=-1
      DO 160 I=1,M
      J=NDX(I)
160   OBJ(J)=B(I)
      RETURN
170   PRINT 190
      STOP 77
180   PRINT 200
      STOP 76
190   FORMAT (1H0* UNBOUNDED SOLUTION *)
200   FORMAT (1H0* MORE THEN 1000 ITERATIONS IN LP *)
      END

```

```

SUBROUTINE PERMUTE (IFIRST,ILAST,IEX,NGOALS)
COMMON /6/ IEXIT(10),ALEVEL(10)
INTEGER REPEAT(10)
DIMENSION IEX(10), ISAVE(10)
IF (IFIRST.NE.3HYES) GO TO 40
NUM=0

```

```

DO 10 I=1,NGOALS
IEXIT(I)=IEX(I)
IF (IEX(I).EQ.0) GO TO 10
NUM=NUM+1
ISAVE(NUM)=I
10  CONTINUE
IFIRST=3HNO
IF (NUM.EQ.1) ILAST=3HYES
ICYCLE=0
DO 20 I=1,NUM
20  REPEAT(I)=ISAVE(NUM+1-I)
DO 30 I=1,NUM
30  ISAVE(I)=REPEAT(I)
RETURN
40  CONTINUE
ICYCLE=ICYCLE+1
DO 50 I=1,NGOALS
50  IEXIT(I)=IEX(I)
II=ISAVE(ICYCLE)
IEXIT(II)=0
IF (ICYCLE.EQ.NUM) ILAST=3HYES
RETURN
END

```

```

SUBROUTINE WRITEX (K,IPOITT)
COMMON /1/ X(10),DFL(10),NX,NN,NC,INQ,VALQ,KK
COMMON /2/ OBJFN,CONSTR(10)
COMMON /5/ Y(6),D(6),Z(6),A(6),AA(6)
IPOITT=IPOITT+1
GO TO (10,120,60,70,80,90), K
10  IF (IPOITT-250) 40,40,20
20  DO 30 I=1,60
    IF (IPOITT-250*I) 120,40,30
30  CONTINUE
40  GO TO (100,50,60,70,80,90), K
50  PRINT 180, IPOITT
    GO TO 100
60  PRINT 190, IPOITT
    PRINT 140
    PRINT 130, Z,Y,A,D
    GO TO 100
70  PRINT 190, IPOITT
    PRINT 200
    PRINT 130, Z,Y,A,D
    GO TO 100
80  PRINT 190, IPOITT
    PRINT 210
    PRINT 130, Z,Y,A,D

```

```

GO TO 100
90  PRINT 190, IPOITT
    PRINT 220
    PRINT 130, Z,Y,A,D
100  PRINT 160, (X(I),I=1,NX)
    KK=1
    INQ=1
    CALL USER
    F=VALQ
    OBJFN=F
    PRINT 150, F
    DO 110 I=1,NC
    INQ=I+1
    KK=1
    CALL USER
    CONSTR(I)=VALQ
110  CONTINUE
    PRINT 170, (CONSTR(I),I=1,NC)
120  RETURN
130  FORMAT (2H0Z6E12.5/2H0Y6F12.5/2H0A6F12.5/2H0D
16E12.5/1H0)
140  FORMAT (/ ,25H THESE ARE FINAL ANSWERS.,/)
150  FORMAT (1H *OBJECTIVE FUNCTION*E13.6,/1H )
160  FORMAT (1H *INDEPENDENT VARIABLES*/1H 3E12.5,/1H )
170  FORMAT (1H *CONSTRAINTS*/1H 7E11.4,/1H )
180  FORMAT (// ,11H *****,/,9H POINT = I4)
190  FORMAT (/// ,21H *****,/9H POINT = I4)
200  FORMAT (1H0*APPARENT INCONSISTENT CONSTRAINT SET*)
210  FORMAT (1H0*PROGRAM MAKING SLOW PROGRESS...PLEASE*
1* EXAMINE SERIES OF POINTS PRINTED*)
220  FORMAT (1H0*POSSIBLE INCONSISTENT CONSTRAINT SET*)
END

```

```

SUBROUTINE USER
COMMON /1/ X(10),DEL(10),N,M,NM,IN,VAL,KK
COMMON /5/ Y(6),D(6),Z(6),A(6),AA(6)
COMMON /6/ IEXIT(10),AL(10)
DIMENSION RANGE(6,2)
DATA AL/3*6.,6.5,2*1.5/,E/1.E-100/
DATA (RANGE(I),I=1,12)/6*0.,3*8.5,7.5,10.,12./
DATA DC12,DC45,DC23,DC13,DC24,DC14,DC25,DC15/
15.68E-5,3.49E-5,2.18E-5,1.31E-5,7.64E-6,4.42E-6,
21.45E-6,8.30E-7/
DATA DN45,DN23,DN13,DN24,DN14,DN25,DN15/7.33E-5,
15.53E-5,3.15E-5,1.60E-5,7.71E-6,1.62E-6,7.25E-7/
ANITRO(CAR)=.39/(1.39-CAR**2)
PANITRO(CAR)=.78*CAR/((1.39-CAR**2)**2)
Z4(CAR)=59/(1.09-CAR**2)-59

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```

Z5(CAR)=532/(1.09-CAR**2)-532
Z6(CAR)=450/(1.09-CAR**2)-450
PZ4(CAR)=118*CAR/((1.09-CAR**2)**2)
PZ5(CAR)=1064*CAR/((1.09-CAR**2)**2)
PZ6(CAR)=900*CAR/((1.09-CAR**2)**2)
DO 10 I=1,6
  AA(I)=AL(I)-RANGE(I,1)
10  A(I)=AA(I)/(RANGE(I,2)-RANGE(I,1))
  Z(1)=4.75+DC12*40000.*(X(1)-.3)
  Z(2)=DC13*40000.*(X(1)-.3)+DC23*128000.*(X(2)-.3)
  1+DN13*28000.*(ANITRO(X(1))- .3)+DN23*48000.*
  2(ANITRO(X(2))- .3)+2
  Z(3)=5.1+DC14*40000.*(X(1)-.3)+DC24*128000.*(X(2)
  1-.3)+DN14*28000.*(ANITRO(X(1))- .3)+DN24*
  248000.*(ANITRO(X(2))- .3)
  Z(4)=(100./5.E6)*(375000.-.6*1000.*(Z4(X(1))))
  Z(5)=1.8E-6*1000.*(Z5(X(2)))
  Z(6)=2.5E-6*1000.*(Z6(X(3)))
DO 20 I=1,6
20  Y(I)=(Z(I)-RANGE(I,1))/(RANGE(I,2)-RANGE(I,1))
  D(1)=A(1)/(Y(1)+E)
  D(2)=A(2)/(Y(2)+E)
  D(3)=A(3)/(Y(3)+E)
  D(4)=A(4)/(Y(4)+E)
  D(5)=Y(5)/A(5)
  D(6)=Y(6)/A(6)
  IF (KK-1) 130,30,130
30  GO TO (40,60,70,80,90,100,110,120), IN
40  VAL=0
  DO 50 I=1,6
50  VAL=VAL+IEXIT(I)*D(I)
  RETURN
60  VAL=DC15*40000.*(X(1)-.3)+DC25*128000.*(X(2)-.3)+
  1DC45*95711.*(X(3)-.3)+DN15*28000.*(ANITRO(X(1))- .3)
  2+DN25*48000.*(ANITRO(X(2))- .3)+DN45*35714*
  3(ANITRO(X(3))- .3)-3.5+1.
  RETURN
70  VAL=(1-IEXIT(1))*(Z(1)-AL(1))
  RETURN
80  VAL=(1-IEXIT(2))*(Z(2)-AL(2))
  RETURN
90  VAL=(1-IEXIT(3))*(Z(3)-AL(3))
  RETURN
100 VAL=(1-IEXIT(4))*(Z(4)-AL(4))
  RETURN
110 VAL=(1-IEXIT(5))*(AL(5)-Z(5))
  RETURN
120 VAL=(1-IEXIT(6))*(AL(6)-Z(6))

```

```

RETURN
130 DO 140 I=1,N
140 DEL(I)=0.
GO TO (150,160,170,180,190,200,210,220), IN
150 DEL(1)=IEXIT(1)*A(1)*(RANGE(1,2)-RANGE(1,1))*
1(-DC12*40000.)/((Z(1)-RANGE(1,1))**2+E)+
2IEXIT(2)*A(2)*(RANGE(2,2)-RANGE(2,1))*(-DC13*40
3000.-DN13*28000.*PANITRO(X(1)))/((Z(2)-RANGE
4(2,1))**2+E)+IEXIT(3)*A(3)*(RANGE(3,2)-RANGE
5(3,1))*(-DC14*40000.-DN14*28000.*PANITRO(X(1
6))) /((Z(3)-RANGE(3,1))**2+E)+IEXIT(4)*A(4)*(RANGE
7(4,2)-RANGE(4,1))*(100./5.E6)*(-.6*1000.)*(PZ4
8(X(1)))*(-1)/((Z(4)-RANGE(4,1))**2+E)
DEL(2)=IEXIT(2)*A(2)*(RANGE(2,2)-RANGE(2,1))*
1(-DC23*128000.-DN23*48000.*PANITRO(X(2)))/
2((Z(2)-RANGE(2,1))**2+E)+IEXIT(3)*A(3)*(RANGE
3(3,2)-RANGE(3,1))*(-DC24*128000.-DN24*48000.*
4PANITRO(X(2)))/((Z(3)-RANGE(3,1))**2+E)+IEXIT(5)*
5(1.8E-6*1000.)*(PZ5(X(2)))/(RANGE(5,2)-RANGE
6(5,1))/A(5)
DEL(3)=IEXIT(6)*(2.5E-6*1000.)*(PZ6(X(3)))/(RANGE
1(6,2)-RANGE(6,1))/A(6)
RETURN
160 DEL(1)=DC15*40000.+DN15*28000.*PANITRO(X(1))
DEL(2)=DC25*128000.+DN25*48000.*PANITRO(X(2))
DEL(3)=DC45*95711.+DN45*35714*PANITRO(X(3))
RETURN
170 DEL(1)=(1-IEXIT(1))*DC12*40000.
RETURN
180 DEL(1)=(1-IEXIT(2))*(DC13*40000.+DN13*28000.*
1PANITRO(X(1)))
DEL(2)=(1-IEXIT(2))*(DC23*128000.+DN23*48000.*
1PANITRO(X(2)))
RETURN
190 DEL(1)=(1-IEXIT(3))*(DC14*40000.+DN14*28000.*
1PANITRO(X(1)))
DEL(2)=(1-IEXIT(3))*(DC24*128000.+DN24*48000.*
1PANITRO(X(2)))
RETURN
200 DEL(1)=(1-IEXIT(4))*(100./5.E6)*(-.6*1000.)*
1(PZ4(X(1)))
RETURN
210 DEL(2)=(1-IEXIT(5))*(-1)*(1.8E-6*1000.)*(PZ5(X(2)))
RETURN
220 DEL(3)=(1-IEXIT(6))*(-1)*(2.5E-6*1000.)*(PZ6(X(3)))
RETURN
END

```

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