RC Distributed Network Modeling

by

Stephen Paul Johnson

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SIGNED: Stephen P. Johnson

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

L P Huelsman
LAWRENCE P. HUELSMAN
Professor of Electrical Engineering

May 15, 1967
Date

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ABSTRACT

A method for modeling the general three terminal distributed network is presented here. First the theoretical values of the admittance parameters for uniform distributed networks and distributed networks with an exponential taper are derived. The modeling technique is then described. Two models for the distributed RC network are introduced, and these two models are used to approximate the admittance parameters of uniform and exponentially tapered distributed networks. The values for the parameters as obtained from the models are compared with the theoretical values, and the models are shown to give an accurate representation of the distributed network.

CHAPTER 1

INTRODUCTION

With the advent of micro-electronics, distributed RC networks are becoming increasingly important in electrical engineering. When the engineer is called upon to analyze a circuit which may contain distributed networks, he is confronted with two formidable obstacles. First, immittances associated with distributed RC networks will, in general, involve transcendental functions. This makes application of standard network analysis techniques difficult, to say the least. Second, analytical expressions for these immittances are available for only a few distributed networks and particular geometries, and these expressions are completely different for each particular geometry. For example, the admittance parameters associated with a distributed network with an exponential taper have a form entirely different from the admittance parameters of a distributed network with a linear taper (1). The use of analysis techniques which employ the digital computer will allow the engineer to surmount the numerical difficulties associated with the analysis of networks which contain distributed RC networks, and the use of a modeling technique to be presented here will allow him to perform this analysis regardless of the geometrical configuration of the distributed networks.

Two models for the three terminal distributed network have been developed by Dr. L. P. Huelsman. These models employ an iterative computation technique to approximate the admittance parameters of the distributed network. The models may be applied to any three terminal distributed network, regardless of the geometrical configuration of that network.

The models will be presented, and it will be shown that they give an accurate approximation to the admittance parameters of uniform distributed networks and distributed networks with an exponential taper. A derivation of the admittance parameters of a uniform distributed network will be given first. The operation of the models will then be explained. The results for the admittance parameters of uniform distributed networks and distributed networks with an exponential taper obtained from the models will be compared with the theoretical values. These results will indicate that the models give a good approximation to the admittance parameters for these two special cases of uniform and exponentially tapered distributed networks. Finally, certain conclusions will be reached regarding the accuracy of each of the two models, and one of the models will be shown to give the more accurate overall approximation.

CHAPTER 2

THEORETICAL VALUES OF THE ADMITTANCE PARAMETERS

In this chapter, the admittance parameters of a uniform distributed network will be derived by consideration of a differential section of the basic structure. The admittance parameters of a distributed network with an exponentially tapering geometry will also be presented.

2.1 Analysis of the Incremental Model

Consider the uniform distributed network shown in Fig. 2.1. The resistance per unit length and capacitance per unit length are both constant along the entire length of the network. A differential section of the network may be represented by the circuit shown in Fig. 2.2. For sinusoidal steady state conditions, and neglecting second order differential terms, we obtain equations for di and dv:

$$di = (v+dv) \cdot (-j\omega c dx) \cong -j\omega c v dx \qquad (2.1)$$

$$dv = -irdx \qquad (2.2)$$

Where i, v, di, and dv are phasors, ω is the angular frequency, and r and c are respectively resistance and capacitance per unit length. Thus, for the differential section which has length dx we have:

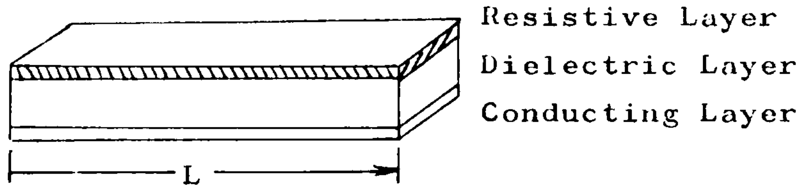

Fig. 2.1. Uniform RC Distributed Network

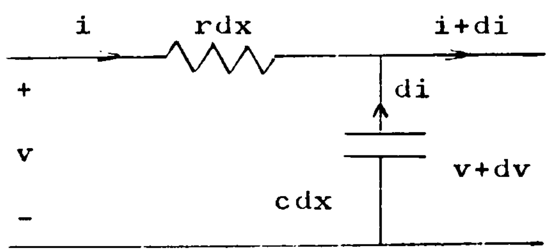

Fig. 2.2. Equivalent Circuit for a Differential Section of the Uniform RC Distributed Network

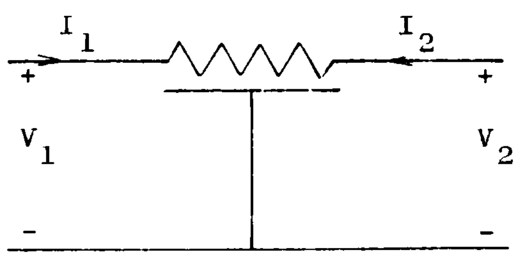

Fig. 2.3. Two Port Representation of the Uniform RC Distributed Network

$$\frac{di}{dx} = -j\omega cv \tag{2.3}$$

$$\frac{d^2v}{dx^2} = j\omega rcv \tag{2.4}$$

This analysis neglects any possible edge effects. The solutions to equations (2.3) and (2.4) are (2):

$$v = Ae^{\beta x} + Be^{-\beta x} \tag{2.5}$$

$$i = \frac{-j\omega c}{\beta}(Ae^{\beta x} - Be^{-\beta x}) + D \tag{2.6}$$

where A, B, and D are arbitrary constants, and $\beta = (j\omega rc)^{1/2}$.

2.2 Analysis of the Complete Network

The equations describing the terminal voltages and currents for the complete uniform distributed network may now be obtained. Let us define:

L = total length of the network

R = rL = total resistance of the network

C = cL = total capacitance of the network

$\Theta = \beta L = (j\omega RC)^{1/2}$

The voltages and currents of the network as shown in Fig. 2.3 are given by:

$$V_1 = A + B \tag{2.7}$$

$$V_2 = Ae^{\Theta} + Be^{-\Theta} \tag{2.8}$$

$$I_1 = \frac{-\Theta}{R}(A-B) - \frac{G}{R} \tag{2.9}$$

$$I_2 = \frac{\theta}{R}\,(Ae^{\theta} - Be^{-\theta}) + \frac{G}{R} \tag{2.10}$$

Where G is a constant related to the arbitrary constant D of equation (2.4). Although equations (2.7) through (2.10) give the most general solution for the voltages and currents of the circuit shown in Fig. 2.3, these equations do not lend themselves to practical circuit applications. The equations are sufficient to obtain the admittance parameters of the network, which are of considerable importance in the chapters to follow.

The constants A, B, and G in equations (2.7) through (2.10) may be eliminated by giving the port currents in terms of the port voltages. This gives the admittance matrix as expressed in equation (2.11).

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{R}\begin{bmatrix} \dfrac{\theta}{\tanh\theta} & \dfrac{-\theta}{\sinh\theta} \\ \dfrac{-\theta}{\sinh\theta} & \dfrac{\theta}{\tanh\theta} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2.11}$$

The admittance matrix gives an exact representation of the behavior of the uniform distributed network at a given frequency.

2.3 The Exponentially Tapered Network

An exponentially tapered RC distributed network is shown in Fig. 2.4. The width of the network is exponentially dependent upon the position from the end. In this case $W(x) = W_o e^{-\alpha x}$ where $W(x)$ is the width at any point

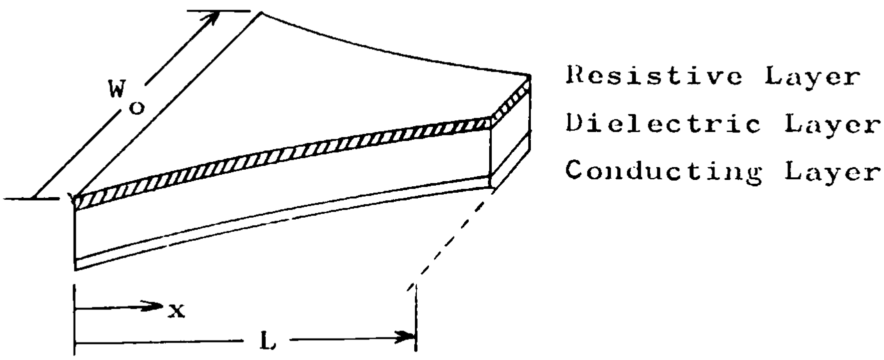

Fig. 2.4. An Exponentially Tapered RC Distributed Network

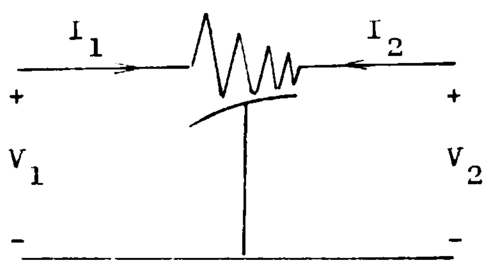

Fig. 2.5. Two Port Representation of the Exponentially Tapered RC Distributed Network

along the network, and W_o is the width at the end point. In such a case we will have:

$$r(x) = r_o e^{\alpha x} \tag{2.12}$$

$$c(x) = c_o e^{-\alpha x} \tag{2.13}$$

where r(x) and c(x) are the resistance and capacitance per unit length at some point along the line. Kaufman and Garrett (1) have derived the admittance parameters of the exponentially tapered network. If we define the total length of the network as L, the admittance parameters of the network shown schematically in Fig. 2.5 are given by:

$$\begin{bmatrix} I_1 \\ \\ I_2 \end{bmatrix} = \frac{1}{r_o L} \begin{bmatrix} \dfrac{\theta}{\tanh\theta} - \dfrac{\alpha L}{2} & \dfrac{-\theta e^{-\alpha L/2}}{\sinh\theta} \\ \\ \dfrac{-\theta e^{-\alpha L/2}}{\sinh\theta} & e^{-\alpha L}\left[\dfrac{\theta}{\tanh\theta} + \dfrac{\alpha L}{2}\right] \end{bmatrix} \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix} \tag{2.14}$$

where $\theta = \left(\frac{\alpha^2 L^2}{4} + j\omega r_o c_o L^2\right)^{1/2}$

The admittance matrix of equation (2.14) will, of course, reduce to that given in equation (2.11) if α is set equal to zero.

2.4 Conclusion

In this chapter we have obtained exact expressions for the admittance parameters of the uniform and exponentially tapered distributed networks. These parameters involve hyperbolic functions which are difficult to manipulate in standard network design and analysis techniques. The admittance matrix of the exponentially tapered network given here will serve later as a basis for determining the validity of the models for the generalized distributed network.

CHAPTER 3

PRESENTATION OF THE MODELS

In this chapter two models are presented which may be used to approximate the admittance parameters of a three terminal distributed network at some specified frequency. The basic principles of the models are explained, and each model is examined in detail.

3.1. Explanation of the Principle used in the Approximation

Both of the models used in approximating the admittance parameters of the three terminal distributed RC network utilize the same basic concept. The distributed network to be analyzed is divided into a number of elemental sections, and the resistance and capacitance associated with each section is specified. First the transmission parameters of an elemental section are approximated at the frequency of interest, then an iterative procedure is used to obtain the port voltages and currents of the complete network. The admittance parameters of the network are then derived from the port voltages and currents.

The inaccuracies that arise in the models are caused by the fact that each elemental section is analyzed as a small network composed of discrete components. Since each elemental section has finite length, the sections are actually small distributed networks themselves, and the analysis which assumes they are composed of discrete components will, of necessity, introduce some inaccuracy. However, it will be shown that the inaccuracy can be made quite small by dividing the network into a sufficient number of elemental sections. If a digital computer is used in the iterative computation process, the number of sections may easily be made large enough so that the approximation is very accurate.

3.2. The Difference Equation Approximation

The approximation of the transmission parameters of an elemental section by a difference equation technique will be considered first. A differential section of the distributed network may be approximated by the circuit shown in Fig. 3.1. If we neglect terms involving second order differentials, we have for sinusoidal steady state conditions:

$$\Delta i = -j\omega c\Delta x\ (v + \Delta v) \cong -j\omega c v\Delta x \tag{3.1}$$

$$\Delta v = -ir\Delta x \tag{3.2}$$

where r and c are respectively resistance and capacitance per unit length, ω is the angular frequency, and v, i, Δv, and Δi are phasors, representing the voltages and currents. We may now represent the voltages and currents

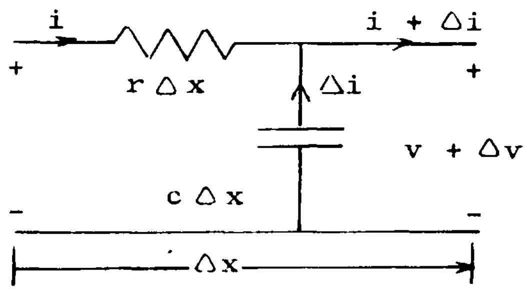

Fig. 3.1. Equivalent Circuit for a Differential Section of the RC Distributed Network

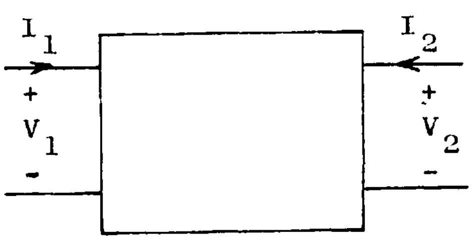

Fig. 3.2. A Two-Port Network

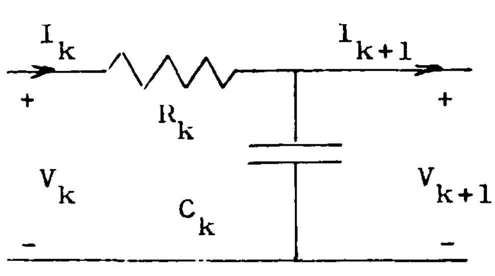

Fig. 3.3. Equivalent Circuit for an Elemental Section of the RC Distributed Network

at the ports of this differential section by a matrix equation:

$$\begin{bmatrix} v + \Delta v \\ i + \Delta i \end{bmatrix} = \begin{bmatrix} 1 & -r\Delta x \\ -j\omega c\Delta x & 1 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \tag{3.3}$$

The inverse transmission parameters of the two port network shown in Fig. 3.2 may be represented by the equation:

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} \mathcal{a} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} \tag{3.4}$$

The matrix equation (3.3), then, is closely related to the inverse transmission matrix of the differential section, the only difference resulting from the fact that the sign convention on the current $i + \Delta i$ is opposite to the standard convention used to define the inverse transmission parameters. It will be convenient to use the sign convention of Fig. 3.1, rather than the standard sign convention of Fig. 3.2.

Now let us suppose we have a distributed network divided into a number of elemental sections. The k^{th} section may be approximated by the circuit shown in Fig. 3.3. We may now relate the port voltages and currents by an equation analogous to equation (3.3):

$$\begin{bmatrix} V_{k+1} \\ \\ I_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -R_k \\ \\ -j\omega C_k & 1 \end{bmatrix} \begin{bmatrix} V_k \\ \\ I_k \end{bmatrix} \tag{3.5}$$

where we again assume sinusoidal steady state conditions, and ω is the angular frequency. R_k is the mean value of resistance per unit length of the section, multiplied by the length of the section. Similarly C_k is the mean value of the capacitance per unit length multiplied by the length of the section. I_k, I_{k+1}, V_k and V_{k+1} are phasors representing the voltages and currents.

Let the number of sections into which the network is divided be KK. If we now set $V_1 = 0$ and $I_1 = 1$, after KK iterative computations using equation (3.5) we obtain V_{KK+1} and I_{KK+1}. We now have the port voltages and currents necessary to obtain two of the admittance parameters of the complete network:

$$y_{22} = -I_{KK+1} / V_{KK+1} \tag{3.6}$$

$$y_{12} = I_1 / V_{KK+1} = 1 / V_{KK+1} \tag{3.7}$$

The matrix of equation (3.5) may be inverted to obtain a matrix relation for I_k and V_k in terms of I_{k+1} and V_{k+1}:

$$\begin{bmatrix} V_k \\ I_k \end{bmatrix} = \frac{1}{1 - j\omega C_k R_k} \begin{bmatrix} 1 & R_k \\ j\omega C_k & 1 \end{bmatrix} \begin{bmatrix} I_{k+1} \\ V_{k+1} \end{bmatrix} \qquad (3.8)$$

The transmission parameters of the network shown in Fig. 3.2 may be represented by the equation:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad (3.9)$$

The matrix of equation (3.8), then, is the transmission matrix of the elemental section. Letting $I_{KK+1} = 1$ and $V_{KK+1} = 0$, we can obtain I_1 and V_1 after KK iterative computations using equation (3.8). This will give us the information needed to calculate the remaining two admittance parameters:

$$y_{11} = I_1 / V_1 \qquad (3.10)$$

$$y_{21} = I_{KK+1} / V_1 = 1/V_1 \qquad (3.11)$$

Although the network is reciprocal, inaccuracies introduced by the approximation will cause a slight discrepancy in values for y_{12} and y_{21}. A flow chart for the computation process is shown in Fig 3.4. The process is equivalent to multiplying the transmission matrices of all the elemental sections together, and then using the resulting transmission matrix to obtain the admittance parameters.

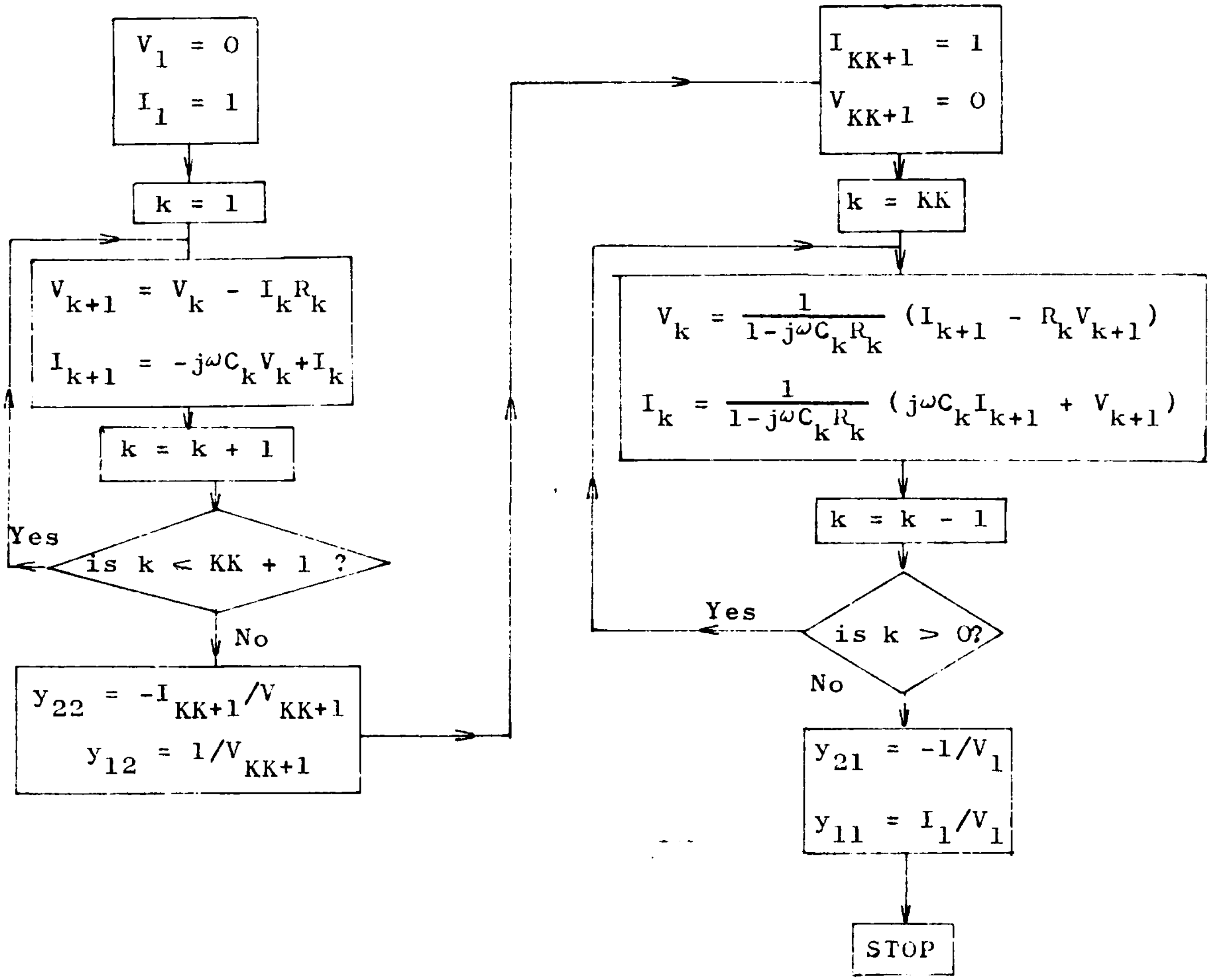

Fig. 3.4. Computation Process for Difference Equation Model

3.3. Lumped Element Approximations

A different approximation for the admittance parameters is obtained if we consider R_k and C_k in the network shown in Fig. 3.3 as elements in an ordinary lumped RC network and obtain matrix equations analogous to equations (3.5) and (3.8). We have:

$$V_{k+1} = V_k - I_k R_k \tag{3.12}$$

$$I_{k+1} = I_k - V_{k+1}(j\omega C_k) \tag{3.13}$$

substituting equation (3.12) for V_{k+1} into equation (3.13) results in:

$$I_{k+1} = I_k(1 + j\omega C_k R_k) - V_k(j\omega C_k) \tag{3.14}$$

we then have:

$$\begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -R_k \\ -j\omega C_k & 1 + j\omega C_k R_k \end{bmatrix} \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{3.15}$$

Inverting the matrix of equation (3.15) to obtain the transmission parameters of the elemental section we have:

$$\begin{bmatrix} V_k \\ I_k \end{bmatrix} = \begin{bmatrix} 1 + j\omega C_k R_k & R_k \\ j\omega C_k & 1 \end{bmatrix} \begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} \tag{3.16}$$

The computation of the admittance parameters from equation (3.15) and equation (3.16) follows exactly the method outlined in Section 3.2 which used equations (3.5) and (3.8).

3.4 Conclusion

In this chapter we have presented two models which may be used to provide an approximation to the admittance parameters of a distributed RC network. The two models differ only in the method used to approximate the transmission parameters of an elemental section of the network. The accuracy of the approximation depends upon the number of elemental sections into which the network is divided, the accuracy improving as the number of sections is increased.

CHAPTER 4

RESULTS WITH A UNIFORM DISTRIBUTED NETWORK

In this chapter, we compare the results for the admittance parameters of the uniform distributed network with the theoretical values. Frequency and magnitude normalization is discussed, and a performance criterion is introduced to compare the accuracies of the models. Certain patterns in the approximation error associated with the models are observed.

4.1 Remarks Concerning Normalization

In Chapter 2, the admittance parameters of a uniform three terminal RC distributed network were derived. They are repeated here for reference:

$$\begin{bmatrix} I_1 \\ \\ I_2 \end{bmatrix} = \frac{1}{R} \begin{bmatrix} \dfrac{\theta}{\tanh \theta} & \dfrac{-\theta}{\sinh \theta} \\ \\ \dfrac{-\theta}{\sinh \theta} & \dfrac{\theta}{\tanh \theta} \end{bmatrix} \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix} \tag{4.1}$$

where $\theta = \sqrt{j\omega RC}$ (4.2)

and R anc C are respectively the total resistance and capacitance of the network. Notice that the magnitude of the admittance parameters varies inversely as the resistance R when the product ωRC remains constant, and that the parameters depend only upon the product ωRC when R is held constant. In other words, the uniform distributed network has the same magnitude and frequency normalization characteristics as a lumped RC network. We would expect this to be the case if our models, which assume the network to be composed of many discrete lumped elements, were to have any validity at all. This means that if we investigate the validity of our models when compared with the uniform distributed network for unity values of R and C, the conclusions will apply for all values of R and C. All results in the following sections are given for unity values of R and C.

4.2 Results Comparing the Accuracy of Each of the Parameters

A computer program was developed to compare the results for the admittance parameters of the uniform distributed network using both the Difference Equation Model and the Lumped Elements Model with the theoretical values of the parameters obtained from equation (4.1). The results comparing the theoretical values of the y_{11} and y_{12} parameters with the values obtained using the models are shown in the graphs of Fig. 4.1 through Fig. 4.4. Both of the models used 40 elemental sections in the approximation. The graphs show that both models approximate the network to exactly the same degree of accuracy at low frequencies, but that the Difference Equation Model gives better results for

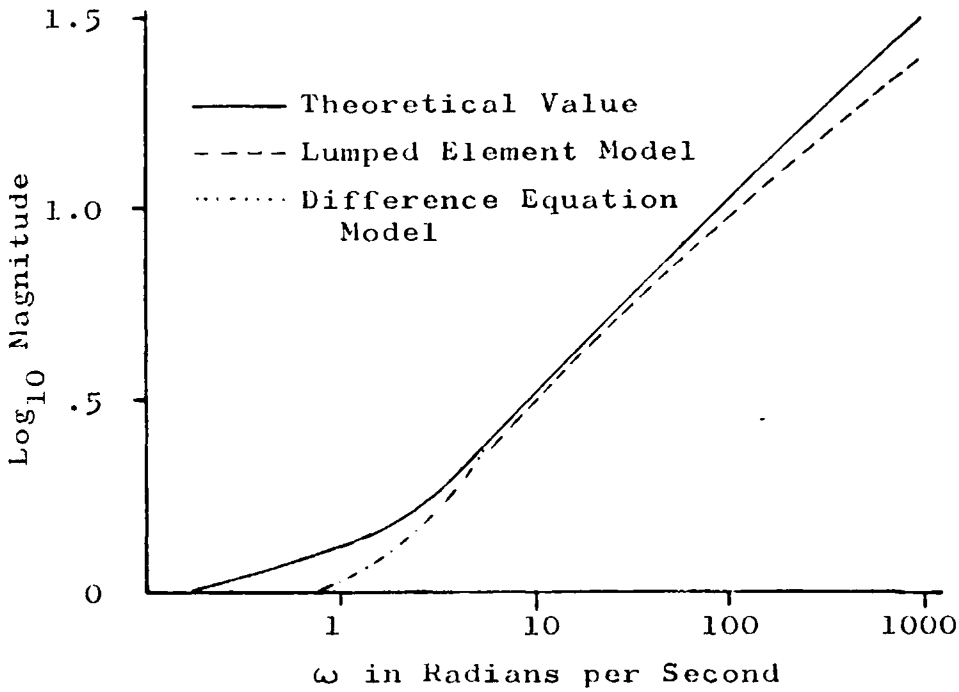

Fig. 4.1. Magnitude of the y_{11} Parameter

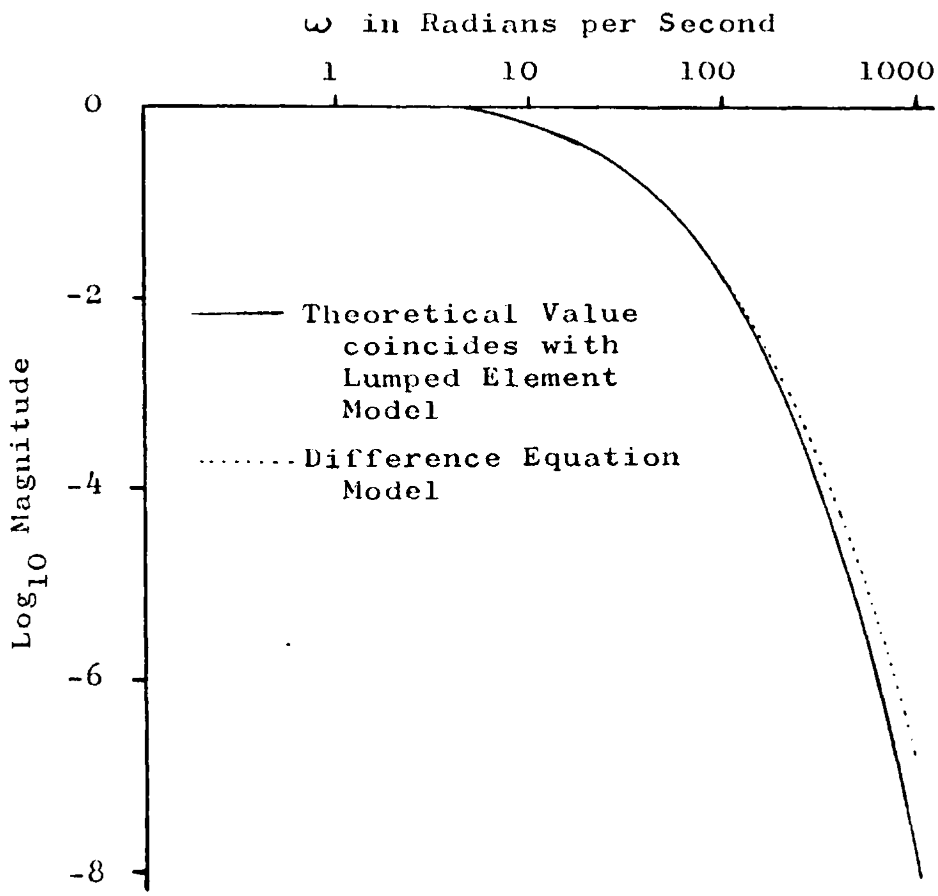

Fig. 4.2. Magnitude of the y_{12} Parameter

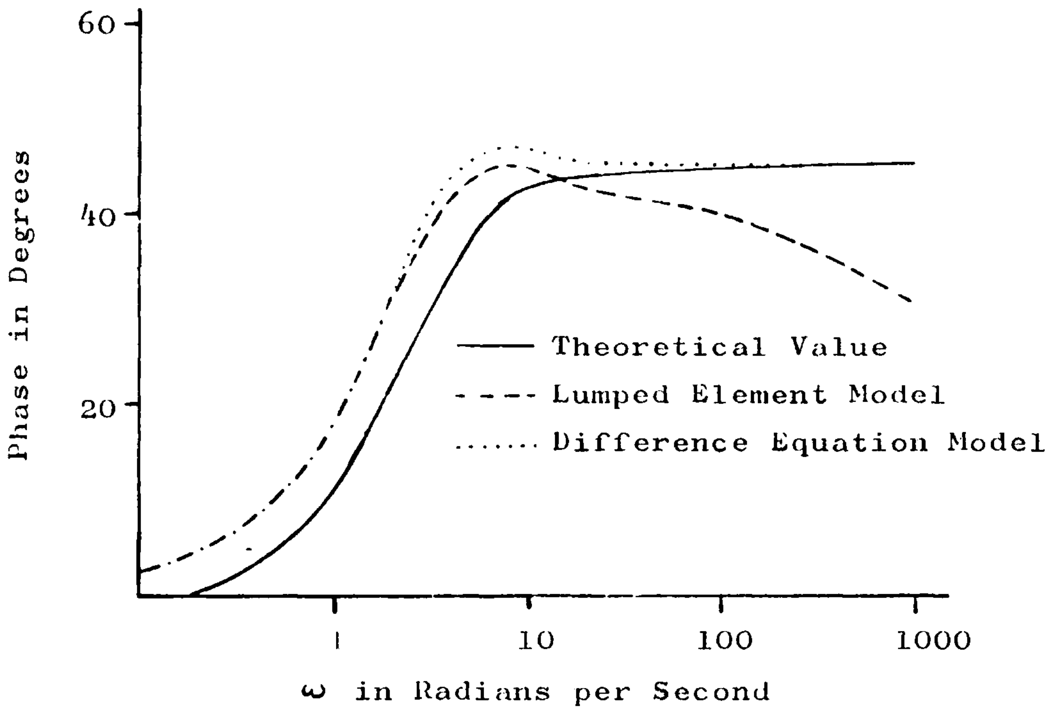

Fig. 4.3. Phase of the y_{11} Parameter

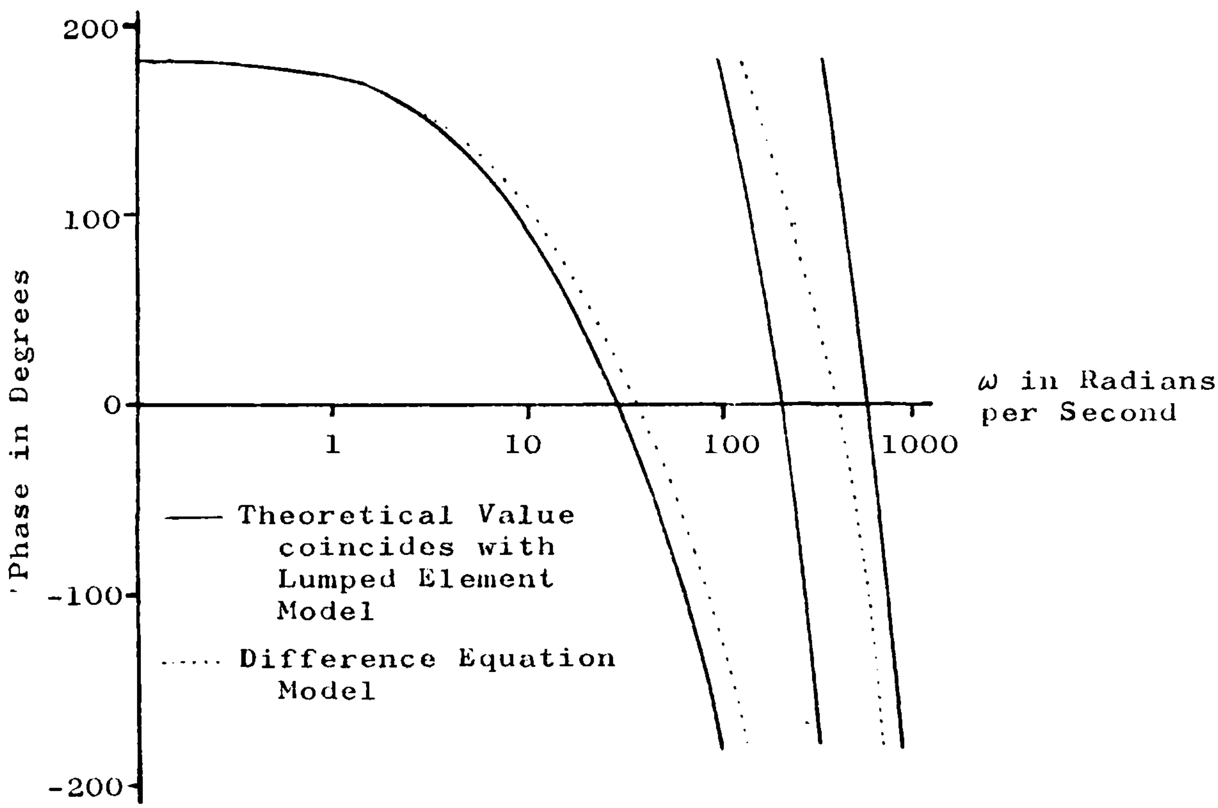

Fig. 4.4. Phase of the y_{12} Parameter

y_{11} at high frequencies, and the Lumped Element Model gives better results for y_{12} at high frequencies. It was found that the results for the y_{21} parameter were significantly better using the Lumped Element Model, and that results for the y_{22} parameter were better using the Difference Equation Model. The magnitudes of the error were of the same order as for the y_{21} and y_{11} parameters, so additional graphs were not included. As a general rule, then, the Difference Equation Model gives better results for the input and output admittance parameters, and the Lumped Element Model gives better results for the transfer admittance parameters.

The graphs of Fig. 4.1 through Fig. 4.4 also show that there is a much larger error in both the phase and magnitude using the Difference Equation Model for the transfer parameters than there is using the Lumped Element Model for the input parameters. This would seem to indicate that the Lumped Element Model would provide better overall accuracy for the entire network. That this is indeed the case will be shown in the following sections.

4.3 Comparison of Average Error vs. Frequency

The graphs of Fig. 4.5 and Fig. 4.6 show the average error in the magnitude and phase respectively for both models for various values of ω and for various numbers of elemental sections. By average error, we mean the

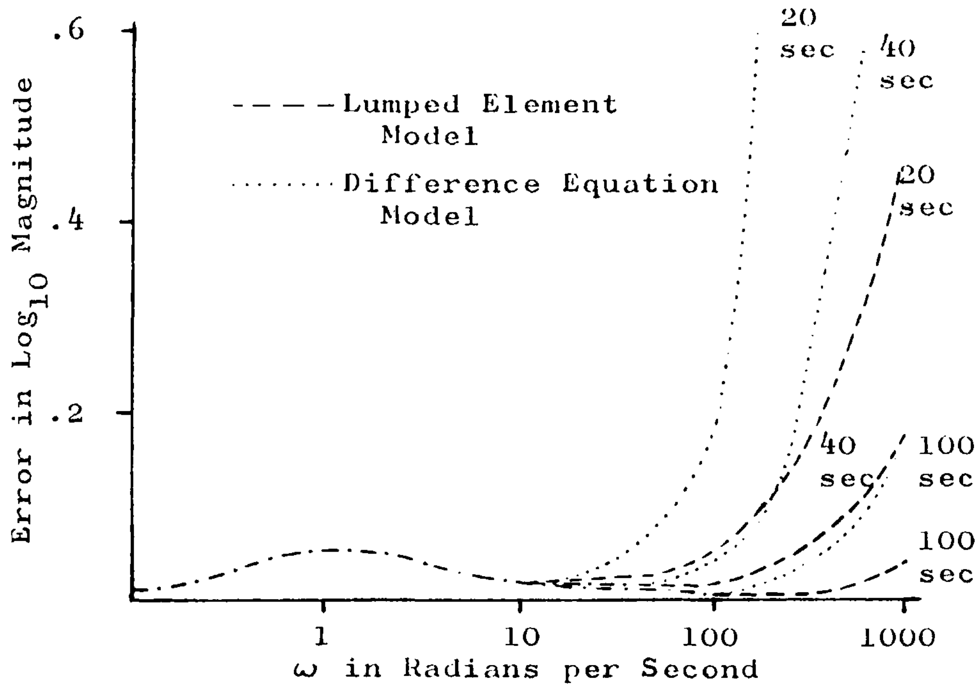

Fig. 4.5. Average Parameter Magnitude Error as a Function of Frequency for Various Numbers of Sections

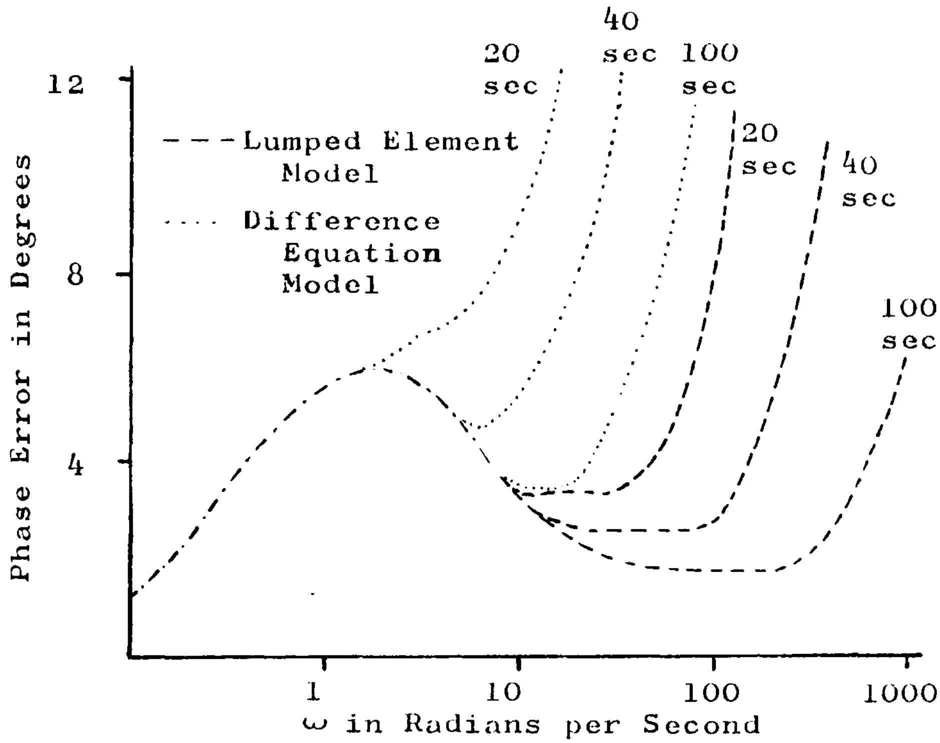

Fig. 4.6. Average Parameter Phase Error as a Function of Frequency for Various Numbers of Sections

magnitudes of the errors in the four parameters were added and normalized by dividing by four.

The graphs show clearly that the error in both magnitude and phase increases rapidly at high frequencies. A comparison shows that the error in both the magnitude and phase is smaller for the Lumped Element Model when the same number of sections is used. As expected, the graphs show that the error decreases for both models as the number of sections is increased.

Notice that the average error in magnitude for the Lumped Element Model using 40 sections is nearly as low as the error of the Difference Equation Model at all frequencies, and the average error in phase for the Lumped Element Model using 20 sections is lower than the error for the Difference Equation Model using 100 sections at any frequency above 20 radians per second.

In summary, the graphs of Fig. 4.5 and Fig. 4.6 show that the Lumped Element Model provides a much more accurate overall approximation to the admittance parameters at high frequencies.

4.4 Introduction of an Overall Performance Criterion

Although the graphs in Fig. 4.1 through Fig. 4.6 indicate that the error in approximation is lower using the Lumped Element Model at high frequencies, they fail to give a meaningful representation of the error we might expect in a typical application of the distributed network, or of the additional benefits as the number of sections used in the models is increased. An overall performance criterion that indicates the expected accuracy of the approximation of a network in a typical circuit application was therefore developed.

Kaufman and Garrett (1) have shown that when a uniform distributed network is used as a notch filter, the frequency of the notch is about 10 radians per second if unity values of R and C are used. The performance criterion developed assumes that in most circuit applications, the frequencies of interest will range from 1 to 100 radians per second (assuming R and C are normalized to unity), and that in most cases the frequency range of interest will center around 10 radians per second.

The overall performance criterion gives the most probable average error in an admittance parameter if the most probable frequencies of interest are normally distributed logarithmically about 10 radians per second. That is, the average error at any frequency is multiplied by the density function $\varphi(\omega)$, and multiplied by the interval between frequency sample points Δ, where the quantities $\varphi(\omega)$ and Δ are defined by the following relations:

$$\varphi(\omega) = \emptyset(\log(\omega/10)) \tag{4.3}$$

where $$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{4.4}$$

$$\Delta = \log \omega_2 - \log \omega_1 \tag{4.5}$$

where ω_1 and ω_2 are successive frequency sample points, ω_2 is greater than ω_1, and ω is the geometric mean of ω_2 and ω_1.

The function $\phi(x)$ is the normal distribution function centered at zero with variance equal to one. (4)

The overall performance criterion is not intended primarily to indicate exactly the numerical value of the most probable average error in an admittance parameter, but rather to provide some indication of the order of error that one might expect in a typical application, and to provide a meaningful basis for comparison of the two models. The criterion is also used to show improvement in performance as the number of sections used is increased.

4.5 Comparison of the Models on the Basis of the Overall Performance Criterion

The graphs of Fig. 4.7 and Fig. 4.8 show the average error to be expected respectively in magnitude and phase for the two models using various numbers of sections. The average error to be expected is calculated on the basis of the overall performance criterion defined in Section 4.4. The graphs show that the overall performance of the Difference Equation Model approaches that of the Lumped Element

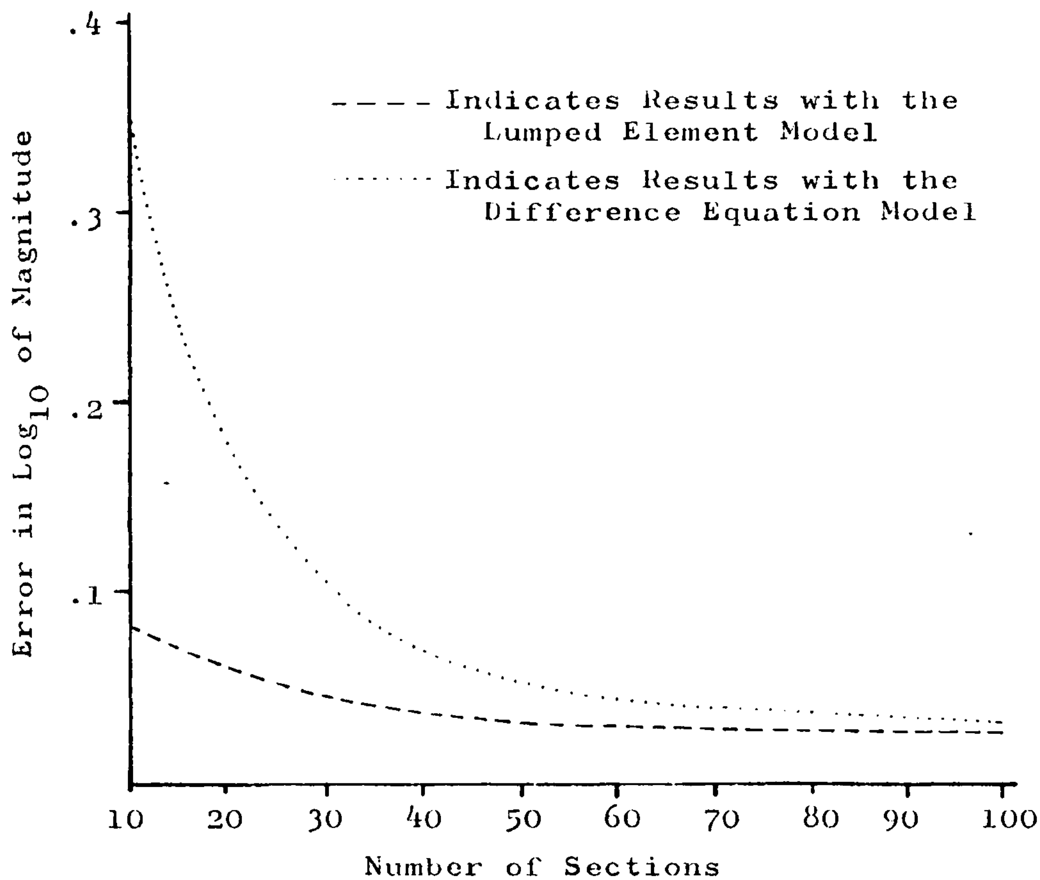

Fig. 4.7. Typical Parameter Magnitude Error to be Expected in Application of a Uniform Distributed Network

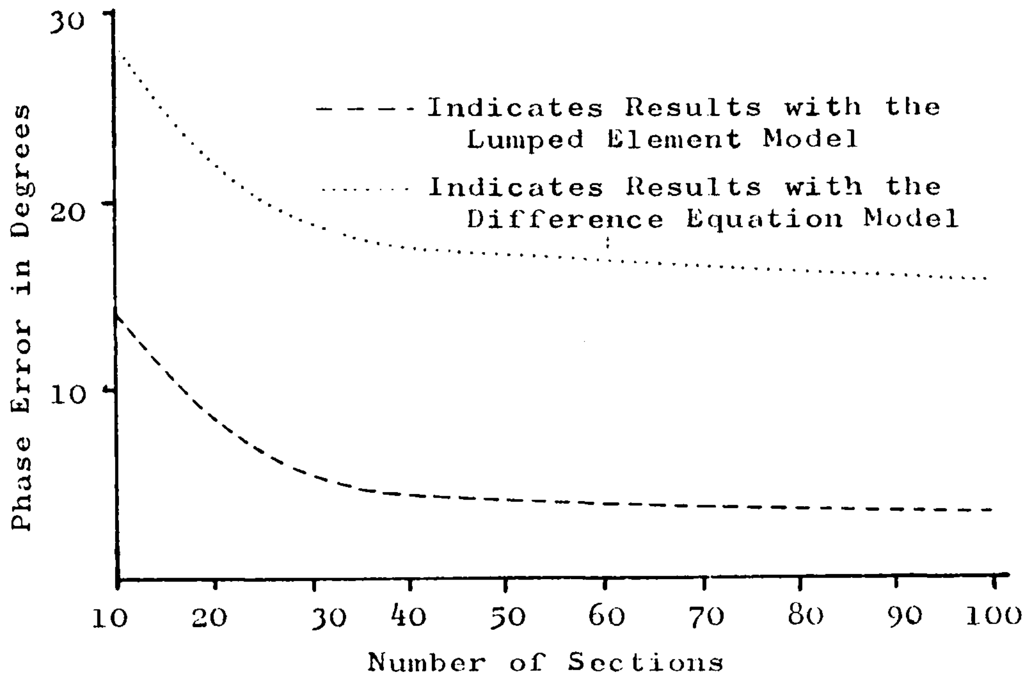

Fig. 4.8. Typical Parameter Phase Error to be Expected in Application of a Uniform Distributed Network

Model with respect to magnitude error as the number of sections is increased to 100. On the other hand, the overall performance with respect to phase error is much better using the Lumped Element Model, even as the number of sections is increased to 100. These two graphs provide the final evidence needed to declare that the Lumped Element Model gives the better overall performance of the two for the case of the uniform distributed network.

The graphs also show that, for the case of the uniform distributed network, the average error to be expected does not decrease significantly with an increasing number of sections after about 60 sections. This would indicate that most of the improvement in accuracy occurs at frequencies above 100 radians per second after the number of sections is increased above 60.

4.6 Conclusion

When the results for the admittance parameters of the uniform distributed network as obtained from the two models are compared with theoretical results, certain patterns are present in the approximation error. The Difference Equation Model gives better results for input and output admittance parameters, while the Lumped Element Model gives better results for the transfer parameters. For both models, error increases rapidly at high frequency, and the average error in both magnitude and phase is less at higher frequencies using the Lumped Element Model. The expected average error, calculated using an overall performance criterion, does not decrease significantly after the number of sections is increased past about 60 for both models, but a comparison of this error provides a final indication that the Lumped Element Model has a greater degree of overall accuracy.

CHAPTER 5

RESULTS WITH AN EXPONENTIALLY TAPERED DISTRIBUTED NETWORK

In this chapter, we compare the results for the admittance parameters of the exponentially tapered network obtained from the models with the theoretical values. Normalization is discussed, and overall performance of the models is compared on the basis of the overall performance criterion introduced in the last chapter.

5.1 Remarks Concerning Normalization

In Chapter 2, the admittance parameters of an exponentially tapered distributed network were presented. They are repeated here for reference:

$$\begin{bmatrix} I_1 \\ \\ I_2 \end{bmatrix} = \frac{1}{r_o L} \begin{bmatrix} \dfrac{\theta}{\tanh \theta} - \dfrac{\alpha L}{2} & \dfrac{-\theta e^{-\alpha L/2}}{\sinh \theta} \\ \\ \dfrac{-\theta e^{-\alpha L/2}}{\sinh \theta} & e^{-\alpha L}\left[\dfrac{\theta}{\tanh \theta} + \dfrac{\alpha L}{2}\right] \end{bmatrix} \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix} \qquad (5.1)$$

where $\theta = (\frac{\alpha^2 L^2}{4} + j\omega r_o c_o L^2)$ (5.2)

$$r(x) = r_o e^{\alpha x} \qquad (5.3)$$

$$c(x) = c_o e^{-\alpha x} \qquad (5.4)$$

and r_o, $r(x)$, c_o, and $c(x)$ are resistances and capacitances per unit length. Notice that the magnitude of the admittance parameters varies inversely with the resistance r_o when the quantities $\frac{\alpha L}{2}$ and $\omega r_o c_o L^2$ remain constant, and that the parameters depend only upon the product $\omega r_o c_o L^2$ when r_o and $\frac{\alpha L}{2}$ are held constant. Thus, when $\frac{\alpha L}{2}$ remains constant, the magnitude and frequency normalization characteristics resemble those of a lumped RC network, since under this condition the product $\omega r_o c_o L^2$ is proportional to the product RC, where R and C are respectively the total resistance and capacitance of the network. The parameter $\frac{\alpha L}{2}$ is defined as the degree of taper of the network (1). If the validity of the models is investigated for unity values of r_o, c_o, and L, the conclusions reached will apply for any exponentially tapered network with the same degree of taper. An exponentially tapered distributed network becomes a uniform distributed network if α equals zero, and under this condition the normalization conditions presented here reduce to the normalization conditions given in Section 4.1. The results given in the following sections are for unity values of r_o, c_o, and L.

5.2 Results Comparing the Accuracy of the Parameters

A computer program was developed to compare the results for the admittance parameters from equation (5.1) to the results obtained from both models. The results were compared for $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$. In all cases, the conclusions reached were the same as in the case of the uniform distributed network. Both models approximate the results to the same degree of accuracy at low frequencies, but the Difference Equation Model gives better results for y_{11} and y_{22} at high frequencies. The Lumped Element Model gives better results for the y_{12} and y_{21} parameters at high frequencies. As with the uniform distributed network, the average error in phase and magnitude both increase rapidly with increasing frequency, but if both models use the same number of elemental sections, the Lumped Element Model provides a more accurate approximation at high frequency.

5.3 Comparison of the Models with Respect to Various Numbers of Sections using the Overall Performance Criterion

Kaufman and Garrett (1) have shown that when the exponentially tapered distributed network is used as a notch filter, the frequency of the notch is about 10 radians per second if unity values of r_o, c_o, and L are used. The frequency is nearly the same for α equal to zero, one, two, and three. We may then use the overall performance criterion introduced in the last chapter for the exponentially tapered network as well as for the uniform network.

The graphs of Fig. 5.1 and Fig. 5.2 show the average expected error in magnitude and phase respectively

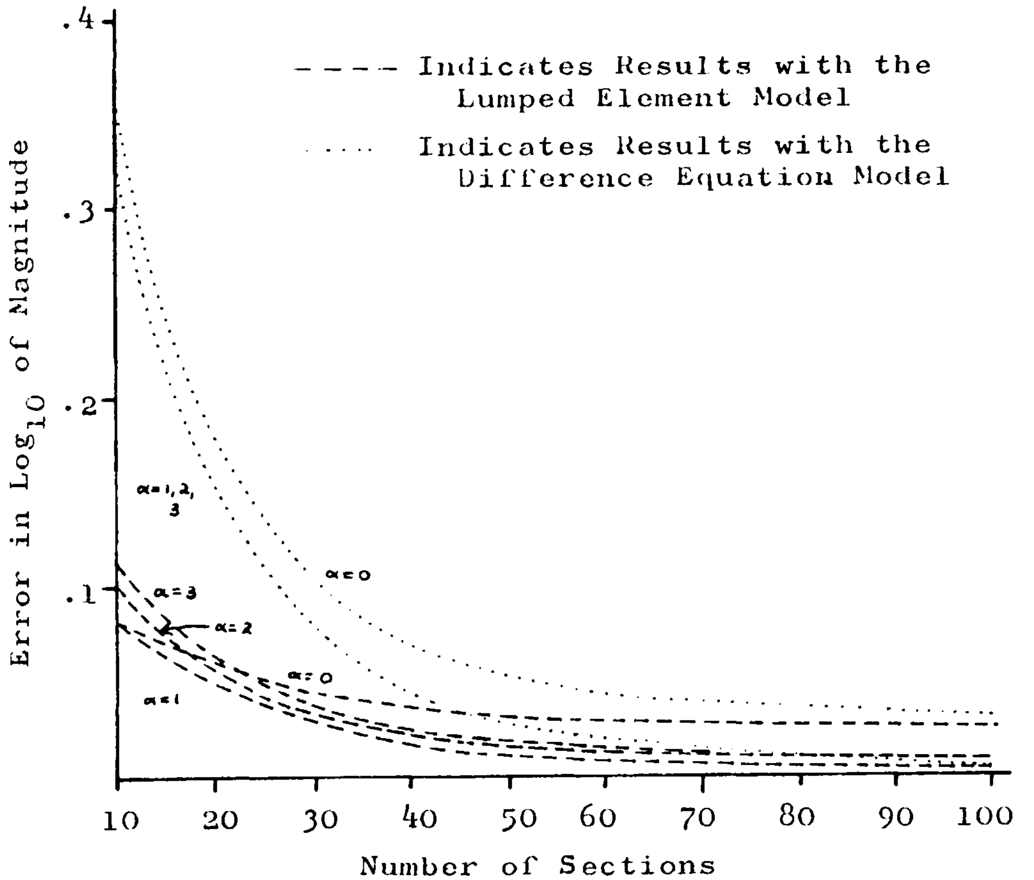

Fig. 5.1. Typical Parameter Magnitude Error to be Expected in Application of an Exponentially Tapered Distributed Network

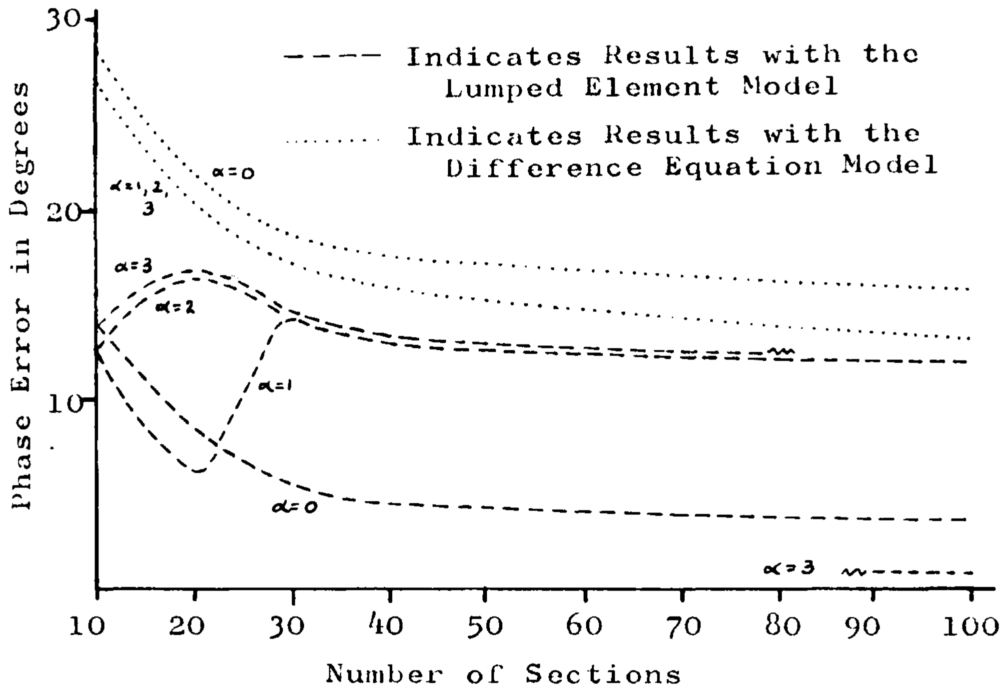

Fig. 5.2. Typical Parameter Phase Error to be Expected in Application of an Exponentially Tapered Distributed Network

for various numbers of sections and for various values of α. The graphs of Fig. 4.7 and Fig. 4.8 for the uniform network ($\alpha = 0$) are shown on the same scales for reference. The graph of Fig. 5.1 shows that the expected average error in magnitude decreases uniformly with increasing number of sections for both models and for all values of α. The error in the Difference Equation Model is virtually the same for α equal to one, two, or three, and runs slightly less than the error for α equal to zero. In the case of the Lumped Element Model, the error for low numbers of sections increases as α increases from one to three, and the error is again less than for α equal to zero for more than 30 sections.

The expected average error in phase shown in the graph of Fig. 5.2 shows that for the Difference Equation Model, the phase error decreases uniformly as the number of sections is increased for all values of α, and the graphs for α equal to one, two, and three coincide as in the case of the magnitude error. The apparently strange behavior in the graphs for the Lumped Element Model requires an explanation. One of the frequencies used for comparison of the models was 100 radians per second. As in the case of the uniform network, there is a change in the phase of the y_{12} and y_{21} parameters from a value of -180 degrees to +180 degrees near 100 radians per second. This effective change in the value of phase for the uniform network is shown in the graph of Fig. 4.4. This change in phase resulted in nominal phase errors of nearly 360 degrees at the frequency of 100 radians per second for most cases using the Lumped Element Model. This large error at 100 radians per second, which is the direct result of defining the phase in such a way as to make it a single valued function of frequency, causes the high phase error for the exponentially tapered distributed network as shown in the graphs. The low phase error for 100 sections when α equals three provides a more accurate indication of the performance of the Lumped Element Model. The phase error then, is actually even less for the Lumped Element Model than might be apparent from consideration of the graph alone. We may then conclude that the Lumped Element Model gives better overall results than the Difference Equation Model for an exponentially tapered network.

5.4 Conclusion

Comparison of the results from the two models with the theoretical values of the admittance parameters of an exponentially tapered distributed network substantiates the conclusions reached from consideration of the uniform distributed network. The expected average error in both magnitude and phase obtained from the overall performance criterion shows that the performance of the Lumped Element Model is significantly better than that of the Difference

Equation Model when each model utilizes the same number of elemental sections.

CHAPTER 6

CONCLUSION

In this chapter, conclusions regarding the validity of the Lumped Element Model and the Difference Equation Model are presented. Application of these models to network analysis and synthesis problems is also discussed.

6.1 Validity of the Models

The results as presented in the preceding chapters show that both models give an accurate approximation to the admittance parameters of a uniform or exponentially tapered distributed network. Though only these two special cases were tested, the fact that the derivation of the models as presented in the third chapter made no assumptions regarding the geometry of the distributed network supports the claim that the models will give an accurate approximation for any three terminal distributed RC network. The Lumped Element Model was shown to give the better overall accuracy, so this model would be the one to use in most applications. The Difference Equation Model gave slightly better results for the y_{11} and y_{22} parameters, so one might use this model in applications where only driving point immitances were of concern.

6.2 Application of the Models to Network Analysis and Synthesis

The primary importance of the models lies in the fact that they provide a technique that may be applied to any three terminal distributed network, regardless of the geometry of that network. Previous techniques of analysis depend upon the geometry of the network, and there are many geometries which remain unexplored for the reason that no analysis technique has been available. The models presented here may easily be incorporated in a network analysis computer program that could be used in the analysis of circuits which employ distributed networks in addition to lumped elements and active devices. Such a program would be invaluable in the accurate analysis of monolithic integrated microcircuits.

Present distributed network synthesis techniques have only touched upon the possibilities of the use of distributed networks in circuits. Techniques have been developed where one may use distributed networks of a particular geometry in a circuit design problem, and a method has been developed (5) whereby one may use standard synthesis techniques on a distributed network. Use of the models presented here provides a new technique for synthesis of circuits which may contain distributed networks. An optimization routine, such as the gradient search method, when applied in conjunction with a circuit analysis computer program employing the model for the distributed network, would result in a synthesis program. The program would give the resistance and capacitance associated with each elemental section of the distributed network. The resistance and capacitance associated with each section would provide the information necessary to construct the network. The optimization routine would have to be constrained to prevent large changes in the resistance and capacitance parameters from section to section.

6.3 Conclusion

The models have been shown to provide an accurate approximation to a three terminal distributed RC network if a sufficient number of sections is used. The models provide a method to analyze a distributed network, regardless of the geometry of the network. This analysis technique, when coupled with a computerized optimization routine, results in a new method for synthesizing circuits which contain distributed RC networks.

SELECTED BIBLIOGRAPHY

1. Kaufman, W. M. and S. J. Garrett. Tapered distributed filters, IRE Trans. on Circuit Theory, Vol. CT-9, December, 1962, pp. 329-336.

2. Castro, P. S. and W. W. Happ. Distributed parameter circuits and microsystem electronics, Proc. NEC, Vol. 16, 1960, pp. 448-452.

3. Huelsman, L. P. Circuits Matrices and Linear Vector Spaces. McGraw-Hill, New York, 1963.

4. Fisz, M. Probability Theory and Mathematical Statistics, Third Edition. John Wiley and Sons, Inc., New York, 1963.

5. Heizer, K. W. Distributed RC networks with rational transfer functions, IRE Trans. on Circuit Theory, Vol. CT-9, December, 1962, pp. 356-362.

6. Dutta Roy, S. C. Theory of exponentially tapered RC transmission lines for phase shift oscillators, Proc. IEEE (London), Vol. 110, October, 1963, pp. 1764-1770.

7. Dutta Roy, S. C. On some three terminal lumped and distributed RC null networks, IEEE Trans. on Circuit Theory, Vol. CT-11, March, 1964, pp. 96-103.

8. Edson, W. A. Tapered distributed RC lines for phase-shift oscillators, Proc. IRE, Vol. 49, June, 1961, pp. 1021-1024.

9. Ekstrom, J. L. The Z matrix parameters of tapered transmission lines, IRE Trans. on Circuit Theory, Vol. CT-9, June, 1962, pp. 132-135.

10. Gruner, L. Synthesis of non uniform transmission lines, Proc. IRE, Vol. 49, March, 1961, pp. 627-628.

11. Gruner, L. The steady-state characteristics of non uniform RC distributed networks and lossless lines, IEEE Trans. on Circuit Theory, Vol. CT-12, June, 1965, pp. 271-247.

12. Happp, W. W. Synthesis of distributed parameter RC networks, Proc. IRE, Vol. 50, April, 1962, pp. 483-484.

13. Hellstrom, M. J. Symmetrical distributed RC networks or transmission lines, IRE Trans. on Circuit Theory, Vol. CT-9, September, 1962, pp. 247-251.

14. Jacobs, I. A generalization of the exponential transmission line, Proc. IRE, Vol. 47, January, 1959, pp. 97-98.

15. Kelly, J. J. and M. S. Ghausi. Tapered distributed RC networks with similar immittances, IEEE Trans. on Circuit Theory, Vol. CT-12, December, 1965, pp. 554-558.

16. Oehler, K. L. and W. C. Duesterhoeft, Jr. A graphical design for exponentially tapered RC circuits, IEEE Trans. on Circuit Theory, Vol. CT-12, June, 1965, pp. 285-290.

17. O'Shea, R. P. Synthesis Using Distributed RC Networks, IEEE Trans. on Circuit Theory, Vol. CT-12, December, 1965, pp. 546-554.

18. Su, K. L. Analysis of the trigonometric RC line and some applications, IEEE Trans. on Circuit Theory, Vol. CT-11, March, 1964, pp. 158-160.

19. Sugar, I. A new exact method of nonuniform transmission lines, Proc. IRE, Vol. 49, March, 1961, pp. 627-628.

20. Wilson, B. L. H. and R. B. Wilson. Shaping of distributed RC networks, Proc. IRE, Vol. 49, August, 1961, pp. 1330-1331.