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# A SIMPLE MODEL FOR STUDYING FRAGMENTATION IN ASTROPHYSICAL SYSTEMS

by

Thomas Travis Arny

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I hereby recommend that this dissertation prepared under my direction by Thomas Travis Arny entitled "A Simple Model for Studying Fragmentation in Astrophysical Systems" be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Ray J. Weymann
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This Work is Dedicated

to

My Father

who gave me curiosity

to

Dr. R. J. Weymann

who made me want to understand and has
shown me standards of excellence which,
though I may never meet them, must try

and

to

C. A. B.

who made me love my work.

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## ABSTRACT

The growth of a density perturbation in a spherically symmetric collapsing gas cloud has been studied by following the motion of points on the periphery of the perturbation. By such a technique, it has been possible to obtain solutions to the equations of motion including the effects of nonlinearity, nonuniformity, radiation and rotation. Magnetic effects have been ignored.

A comparison with the linearized theories of Hunter (1962) and Savedoff and Vila (1962) shows excellent agreement in the linear regime. It is found that nonlinearity enhances the growth rate relative to the linearized solutions. In the zero-pressure case it has been found that nonuniformity retards the growth.

The inclusion of radiation is found to give a roughly isothermal collapse both for temperatures of 100 and densities of 10 particles / cc, and for temperatures of $1.5 \times 10^{4}$ and densities of $10^{-3}$ particles / cc.

It is concluded that rotational forces can prevent collapse if they are sufficiently large. Pressure effects and the inclusion of background stars are both found to inhibit growth.

A calculation of the importance of collisions as a way of destroying fragments suggests that the effects will be most pronounced in high angular momentum systems.

It is suggested that a second generation of fragments is unlikely to form in low angular momentum systems. In high angular momentum systems, secondary fragmentation can probably occur in localized regions.

# I - INTRODUCTION

## A) EARLY STUDIES OF FRAGMENTATION

Since prehistoric times man has speculated on the origin of the heavens and the world about him. Most of this thinking remained steeped in mythology until the rise of science in the 17th and 18th centuries. Newton suggested that objects might cluster together under their own self-gravitational force to form bodies of cosmic dimensions. Laplace and Kant considered the origin of the solar system from a similar point of view.

The formation of cosmical bodies as the result of the instability of a background medium, the essence of many current studies in star and galaxy formation, was first suggested by Jeans and is described in his book Astronomy and Cosmogony (1929). Jeans showed that in a uniform medium at rest, disturbances with a scale $\lambda$ are unstable if

$$(1) \qquad \lambda^2 > \frac{RT}{\frac{4}{3}\pi G\rho}$$

or in terms of masses,

$$2) \qquad M > \frac{8.24\, T^{3/2}}{\sqrt{n}}$$

where T is the temperature in degrees kelvin, R is the gas constant, G the gravitational constant, $\rho$ the density in gms/cc, and n is the particle density in particles/cc. $\lambda$ and M are then in cm. and solar masses (M☉) respectively.

Jeans reasoned that an instability results if the compression due to the passage of a sound wave decreases the gravitational potential more than it increases the thermal energy. The state of lower potential is the more stable and thus the gas tends to form regions of higher than average density.

The critical value for $\lambda$ in eq(1) results from the linear analysis of the behavior of a self-gravitating sound wave moving in a uniform infinite medium at rest. Disturbances of wavelength smaller than $\lambda$ oscillate, while those larger grow exponentially in time.

If one inserts values for T and n representative of what are thought for the interstellar and intergalactic medium, say $n = 10$, $T = 100$ and $n = 10^{-3}$, $T = 10^{6}$, one obtains $M = 2 \times 10^{3}$ M☉ and $M = 2 \times 10^{11}$ M☉ . While the latter is approximately what one finds for galactic masses, the former is nearer the mass of a star cluster than that of an individual star. Thus either some mechanism intervenes in star formation or Jeans' analysis is too idealized.

As has been pointed out by several authors, the initial state assumed by Jeans is inconsistent, for one cannot satisfy both Poisson's equation and have an infinite, uniform medium at rest.

A more extensive analysis of the instabilities of an astrophysical system was made by Hoyle (1953) in which the effects of radiation were included. The model considered was a cloud of $3.9 \times 10^{9}$ Mo with an initial temperature of $1.5 \times 10^{4}$ and a density of $10^{-27}$ gm/cc. By comparing the gravitational and thermal energy content and using the virial theorem, one finds that such a mass is unstable against collapse. As the cloud radius shrinks, the density rises. The increasing density also increases the rate of radiation, offsetting the heating due to compression and maintaining the collapse isothermal. The increased density enhances still more the instability. The cloud begins to break up or fragment. Simultaneously the density of the fragments is rising, making them unstable also. This initiates another stage of fragmentation.

When the density of the fragments has risen to the point that they are opaque, preventing the escape of radiation, the temperature rises. Eq(2) shows that a larger T will raise the critical mass offsetting the instability due to the increased density. Hence, fragmentation stops.

In a later paper, Hoyle and Fowler (1963) make a similar analysis of cooler clouds including in a rough way the effects of angular momentum and magnetic fields. These papers are exceedingly valuable in that they outline a plausible sequence of events leading to star formation and because they offer a good explanation of why collapse stops.

In addition to these rather general and semiquantitative approaches, more specific investigations have been made of instabilities in polytropes (Ebert (1955), Bonner (1956), McCrea (1957), Unno and Simoda (1963) and McNally (1964)), the effect of magnetic fields (Mestel and Spitzer (1956)), the interplay of thermal radiation and mechanical stability (Hatanaka et al. (1961)), and the role of opacity (Gaustad (1963), and Gould (1964)). However, in all of these papers the actual development of fragments has been more a reasonable supposition than a proven fact.

The actual growth and formation of fragments was first investigated by Hunter (1962) and Savedoff and Vila (1962). These papers form the basic literature for an exact treatment of fragmentation. The model studied in both cases is that of a uniform cloud collapsing under its own gravitational force. The perturbations are assumed to satisfy a polytropic equation of state. The pressure in the main cloud is taken as uniform. With these assumptions it is possible to write down the linearized equations of motion and continuity and obtain from these a single differential equation for the density of the perturbation. The solutions of this equation are obtainable for a number of special cases and while Savedoff and Vila also study an expanding case, the results of the two papers are the same. While objections have been raised about the applicability of this work to star formation (Layzer (1963, 1964)), the conditions under which gravitational instabilities produce density fluctuations that grow in time are well determined.

Layzer has criticised this work on several grounds. He states that nonlinear processes will intervene to prevent the growth of fragments and that torques induced on them in the early stages of growth will increase the angular momentum to the point where they are unable to grow. Since the main cloud continues collapsing, they will be reabsorbed in the subsequent stages of collapse.

Hunter (1964) presents arguments in a later paper based on the extension of the linear theory which purport to controvert Layzer's objections. However, one may question his conclusion that angular momentum does not play an important role because it is intrinsically nonlinear due to the $v^2$ dependence of the centrifugal force.

Any theory of fragmentation meets its ultimate test in predicting the mass function of stars and galaxies. In principle, from the full equations, if one had the opacities, the magnetic fields, and the turbulence spectrum for each set of initial conditions, such a prediction might be made. However, in view of the basic objections raised by Layzer to the fragmentation concept and the idealizations in the linear theories to this stage, it seems more expedient to study in more detail even the simplest model of a collapsing cloud. Thus in the work discussed here, magnetic fields are not included. Furthermore, "turbulence" is treated only in terms of an average over the mass elements being studied. (This point will become clearer when the model is described in part II.) Despite these simplifications, it is hoped that by clarifying the importance of nonlinearity, nonuniformity, pressure and rotation, a step will have been made toward a fuller understanding of the collapse of gas clouds.

## B) METHODS FOR STUDYING INSTABILITIES

The way the problem should be resolved is to write down the full equations of hydrodynamics including the proper energy equation. As was mentioned, Hunter and Savedoff and Vila eliminated the energy equation by assuming a polytropic equation of state. Radiation effects might be taken into account by using a variable and complex gamma.

The difficulty with the hydrodynamic approach is that an unperturbed solution is necessary in order to use first order perturbation theory. This is not easy to obtain except in a few simple cases, such as the uniform sphere. In this case it is possible to write a separable partial differential equation for the density for the linearized problem. When nonlinear effects are included, or when nonuniformity, for example, is added to the sphere, it is not clear whether an equation can be written that separates the density and velocity fields.

One method that has been used with success by Chandrasekhar (1961), Lynden-Bell (1964), Michie (1963) and others is to take moments of the hydrodynamic equations. In this case, particularly the work of Michie, by choosing the averages properly, nonuniformity and nonlinearity may be included in a very reasonable fashion.

While there is a loss of information in moment methods, it must be remembered that the hydrodynamic equations are themselves moments of the Boltzmann equation, and the taking of moments in this case is not only physically reasonable, but a virtual necessity if solutions are to be obtained. The connection with the Boltzmann equation suggests another possible approach.

In a system with spherical or plane symmetry it is only necessary to study the motion of one point in each surface in order to follow the development of the system. Thus in a contracting sphere with no perturbation, for a particular shell, all points move in the same fashion. If asymmetries are introduced, for example, a perturbation, then while one point is no longer adequate, one can imagine locating points all about the periphery of the perturbation and studying their motion. If the perturbation itself possesses a high degree of symmetry, say spheroidal, one might think that following a point on its pole and another on its equator would describe the motion with some accuracy. One thus replaces a set of continuum equations by a set of equations describing the motion of points. In the work described in the rest of this paper, extensive use is made of this method which will be referred to hereafter as the "two-point model". The results obtained with this method will be compared with those from the continuum model where the latter are available.

## II - TWO-POINT MODEL

### A) EQUATIONS OF MOTION FOR THE ZERO-PRESSURE CASE

In order to study the stability of a collapsing gas cloud, a point model such as described above in part I has been used. To illustrate the method, the full equations for the zero-pressure case are derived below. A step by step comparison of them to the hydrodynamic equations with pressure is given in appendix I.

The unperturbed model is a spherically symmetric cloud with some arbitrary density distribution. Hence, all particles in a particular shell feel the same gravitational attraction. Furthermore, the mass that is initially in the shell remains within it so that the equation of mass conservation can be immediately written

$$3) \qquad \rho_L(r)\,\Delta r\, r^2 = \rho_{Lo}\,\Delta r_o\, r_o^2$$

where $\Delta r$ represents the thickness of a shell, $\rho_L$ is the local density and r is the distance from the cloud center. The subscript "o" refers to the initial values of the quantity.

**Next, a small density perturbation is introduced, as shown in fig(1), whose center lies at some initial distance $r_o$ from**

the cloud center. It is assumed to be spheroidal. The excess mass, assumed to be distributed uniformly throughout it, is to be small compared to the mass of the main cloud, in the sense that the rest of the cloud is not significantly influenced by the perturbation. Note that this is not the same as assuming that the density of the perturbation is small. In addition, the perturbation is oriented so that its axis of symmetry lies along a cloud radius. If "a" and "b" are the equatorial and polar dimensions, its volume is

$$V_p = 4\pi a^2 b/3 .$$

If one then assumes that the total mass within it remains constant,

$$\rho_p V_p = \text{constant}$$

where $\rho_p$ is the total density of the perturbation. Since it is the density relative to the background that determines whether the perturbation will survive the collapse, the quantity $\xi$ is introduced, defined as

$$\xi = \frac{\rho_E}{\rho_L}$$

where $\rho_E$ is the excess density. One can then write

$$\rho_p = \rho_L (1 + \xi) .$$

Substituting now for $V_p$ and $\rho_p$ one obtains

$$4) \qquad \tfrac{4}{3}\pi\rho_L(1+\xi)\,a^2b = \text{constant} .$$

Eliminating $\rho_L$ with the help of eq(3), one finds

$$5) \quad \rho_L(r=r_o)\,\frac{r_o^2}{r^2}\,\frac{\Delta r_o}{\Delta r}\,(1+\xi)\,a^2b = \text{constant}$$

which is the equation of continuity for the perturbed flow. The only difficulty that arises in using this expression is evaluating $\frac{\Delta r}{\Delta r_o}$ which is done in appendix II. Physically this quantity determines how the thickness of a shell varies in time compared to its original value. It is hence a function of position as well as the initial mass and velocity distribution in the medium. To summarize, the assumptions made are:

1) Spherical symmetry in the main flow.
2) A uniform spheroidal perturbation.
3) A constant mass interior to the perturbation.
4) The mass excess is small compared to the total cloud mass.

To determine how the density of the perturbation varies in time, it is necessary to know how the distances "a" and "b" vary. From fig(1), one can see that they depend on the motion of the points A, B, and C. Since C is the center of the perturbation, its motion is unaffected by the presence of the perturbation provided that there are no pressure gradients in the main cloud. One can thus write

$$6) \qquad \ddot{\vec{r}}_c = - \frac{GM_c}{r_c^3} \vec{r}_c$$

Point B experiences an acceleration due to the mass $M_b$ interior to it contributed by the main sphere, as well as an acceleration due to the perturbation. Its equation of motion is thus

$$7) \qquad \ddot{\vec{r}}_b = - \frac{GM_b}{r_b^3} \vec{r}_b - \left.\frac{\partial V}{\partial r}\right|_p \hat{r}_b$$

where $\left.\frac{\partial V}{\partial r}\right|_p$ is the acceleration of the perturbation.

In determining the acceleration of A, it is simplest to introduce a new variable $\pi = r \tan\theta$ where $\theta$ is defined in fig(1). Differentiating twice with respect to time yields

$$\ddot{\pi} = \frac{R^2}{r^2} (r\ddot{\theta} + 2\dot{r}\dot{\theta}) + 2\frac{R^2}{r^2} \dot{\theta}^2 + \frac{\ddot{r}\pi}{r}$$

If $u = \pi/r$, then

$$\ddot{u}r = -2\dot{r}\dot{u} - \frac{R^2}{r^2} \left.\frac{\partial V}{\partial r}\right|_p + 2r\cos^2\theta \, \dot{\theta}^2 u$$

Since $u = \tan\theta = \theta(1 + \theta^2/6)$, if u is small (the linear size of the perturbation is small compared to the cloud radius)

$$8) \qquad \ddot{u}r + 2\dot{r}\dot{u} = - \left.\frac{\partial V}{\partial r}\right|_p .$$

Since there is more mass interior to R than r, a torque is exerted on point A which tends to distort the perturbation. This has been ignored as it can be shown that it is of second order in $\theta$. This allows one to write

$$9) \qquad 2r\dot{\theta}^2 + r\ddot{\theta} = -\left.\frac{\partial V}{\partial r}\right|_p .$$

Now a = u and $b = \left|\vec{r}_c - \vec{r}_b\right|$. Thus

$$10) \quad \ddot{b} = \ddot{r}_b - \ddot{r}_c = -\frac{GM_b}{r_b^2} + \frac{GM_c}{r_c^2} - \left.\frac{\partial V}{\partial r}\right|_{pole}$$

$$11) \quad \ddot{a} = \ddot{u} = -\left.\frac{\partial V}{\partial r}\right|_{equator} \cdot \frac{1}{r} - \frac{2\dot{r}\dot{u}}{r}$$

To facilitate computations and to indicate more clearly the functional dependence of the equations on the parameters, dimensionless variables have been introduced with $r_c(t=0)$ as the scale of length and $\frac{1}{\sqrt{\frac{4}{3}\pi G\rho_o}}$ the unit of time, where $\rho_o$ is the initial density of the main cloud.

If one then defines

$$x = \frac{r}{r_o}$$

$$y = \frac{b}{r}$$

$$z = u$$

one finds

$$\ddot{x} = -\frac{1}{x^2}$$

$$\ddot{y} + \frac{2\dot{x}\dot{y}}{x} + y\frac{\ddot{x}}{x} = -\frac{1}{x^3}\left[\frac{M_b}{M_c(1+y)^2} - 1\right] - \frac{\left.\frac{\partial V}{\partial r}\right|_{pole}}{\frac{4}{3}\pi G \rho_o r_o x}$$

$$\ddot{z} + \frac{2\dot{x}\dot{z}}{x} = -\left.\frac{\partial V}{\partial r}\right|_{equator} \cdot \frac{1}{\frac{4}{3}\pi G \rho_o r_o x}$$

The expressions for the potential are taken from McMillian (1958) and differentiating them one obtains

$$\frac{GM_p}{r_p^2} f(e) \quad ,$$

or an attraction due to a spherical mass times a correction factor f(e) where e is the eccentricity. (See appendix III for the explicit expressions.)

$M_b$ is eliminated as follows:

$$M_b = M_c + 4\pi \int_{r_c}^{r_b} \rho(r)\, r^2\, dr$$

$$= M_c + 4\pi \overline{\rho}_L \frac{r_b^3 - r_c^3}{3}$$

where $\overline{\rho}_L$ is the average value of $\rho$ across the shell containing the perturbation.

Now $r_b = r_c + b = r(1+y)$. Since

$$M_c = \frac{4}{3}\pi r^3 \overline{\rho}$$

$$M_b = M_c \left\{1 + \eta\left[(1+y)^3 - 1\right]\right\} \qquad \text{where}$$

$$\eta \equiv \frac{\bar{\rho}_L}{\bar{\rho}}$$

substituting this into eq(13) and assuming $y \ll 1$, as was done in deriving eq(8), one finds

$$\begin{array}{lllll} & \text{A} & \text{B} & \text{C} & \text{D} \\ 15) \quad \ddot{y} = & -2\frac{\dot{x}\dot{y}}{x} & + \frac{3y(1-\eta)}{x^3} & - & \xi\eta\frac{f(e)y}{x^3} \end{array}$$

$$16) \qquad \ddot{z} = -2\frac{\dot{x}\dot{z}}{x} - \xi\eta\frac{g(e)\,z}{x^3}$$

$$17) \qquad \ddot{x} = -\frac{1}{x^2}$$

$$18) \quad yz^2\eta(1+\xi) = y_o z_o^2 \eta_o (1+\xi_0) \ .$$

Eq(18) comes from eq(4) with $\rho_L$ eliminated in terms of $\eta$.

The equations of motion thus involve four terms describing the time rate of change of the lengths (A); a term describing the change in y due to the fact that the main cloud radius x which is the unit in which y is measured is changing (B); a tidal term for the main flow absent in the z equation (C); and a term describing the self-gravitation and the tidal force on the perturbation itself (D).

Three points should be noted regarding the equations:

1) Aside from the initial values of x, y, and z and their velocities, only the parameters $\xi_o$, $e_o$, $\eta_o$, and $\eta$ enter.

2) The equations for the zero-pressure case are homogeneous in y and z and hence the behavior of the perturbation is independent of its linear size, provided this is small (See eq(15,16)).

3) No restriction is placed upon the value of the density, $\xi$. Thus the equations derived should be a valid representation of the nonlinear results, and represent therefore an extension beyond the work of Hunter and Savedoff and Vila.

B) RESULTS FOR THE TWO-POINT ZERO-PRESSURE MODEL

In principle, it is possible to obtain an analytic solution to the two-point equation for the zero-pressure uniform cloud case. As is shown in appendix IV, a perturbation in a uniform sphere is independent of the gravitational field of the main cloud and hence can be treated as an isolated body. It is then simple to obtain "t" as a function of the perturbation radius and hence the density, but the inversion to find the density as a function of time is not so simple. Since this solution is available only for this special case, the other equations must be integrated numerically anyhow. Hence, all work was done numerically.

Before even numerical work can be done, $\eta$ must be evaluated. Since it is a measure of the central concentration of the main cloud, it is in general a function of time. For the uniform case, $\eta = 1$. The evaluation for other cases is described in appendix II. One can say that in general $\eta$ decreases indicating an increase in the central cloud density with time relative to the mean density. A plot of $\eta$ versus time is shown in fig(2).

The equations were integrated numerically with a Runge-Kutta scheme on the IBM 1401-7072 computer of the University of Arizona Numerical Analysis Laboratory. $\mathcal{E}$ was computed from $\eta$, y, and z with the use of eq(18). The results are plotted in fig(3) and fig(4) showing the effect of nonlinearity ( $\mathcal{E}_o$) and central condensation ( $\eta_o$) on the growth. One sees that nonlinearity enhances the growth relative to the linear and that larger central concentrations retard the growth relative to that for the uniform sphere. Fig(5) shows how the particle density, n, varies both in the perturbation and the main cloud for typical values of the interstellar medium.

An interesting point arises in that the perturbations are unstable with respect to nonradial perturbations. If the surface is deformed slightly, the surface accelerations are increased accentuating the deformation. Thus perturbations tend to become pancakes or cigars. Pressure forces obviously stabilize a perturbation against such distortion. One should note, however, that the density growth is not seriously affected and cannot be discerned on the plots shown in fig(3) or fig(4). One thus concludes that the initial eccentricity is not an important parameter for a zero-pressure cloud.

For a uniform cloud, if a perturbation starts spherical, it remains spherical since there is no distorting force. In a nonuniform cloud, however, the tidal forces rapidly pull a perturbation into a highly prolate form.

The parameter $\eta$ plays a very important role. As was mentioned, this is related to the tidal forces on the perturbation in the sense that a small $\eta$ implies a large gradient in the force across the perturbation. It may seem curious that this gradient only retards the growth of a perturbation. However, the perturbation is a density excess and thus one should compare its growth with that of the surroundings. In this case, it is easy to see that any tidal force due to a central condensation in the main cloud will affect the material in the perturbation and the surroundings identically. As far as the excess mass is concerned, while contraction along the direction of the tidal force gradient is retarded, that perpendicular to it is not affected. Hence, the excess density can still increase.

One additional set of computations has been made in order to study the effect of background stars on the density growth of fragments. The stars are taken to be distributed in a uniform sphere about the center of the gas cloud in a static configuration. The gas then collapses to the cloud center through the swarm. While the stars contribute mass to the cloud and thus accelerate the collapse, they do not aid the self-gravitation. Thus the growth of perturbations is retarded by their presence. Calculations were made for the density ratio of stars to gas as 1, 10, and 100. The results are shown in fig(6) along with a solution illustrating the growth when no stars are present.

It is interesting to compare the results for the two-point model in a uniform cloud to those of Hunter and Savedoff and Vila for the linear case. As is shown in appendix I, eq(6), the differential equations for the hydrodynamic case are identical to those for the linearized two-point models. Thus it is not surprising that a comparison of the Hunter solutions with the two-point results gives excellent agreement. This is plotted in fig(3). One does see though that unless the initial amplitudes are taken small for the two-point model, the density very quickly grows much larger than the linear theory would predict. This is a consequence of the fact that a larger initial amplitude increases the self-gravitational terms. The excellent agreement with the solutions for the hydrodynamic equations leads one to hope that the two-point model gives equally good results in those cases where the hydrodynamic treatment cannot readily be used.

To summarize the results for the zero-pressure case, one can say the following:

1) The agreement with the linear theory suggests that the two-point model is a good approximation to the full hydrodynamic equations.

2) Nonlinear effects accelerate the growth rather than retard it in the sense that denser perturbations grow faster.

3) Nonuniformity of the main cloud induces a tidal disruption that hinders the growth of perturbations but does not prevent it. It furthermore accentuates the central condensation in the main cloud.

4) Background stars reduce the growth rate of a perturbation by accelerating the collapse of the main cloud.

## III - CASE OF FINITE-PRESSURE

### A) DISCUSSION OF THE LINEARIZED EQUATIONS

The analysis of the flow in the finite-pressure case for a uniformly collapsing sphere is possible provided pressure gradients are allowed only in the perturbation. If pressure gradients are introduced in the main flow as well, it would appear no longer possible to obtain an analytic solution for the unperturbed flow. If one is content, though, to allow the main sphere to collapse in free-fall, but to include pressure gradients in the perturbation, some solutions can be found. This has been pointed out by Hunter (1962) and Savedoff and Vila (1962), hereafter referred to as HSV. These solutions will be briefly discussed because they afford an additional check on the two-point model as well as a more realistic representation of a collapsing cloud.

An inspection of the partial differential equation for the density derived by HSV in the linearized case shows that it can be separated into a space and time portion. The space portion can be further separated into an angular and radial equation yielding spherical harmonics and spherical Bessel functions for the solutions. The density field is thus decomposed into a series of normal modes, and a general solution requires the superposition of many different wavelengths.

The time part is a differential equation of the second order in two parameters. One parameter is the ratio of the size of the perturbation to the Jeans' length. The other is the ratio of the specific heats of the gas--the gamma of the polytropic relation. If one thinks of the perturbation as being described by a dominant mode, its behavior is then a function of its size relative to some critical size and gamma. This is physically reasonable since small disturbances might be thought to behave somewhat like sound waves, while longer ones would be dominated by self-gravitation. The gamma enters as a measure of the elasticity, so to speak, of the medium.

Since two parameters appear, it is surprising that except in the case gamma = 4/3, the asymptotic behavior is a function of gamma alone. This is quite at odds with the classical Jeans result, though perhaps not so surprising considering that he was solving a different problem. The explanation for this lies in the fact that for gamma less than 4/3, the pressure forces diminish relative to the gravitational forces as the cloud collapses causing $\lambda_J$ to go to zero. For gamma greater that 4/3, $\lambda_J$ goes to infinity since the pressure forces rise relative to the gravitational forces. Thus the Jeans' length enters in determining the growth rate but not the stability. Disturbances large with respect to the Jeans' length grow more rapidly than those small with respect to it.

The asymptotic behavior of the solutions has been obtained by Savedoff and Vila from an analysis of the linearized equations. The results can be summarized as follows:

Gamma less than 4/3 . . . all modes grow in time

Gamma greater than 4/3, less than 5/3 . . . all modes oscillate in time with increasing amplitude

Gamma = 5/3 . . . all modes oscillate with constant amplitude

Gamma greater than 5/3 . . . all modes decay and oscillate

Gamma = 4/3 . . . modes grow if $\lambda_J$ greater than $\lambda$ critical, oscillate with increasing amplitude otherwise.

The simple order of magnitude calculation below indicates why, for gamma less than 4/3, pressure forces diminish relative to gravitational forces and allow the perturbation to collapse in essentially free-fall.

19)

$$A_g = -\frac{GM_p}{r_p^2}$$

$$A_p = \frac{1}{\rho}\frac{dp}{dr} \sim \frac{k\rho^{\gamma-1}}{r_p} \qquad \rho \sim \frac{M_p}{r_p^3}$$

$$\frac{A_p}{A_F} \sim r_p^{-3(\gamma-4/3)}$$

$$\text{for} \quad \gamma < 4/3 \quad \text{as} \quad r \to 0$$

$$\therefore \quad \frac{A_p}{A_g} \rightarrow 0$$

One thus concludes that for gamma less than 4/3, the collapse is asymptotically pressure independent.

The explanation of the oscillating modes is less apparent. One can argue that in these cases the pressure gradients build up more rapidly than the gravitational forces causing the perturbation to reexpand. This is not strictly true however as will be seen later. The fact that the perturbation oscillates as a standing wave is particularly strange since one might think that a disturbance would generate a sound wave that would simply move out of the disturbed area and disperse. The standing wave character of the solution is a consequence of the spherical nature of the system and the fact that the boundary conditions demand that the amplitude of the oscillations always remain finite. In general, there can be no traveling wave solutions in a steady state bounded spherical system (Landau and Lifschitz, (1958)).

## B) TWO-POINT FORMULATION OF THE FINITE-PRESSURE CASE

In order to understand the effects that nonlinearity will have on the finite-pressure case, the two-point model has been modified to include pressure gradients. As in the linearized theories, the main flow is assumed to be uniform, but the perturbation is assumed to have a pressure gradient and to satisfy a polytropic equation of state. The zero-pressure case showed that the shape of the perturbation was not important in determining the results. Hence, a spherical perturbation has been assumed here.

A question now arises as to how $\frac{1}{\rho}\frac{dP}{dr}$ should be evaluated when only two points are used. The most straightforward scheme would be to replace $\frac{1}{\rho}$ by $\frac{1}{(\rho_L + \rho_P)/2}$ and $\frac{dP}{dr}$ by $\frac{\Delta P}{\Delta r}$. In this case, $\frac{1}{\rho}$ is averaged over the perturbation and the background, and $\frac{dP}{dr}$ is taken as linear across the disturbance. Such a procedure was followed in deriving eq(20b) below.

Suppose, however, that one assumes that the perturbation collapses homologously, as is found to be the case in the linear theory with $\rho_{excess} = \rho_c(t)\,\frac{\sin kr}{kr}$ where $\rho_c$ is the density excess at the center of the perturbation. If one now assumes that this same spatial profile is maintained for the nonlinear regime, it is possible to approximate $\left.\frac{1}{\rho}\frac{dP}{dr}\right|_i$ by $\frac{1}{\rho_i}\,\frac{\Delta P}{\Delta r}\cdot\alpha$ where $\alpha$ is a factor to correct for any error incurred in replacing the derivitive by a difference. $\alpha$ can now be calculated using the expression for $\rho_{excess}$ above. The two-point model, however, involves not only $\rho_{excess}$, but also the excess mass. It is thus necessary to relate the mass excess and the density excess, $\mathcal{E}_c$. This can be done by the use of another factor, $\beta$. Thus, $m_{excess} = \frac{4}{3}\pi r^3(1+\bar{\mathcal{E}}) = \frac{4}{3}\pi r^3\rho_c(1+\beta\mathcal{E}_c)$. It is then possible to evaluate $\beta$ in the same fashion as $\alpha$. (See appendix I for the expressions for $\alpha$ and $\beta$.) In addition, as is seen in in appendix I, by choosing the radius of the perturbation in the two-point model properly, it is possible to obtain a nonlinear differential equation that reduces exactly to the linearized one for small density excesses. (In principle, if $\alpha$ and $\beta$ were treated as functions of time, it should be possible for the two-point model to give an exact representation of the nonlinear equation.) This nonlinear two-point equation for $\xi$ is given in appendix I.

It is interesting to compare the two-point equation derived with the aid of $\alpha$ and $\beta$, to the linear equation of HSV and the two-point equation derived by simply averaging without recourse to $\alpha$ and $\beta$. To do this, x is eliminated from eq(5), appendix I, in favor of $\rho$ so that $\rho = \rho_0/x^3$. x is eliminated in the same fashion. One then obtains eq(20a) or eq(20b) depending on whether $\alpha$ and $\beta$ (eq(20a)) are used, or a simple average (eq(20b)). If the equations are linearized in $\xi$, one obtains eq(20c), identical to that of HSV derived in appendix I, with the provision that in eq(20b), $\lambda_n^2 \Rightarrow 3\lambda_n^2$. The factor three arises from the difference between the simple average and the use of $\alpha$ and $\beta$. If the density $\rho$ is maintained constant, one sees that eq(20c) reduces to eq(20d) which is essentially the equation derived by Jeans.

The term in eq(20a,b) involving $\frac{\lambda_J}{\lambda_n}$ is the ratio of the pressure to the self-gravitational forces. Since the density is measured with respect to the background, one must include the general rise in the density of the main flow. The term $\xi^2$ is a nonlinear term that always acts to increase the growth or retard the decay.

$$20a)\quad \ddot{\xi} = 3(1+\xi)\xi\left\{1 - \frac{\lambda_J^2}{3\lambda_n^2}\left(\frac{\bar{\rho}}{\rho_o}\right)^{\gamma-4/3} + \frac{2\dot{\xi}}{9(1+\xi)\xi}\left(\frac{d\ln\bar{\rho}}{dt} + \frac{2\dot{\xi}}{1+\xi}\right)\frac{\rho_o}{\bar{\rho}}\right\}\frac{\bar{\rho}}{\rho_o}$$

$$b)\quad \ddot{\xi} = 3(1+\xi)\xi\left\{1 - \frac{\lambda_J^2}{\lambda_n^2}\left(\frac{1+\xi}{1+\xi_o}\right)^{2/3}\left(\frac{(1+\xi)^{\gamma-1}}{1+\frac{\xi}{2}}\left(\frac{\bar{\rho}}{\rho_o}\right)\right)^{-4/3} + \frac{2\dot{\xi}}{9(1+\xi)\xi}\left(\frac{d\ln\bar{\rho}}{dt} + \frac{2\dot{\xi}}{1+\xi}\right)\frac{\rho_o}{\bar{\rho}}\right\}\frac{\bar{\rho}}{\rho_o}$$

$$c)\quad \ddot{\xi} = 3\xi\left\{1 - \frac{\lambda_J^2}{3\lambda_n^2}\left(\frac{\bar{\rho}}{\rho_o}\right)^{\gamma-4/3} + \frac{2}{9}\frac{d\ln\bar{\rho}}{dt}\frac{\rho_o}{\bar{\rho}}\frac{\dot{\xi}}{\xi}\right\}\frac{\bar{\rho}}{\rho_o}$$

$$d)\quad \ddot{\xi} = 3\xi\left\{1 - \frac{\lambda_J^2}{3\lambda_n}\right\}$$

As was mentioned in part II, gravitational forces in the uniform cloud do not affect a spherical perturbation, allowing it to be treated as an isolated unit. Since the pressure is assumed to be uniform in the main cloud, the inclusion of pressure does not alter this result with the provision that the isolated fragment must now have an increasing external pressure applied to it. Bonner (1958) has studied the stability of polytropes when an external pressure is applied and found that for gamma less than 4/3 they were unstable. This is exactly the result found from the linearized theories mentioned earlier.

In order to test the consequences of assuming a sin kr/ kr density profile for the perturbation, a numerical integration of a collapsing gas cloud has been made using a program of Weymann's (1964). As was mentioned before, since the background is uniform, the perturbation can be treated as an isolated unit. If the radius of the perturbation is choosen at a maximum or a minimum of the density profile, then the excess mass vanishes, and since the pressure gradient is zero, there is effectively no coupling between the perturbation and the main cloud. Weymann's program integrates the full hydrodynamic equations. It is thus possible to follow in some detail the changes in the physical variables throughout the sphere as collapse proceeds. For the particular case studied, a disturbance with a sin kr / kr profile and an amplitude of .1 at the center was assumed along with an isothermal collapse. It was found that the sin kr / kr profile maintained itself well. The central excess density was computed and compared with that determined from the two-point model (eq(20a)). For $\tau$ = 1.029, or .936 of the free fall time, the exact hydrodynamic model gave an excess density of 30.247 times the initial excess density. The two-point model yielded 30.213. On the other hand, the linear theory predicted a value of 10.3. These results are shown in fig(10).

One would thus conclude that the assumption of an homologous contraction is fairly good, even in the nonlinear regime, and that the pressure gradient in the two-point model can be approximated accurately with the aid of the scale factors $\alpha$ and $\beta$.

C) RESULTS OF THE NONLINEAR ANALYSIS

As before, the equations were integrated numerically to obtain the excess density as a function of time. In the computations, it was found advantageous to decrease the step lengths as the collapse proceeded in order to follow in more detail the oscillations that result for gamma greater than 4/3. Therefore the time steps were calculated at each step as

$$\Delta t = \Delta t_o x^{3/2}$$

where $\Delta t_o = .01\, \tau_{ff}$ and $x = \frac{r}{r_o}$ .

Since the main cloud is uniform, $\eta \equiv 1$. Thus the parameters that enter now are the initial excess density, the polytropic index (expressed in terms of gamma), and the ratio of the Jeans' length to the perturbation size. The latter is a measure of the initial pressure forces to the self-gravitation. Once again, all initial velocities were chosen as zero. The excess density was fixed at .1 as being a physically reasonable value and yet large enough to indicate the effect of nonlinearity. $\frac{\lambda_J^2}{\lambda_n^2}$ was selected as 1.5. Since asymptotically the value of $\lambda_n$

does not matter, a value was chosen that illustrated the effect of the pressure relatively early in the collapse.

Fig(7) shows now the polytropic index affects the solution. The smaller gamma is, the more rapidly the perturbation grows. In fig(8), the effect of nonlinearity is shown. The dotted curve representing the nonlinear point model as well as the results of HSV (the solid curve) is shown. One can see that in the case where growth occurs, the nonlinear terms again accelerate the growth relative to the linear theory. In the case of oscillations, the conclusions are less clear except that the nonlinear effects cause a drastic change in the oscillatory nature and limit the lower density obtained. The crosses on the curve indicate the results obtained from the two-point model when the initial amplitude is .0001. Thus, agreement is again excellent in the linear regime between HSV and this work.

In fig(9), the density of a perturbation and the cloud background are shown for a typical HI cloud assumed to be collapsing isothermally. One sees that the pressure slows down the growth relative to the zero-pressure case and that it takes nearly a free fall time for the density to increase to 100 particles / cc.

Since, for gamma equal to one, the collapse is asymptotically the same as the case of free-fall, $\dot{x} \sim \frac{1}{x}$ or $x^{3/2} \sim t - t_o$. Converting to a density,

$$\rho \sim \frac{\text{const}}{(t - t_o)^2} \, .$$

If this is fitted to the numerical results, one finds for an isothermal collapse with an initial density of n = 10, that

$$n \sim \frac{960}{(t - 16)^2} \quad \text{particles / cc}$$

where n is the particle density and t is the time in millions of years.

D) PHYSICS OF PRESSURE-RETARDED COLLAPSE

The pulsation theory of variable stars shows that a gas with a gamma of 5/3 should oscillate with constant amplitude. It might then at first seem surprising that if one plots the radius of a perturbation in such a cloud as a function of time, it is found to be in general a decreasing function. When one remembers though that the external pressure is constantly increasing due to the main cloud collapse, this becomes very understandable. (I am indebted to Dr. R. Michie for first pointing this out to me.) It is also the increasing pressure that is responsible for the increasing amplitude of the oscillations found for gamma between 4/3 and 5/3. To show this, a WKB type analysis similar to that of HSV is used.

If eq(15) is modified to allow for a pressure gradient in the perturbation following the method of appendix I, and a uniform collapse is assumed ( $\eta = 1$), one obtains

$$\ddot{y} + \frac{2\dot{x}\dot{y}}{x} + \frac{\mathcal{E} y}{x^3}\left[1 - \frac{\lambda^2}{\lambda_o^2\, x^{3(\gamma - 4/3)}}\right] = 0$$

where the perturbation has been assumed spherical. This can be transformed to an equation of the form $\ddot{u} + T(\eta)u = 0$ by means of the substitution $y = fu$ where $\dot{f}/f = -\dot{x}/x$. Thus $f \sim 1/x$. In this case, $T = Q + 1/x^3$ where Q is the coefficient of y in the above equation. It is now possible to use a WKB type solution and obtain

$$u \sim \frac{1}{T^{1/4}} \quad \left[\text{oscillating function}\right] .$$

In the late phases of collapse ($x \rightarrow 0$), the term $\frac{\lambda^2}{\lambda_o^2} x^{4-3\gamma}$ goes to zero if gamma is less than 4/3 and to infinity if gamma is greater than 4/3. Since this term is effectively a weighted ratio of the Jeans' length to the perturbation size, one can say that for gamma less than 4/3, $\lambda_J$ goes to zero, while for gamma greater than 4/3, $\lambda_J$ goes to infinity. This was mentioned in section A of this part.

The case of interest here is gamma greater than 4/3. In this instance, asymptotically, $T \sim \frac{\mathcal{E}\lambda_J^2}{x^3\lambda_o^2} x^{4-3\gamma}$. Thus $u \sim x^{\frac{3\gamma-1}{4}}$ and hence $y \sim x^{\frac{3\gamma-5}{4}}$. Thus gamma equal to 5/3 is also a critical value, as was shown by HSV. If one now converts back to the physical variables, the radius of the perturbation, r, can be

written $$r = xy$$
$$\sim x \cdot x^{\frac{3}{4}(\gamma - \frac{5}{3})}\left[\text{oscillating function}\right]$$

Since gamma is between 4/3 and 5/3, one can see that r goes to zero as x does. Thus the physical size of the perturbation actually is decreasing despite the increasing amplitude of the oscillations. One can further see that the dependence on gamma enters through the pressure term. Thus, as claimed above, it is the pressure that is responsible for the increasing amplitude of the oscillations.

Physically, one sees that while the perturbation starts collapsing, the internal pressure gradient soon steepens, halting the collapse. In the meantime, though, the external pressure is rising, tending to compress the matter and actually raising the external density above that interior to the perturbation. This lowers the pressure gradient across the fragment allowing self-gravitation to again dominate and the collapse to begin again. For a sphere that remains rigorously uniform, this behavior continues to the singularity. This behavior is sketched schematically in fig (11).

It is doubtful, under these circumstances, whether one can speak meaningfully of fragmentation occuring for gamma in this range on the basis of the linear theory despite the fact that large density excesses are acheived. This ambiguity appears to be intrinsic in the model as a result of the unrealistic behavior of the cloud collapsing to a point. In a real cloud, pressure gradients would be likely to develope. In this case, the growth of the perturbation probably depends on other factors. This point will be discussed further in part VII.

To summarize the results for the finite-pressure case, one can say the following:

1) The linearized two-point model as well as the nonlinear two-point model for small amplitudes again agrees extremely well with the hydrodynamic approach.

2) The nonlinear terms increase the growth rate relative to the linear theory and limit the lower densities a perturbation can obtain for the oscillatory solutions.

3) Just as in the linear theory, perturbations with gamma less than 4/3 grow. Those with gamma between 4/3 and 5/3 oscillate with increasing amplitude.

4) The increasing amplitude of the oscillations can be traced to the increasing external pressure forcing another stage of collapse after initial equilibrium is obtained by a fragment.

## IV - THERMODYNAMICS OF FRAGMENTATION

### A) TWO-POINT MODEL INCLUDING RADIATION

While the polytropic equation of state is useful for studying the gross features of fragmentation, it is desirable to know how good an approximation it really is.

Hoyle (1953) has made rough calculations of the thermal evolution of an optically thin gas cloud with a temperature of about $10^5$ and a particle density of $10^{-3}$. He finds that the temperature drops rapidly to about $10^4$ whereupon the reduced ability of the hydrogen to radiate causes the collapse to be isothermal. Mestel and Spitzer (1956) have suggested that for a temperature of about 100 and a particle density of 10, the collapse is again isothermal. Thus, one could describe these systems with a polytropic relation whose gamma was one.

As the density of the cloud rises increasing the optical thickness of the medium, one expects the collapse to become adiabatic so that gamma would then be 5/3. Cameron (1962) has further suggested that in the last phases of collapse to stellar densities, the ionization of the Hydrogen or the disociation of the hydrogen molecules might cause gamma to become $\lesssim 4/3$. There are thus several instances in which the thermodynamics of collapse could be described by a polytropic equation of state.

However, in none of these cases was the energy equation actually solved.

In order to see if the explicit inclusion of radiation with the dynamics of the collapse would modify the results, it was decided to again utilize the two-point model. Since in this case one is interested primarily in the thermodynamics, and in order to have an unperturbed solution, a uniform spherical collapse was again used.

The basic form of the energy equation is

$$21) \qquad \frac{dU}{dr} = \frac{P}{\rho^2}\frac{d\rho}{dt} - F + H$$

where U is the internal energy per gram, P the pressure, $\rho$ the density in grams / cc and F and H are the radiation losses from the cloud and the heat gained from external sources respectively. One can replace U by $3RT / 2\mu$ where $\mu$ is the molecular weight of the gas and R is the gas constant. With this substitution and the definition of gamma in a polytropic relation as

$$\gamma - 1 = \frac{d \ln T}{d \ln \rho}$$

one can see the connection with the polytropic equation of state. One then has

$$22) \qquad \gamma - 1 = \frac{2}{3}\left[1 - \frac{\rho(F-H)}{RT\,\frac{d\rho}{dt}}\right]$$

Note that for F = H = 0, implying no heat gains or losses, gamma is 5/3, as is expected for an adiabatic system.

It is possible to solve eq(22) simultaneously with eq(20) changing gamma as one goes. However it is just as simple to solve the energy equation in the form of eq(21) eliminating, as mentioned above, the internal energy in favor of the temperature. This is the course followed here. One then has, from eq(5) appendix I and eq(21),

$$23)\quad \ddot{\xi} + \frac{2\dot{x}\dot{\xi}}{x} - \frac{4}{3}\frac{\dot{\xi}^2}{1+\xi} = \frac{4\pi G \rho_0 \xi(1+\xi)}{x^3} + \frac{3(1+\xi)\Delta P}{\rho(1+\frac{\xi}{2})r_p^2}$$

$$24)\quad \frac{\mu}{T}\frac{d\left(\frac{T}{\mu}\right)}{dt} = \frac{2}{3}\left[\frac{d\rho}{dt} - \frac{\rho(F-H)}{R\,\frac{T}{\mu}}\right]$$

$\Delta P$ is eliminated from eq(23) by means of the perfect gas relation. One thus has two coupled equations which give the run of the temperature and density in time. $\Delta T$ and $\Delta\rho$ refer to differences between the perturbation center and the cloud background. The appearance of $\Delta T$ means that eq(24) must be solved for both the perturbation and the main flow. Hence, one should worry about differences in the molecular weight and the heating function between the perturbation and its surroundings. For simplicity however, the fragment has been assumed to start in thermal equilibrium with the surroundings, and the molecular weight has been taken as a constant. This latter assumption might be dangerous in some cases, but the additional complications in solving an ionization equation seem unwarranted in this preliminary treatment.

The radiation loss term F must also include conduction losses. Since the latter varies as $\nabla^2 T$, it will only be important, for the densities considered here ($n = 10^{-3}$ to $n = 10^{16}$), when the wavelength of the disturbance is very short or the temperature very high.

The following numerical example may be used to illustrate this. The conduction loss term in ergs / gm-sec can be written as

$$F = \frac{K \nabla^2 T}{\rho} \sim \frac{K \Delta T}{\rho (\Delta r)^2}$$

where

$$K \approx 7 \times 10^3 T^{1/2}$$

for a neutral gas (Chapman and Cowling (1958)). For an ionized gas, Spitzer (1962) gives

$$K \approx \frac{4 \times 10^{-5} T^{5/2}}{\ln \Lambda}$$

where

$$\ln \Lambda \approx - 17.4 + 1.15 \left[ 3 \log T - \log \rho \right]$$

Radiation losses can be written as

$$F = a n g(T)$$

where g(T) is some function of the temperature and n is the particle density. The value of a and g depends on the source of cooling. In appendix V, values are given for certain cases of interest.

One then obtains

$$\frac{F_{cond}}{F_{rad}} \sim 10^{-4}$$

for T = 100 and n = 10 and $\Delta r$ = 1 psc.

For $T = 10^5$ and $n = 10^{-3}$ with $\Delta r$ = 10 kpc, one finds

$$\frac{F_{cond}}{F_{rad}} \sim 10^{-2}$$

Since conduction is such a small part of the energy loss, one can simply use the radiation loss to evaluate the cooling term. (In the numerical calculations, however, conduction was included.) Thus the cooling term becomes

$$F = an g(T)$$

One can now write down the equations governing the collapse and fragmentation.

For the main flow, one has

$$\rho = \frac{\rho_o}{x^3} \quad \text{Eq. of Density Change}$$

$$\frac{d \ln T}{dt} = \frac{2}{3}\left\{\frac{d\rho}{dt} - \frac{\rho\left[a\rho\, g(T) - H\right]}{\frac{RT}{\mu}}\right\} \quad \text{Energy Eq.}$$

$$\frac{d^2x}{dt^2} = -\frac{4\pi G \rho_o}{3x^2} \quad \text{Eq. of Motion}$$

For the perturbation one has

$$\ddot{\xi} + \frac{2\dot{x}\dot{\xi}}{x} - \frac{4}{3}\frac{\dot{\xi}^2}{1+\xi} = \frac{4\pi G\rho_o\xi(1+\xi)}{x^3} + \frac{3RT(1+\xi)}{(1+\frac{\xi}{2})r_p^2}\left[\frac{\Delta\rho}{\rho} + \frac{\Delta T}{T}\right]$$

$$\frac{d\ln T_p}{dt} = \frac{2}{3}\left\{\frac{d\rho_p}{dt} - \frac{\mu\rho_p}{RT_p}\left[a\rho_p g(T_p) - H\right]\right\}$$

where the subscript "p" refers to the perturbation.

As before it is convenient to introduce dimensionless variables. If one defines $T_p = T(1+\theta)$ and $T = T_o\delta$, then one can write

25) a) $$\ddot{x} = -\frac{Q}{x^2}$$

b) $$\rho = \frac{\rho_o}{x^3}$$

c) $$\dot{\delta} = \frac{\mu g(\delta)}{x^3} + \frac{\mu}{\rho_o} - \frac{2\dot{x}\delta}{\rho}$$

d) $$\ddot{\xi} + \frac{2\dot{x}\dot{\xi}}{x} - \frac{4}{3}\frac{\dot{\xi}^2}{1+\xi} = \frac{3a(1+\xi)}{x^3}\left\{\xi - \frac{\lambda_o^2\xi x}{y^2}\left[\delta + \xi(1+\theta)\right]\right\}$$

e) $$\dot{\theta} = \frac{1}{\delta x^3}\left\{\mu_p(1+\xi)g(\theta) - \mu(1+\theta)g(\delta)\right\} - \frac{\mu H(1+\theta)}{\delta\rho_o} + \frac{\mu_p H_p}{\delta\rho_o} + \frac{2\xi}{3}\frac{1+\theta}{1+\xi}$$

where $Q = \dfrac{3\pi GRT_o^2}{a^2\rho_o}$.

Q is proportional to the square of the ratio of the free-fall time to the cooling time.

## B) INITIAL CONDITIONS AND COOLING FUNCTIONS

The cooling functions depend on the environment through the temperature, the density and the composition. One must therefore decide what are reasonable initial conditions. Following Spitzer (1963), one may suppose that the formation of population I stars occurs in regions where the temperature is about 100 and the particle density is near 10. One further takes the abundances to be roughly solar. For galaxy formation, in the absence of any observational evidence, one can assume the conditions suggested by Hoyle (1953) and take the temperature as between $10^4$ and $10^6$ with the particle density as $10^{-3}$.

With the inital conditions defined, the expressions used are given in appendix V. They may be divided, though, into the two following regimes:

1) T greater than $10^4$. Pure hydrogen. Cooling by bound-bound, bound-free, and free-free emission . . . Michie (1963).

2) T less than $10^3$. Population I composition.

   a) Cooling by collisional excitation of Si, Fe, C, etc. Electrons supplied by cosmic ray ionization of H. Unno and Simoda (1963).

   b) Molecular hydrogen cooling . . . Gould (1964).

One can compute cooling times from

$$\tau_{\text{cooling}} \sim \frac{RT}{F}$$

where F is given in appendix V. Comparing these with the free-fall time

$$\tau_{ff} = (\tfrac{4}{3}\pi G \rho_0)^{-1/2}$$

one sees that $\tau_{ff} >> \tau_{\text{cooling}}$ unless T is very large ($10^5$) or n is very small ($10^{-4}$). The short cooling times present a difficulty. If radiation is so effective, what maintains the high observed temperatures of the medium? It has usually been assumed that there is an input of heat due to turbulence, cosmic rays or cloud-cloud collisions. This then gives rise to initial thermal equilibrium.

Objections may be raised about assuming a constant heat input. One might imagine that as the cloud contracts the efficiency of cloud collisions would diminish and that cosmic ray heating would be confined to the surface. (See Hatanaka, Unno and Takebee (1961)) Thus the constant heat input tends to over estimate the heating. Since the heating in turn acts to hinder collapse, one underestimates the growth.

## C) DISCUSSION OF THE TWO-POINT EQUATIONS

An examination of eq(25a) through (25e) shows that the following quantities enter: the excess density, the initial density and the density of the main flow; the excess temperature, the initial temperature and the temperature of the main flow; the heating functions and the molecular weights for both the main flow and the perturbation; the Jeans' length and the size of the perturbation; and finally, the cooling functions, the cloud radius and the time.

The number of parameters actually influencing the equations, though, is much reduced when one recalls the assumptions that have been made. Once the initial values of the temperature, density and the molecular weight are specified, the heating and cooling functions are determined. The excess temperature is assumed zero since one postulates thermal equilibrium. The cloud radius is chosen as ten times the size of the perturbation. The parameters that enter are thus reduced to the initial excess density, the ratio of the Jeans' length to the initial size of the perturbation, and the initial values of the main cloud temperature, density and composition. As in the finite pressure case, the excess density is chosen as .1 and the ratio of the Jeans' length to the perturbation size is picked as 1.5. Thus only the temperature, density and composition appear. The composition is only required to specify the radiation law and the molecular weight and hence is really specified by the choice of environment. Furthermore, the density appears only in the expression for Q and in combination with the heating functions. It is thus necessary to specify only one more parameter than is used in the polytropic relations, as one would expect, since a polytropic equation of state relates the temperature and the density.

D) RESULTS OF THE NUMERICAL INTEGRATIONS

It is now possible to integrate eq(25a) through (25e). The time steps were chosen as .01 times the smaller of the thermal and mechanical time scale. This usually meant that the thermal time scale set the step length. The inclusion of conduction means that the results depend on the linear scale of the system, as was mentioned earlier. In order to check the importance of conduction, values were chosen for the scale which were hoped to be reasonable for the environment considered. Thus, for an HI cloud, 30 psc was chosen, while for the pregalactic cloud, a distance of 1Mpc was taken. The resulting masses of the perturbations were then chosen as 10 M☉ and $10^9$ M☉ respectively.

The results of the integrations are shown in fig(12) and fig(13), in which the perturbation temperature is plotted against the fractional radius of the cloud. One sees that for an HI region ionic cooling leads to a substantial drop in temperature as collapse occurs. Molecular hydrogen produces a more nearly isothermal result. This is directly a result of the cooling laws. Since the ionic cooling varies as $T^{3.32}$ while the molecular hydrogen cooling varies as about $T^{21.8}$, a slight drop in T in the former case causes very little decrease in the amount of radiation. In the latter case, however, a slight drop in T cuts off virtually all radiation.

A further point worth mentioning is that the increased density in the perturbation causes a slightly greater cooling than occurs in the main cloud. Thus a temperature difference builds up with the hotter gas in the main flow able to compress the perturbation.

One can see from fig(12) that for temperatures about $15 \times 10^3$, a gradual drop in T occurs to about $10^4$. For temperatures nearer $10^5$, the effect of the longer thermal time scale means that it takes a significant portion of the free fall time to achieve the $10^4$ level.

The fact that the temperature does drop in collapse suggests that the isothermal assumption actually will underestimate the growth rate as the dropping temperature implies a gamma less than 1. It has not been feasible to follow the density to the point where opacity becomes important and hence no firm conclusions can be reached about the value of gamma appropriate for this region.

In summary one can say the following:

1) If one approximates the thermodynamics of collapse by a polytropic relation, gamma should be set less than or equal to one.

2) For lower temperature clouds, Molecular hydrogen will maintain a more nearly isothermal condition than ionic cooling.

3) The lower temperature in the perturbation may allow the external pressure to play an important role in accelerating collapse of fragments, as pointed out by Michie (1963) and McNally (1964).

## V - THE EFFECT OF ROTATION UPON THE DEVELOPMENT OF FRAGMENTS IN A COLLAPSING CLOUD

### A) PRELIMINARY ARGUMENTS ABOUT THE IMPORTANCE OF ROTATION

The arguments against fragmentation raised by Layzer (1963) have been mentioned in part I. These objections center about the importance of angular momentum as a force retarding collapse. In the study of fragmentation with the two-point model, rotation has been ignored up to this time. To see why rotation may be important in preventing fragmentation, Layzer's arguments are sketched below.

A perturbation of mass m in a contracting gas cloud of mass M will possess a peculiar velocity v due to the interaction with its neighbors. This is roughly given by

$$v^2 = \frac{Gm}{L}$$

where L is a dimension characteristic of the separation between fragments and hence is also a measure of the volume out of which they form. The distance to which a perturbation can approach the cloud center, R, is determined by its energy and angular momentum. The former condition yields

$$v_{max}^2 = \frac{GM}{R} .$$

The latter yields

$$vr = v_{max} R$$

where $v_{max}$ is the velocity at closest approach and r is the initial distance of the perturbation from the cloud center.

Combining these two results one finds

$$\frac{r}{R} = \frac{ML}{(mr)} .$$

Now $M/m = N = r^3/L^3$ = the number of fragments. Thus

$$\frac{\rho_{max}(\text{cloud})}{\rho_o(\text{cloud})} = N^2 .$$

Since N is expected to be fairly large, one expects a large increase in the cloud density.

Turning now to the perturbations, if one says that contraction along the equator ceases when the centrifugal force equals the gravitational force,

$$\frac{Gm}{r_p^2} = \frac{v_{rot}^2}{r_p} ,$$

where $r_p$ is the radius of a perturbation, and $v_{rot}$ is its tangential velocity. This expression can be written in terms of the density as

$$\frac{\rho_{final}}{\rho_{initial}} \doteq \left(\frac{F_{grav}}{F_{rot}}\right)^3$$

where the F's are the initial values of the rotational and gravitational forces. Layzer now shows in a rough way that one might expect the two forces to be about equal and hence that the density ratio will not change. He points out that while collapse can still occur along the rotation axis, the limited growth that one finds for the fragments suggests that they will be obliterated by the rising background density. Thus, even if fragments do separate out initially, they will be subsequently destroyed.

Hunter (1964) has also analyzed the problem. He concludes that unless the rotational part of the velocity field is large compared to the irrotational part, or unless the initial perturbation amplitude is very small, rotation will not influence seriously the growth rate until the density of the perturbation is large compared to the background. However, these conditions are exactly those that Layzer is unwilling to grant. A further difficulty with Hunter's analysis is that it is based on the linearized theory. As has been mentioned before, rotation is intrinsically nonlinear due to the $v^2/r$ term. One must therefore be careful in extending any linear theory to include it.

Since the two-point model is not limited by the assumptions of linearity, it has been modified to include rotational effects in the hope of understanding the difference in the results of Hunter and Layzer.

## B) THE TWO-POINT MODEL WITH ROTATION

It was mentioned earlier that even the two-point approach requires an unperturbed solution for the main flow. For this reason a uniform collapse of the main cloud has again been assumed, but a term has been added to limit the collapse. This term may be considered an artificial angular momentum. Thus, while one pictures the individual fragments as possessing orbital angular momentum, one assumes that the angular momentum vectors are distributed randomly so that their average over the entire cloud is zero. Thus the cloud itself does not rotate.

The perturbation is again assumed to be spheroidal with its axis of symmetry coinciding with a radius vector of the cloud. The equatorial cross-sections are assumed circular, and the perturbations are assumed not to interact with one another except through the potential of the entire cloud. This bypasses Layzer's initial assumption that the perturbations interact. The effect that perturbations have on one another can be imitated, though, by assigning a slightly higher initial spin than one otherwise would.

While it is not obvious that any configuration will be able to collapse uniformly to a nonzero radius and then re-expand maintaining its uniformity, such a collapse is postulated. Since it was shown in part II that only gross departures from uniformity seriously affect the growth rates, this assumption should not vitiate the results.

If one assumes then a uniform collapse of the main cloud to a finite radius and that the perturbations rotate as described above, one can derive the equations for the two-point model. This is done in appendix IV. However, since the gravitational field of the main cloud is unimportant due to the assumption of uniformity, it is possible to give a derivation which shows more clearly the physical picture.

Treating the perturbation as an isolated unit, one can simply add a centrifugal force in the equatorial direction and pressure gradients to eq(10) and (11). One thus has

$$\ddot{r}_p = -\frac{Gm_p f(e)}{r_p^2} + \omega_p^2 r_p - \frac{1}{\rho}\frac{dP}{dr} \qquad \text{for the equator}$$

$$\ddot{s}_p = -\frac{Gm_p g(e)}{s_p^2} - \frac{1}{\rho}\frac{dP}{ds} \qquad \text{for the pole .}$$

Since it is desirable to measure density excesses, one can convert again to dimensionless variables. The collapse of the main cloud of radius R is now modified as well so that collapse is halted at a nonzero radius. Thus

$$\ddot{R} = -\frac{GM}{R^2} + \omega_o^2 R$$

and

$$\rho_{\text{cloud}} = \rho_o\left(\frac{R^o}{R}\right)^3 \equiv \frac{\rho_o}{v^3}$$

While reasons have been given by Hoyle and Fowler (1963) and others for assuming a constant angular velocity, the more stringent requirement of constant angular momentum is chosen.

If one now uses R as the unit of length and writes

$$y = \frac{r_p}{R} \qquad \text{and} \qquad z = \frac{s_p}{R}$$

the two-point equations become

$$26)\quad \ddot{y} + \frac{2\dot{y}v}{v} = -\frac{y}{v^4}\left[B_o - \frac{\nu_o^2 y_o^4}{y^4}\right] - \frac{y\xi}{v^3}\left[f(e) - \frac{\lambda_o^2}{3y_o^2}\right]$$

$$27)\quad \ddot{z} + \frac{2\dot{z}v}{v} = -\frac{zB_o}{v^4} - \frac{z\xi}{v^3}\left[g(e) - \frac{\lambda_o^2}{3z_o^2}\right]$$

$$28)\quad \ddot{v} = -\frac{1}{v^2} + \frac{B_o}{v^3}$$

where $B_o = \omega_c^{o\,2}\tau_o^2 + \nu_o^2 = \omega_p^{o\,2}\tau_o^2$ .

$\omega_c^o$ and $\omega_p^o$ are the angular velocities of the perturbation about the cloud center and about its own axis of rotation respectively. The terms f(e) and g(e) are the corrections to the potential required due to the oblateness of the perturbation. A polytropic equation of state has been used with $\gamma = 1$, as was suggested by the results of part IV. $B_o$ and $\nu_o^2$ are the ratio of the revolution period and rotation period to the free fall time squared. If these quantities are small, angular momentum is unimportant.

A comparison with eq(20) shows that two new terms have entered the equations. The first, involving $B_o$, enters through eq(28) when one converts to dimensionless variables. It is a consequence of the fact that the density of the perturbation is referred to the background density, which in turn is a function of $B_o$ through the latter's effect upon the collapse. The second term, involving $\nu_o$, is the centrifugal force due to the rotation of the fragment.

## C) RESULTS FOR THE TWO-POINT MODEL WITH ROTATION

It is interesting to solve the two equations (26) and (27) be means of a power series expansion. Since the excess density is

$$\xi = \frac{y_o^3}{y^2 z}(1 + \xi_o) - 1$$

one obtains, assuming zero initial velocities and an initially spherical perturbation,

$$\text{28)} \quad \xi = \xi_o \left\{ 1 + \frac{3}{2}(1 + \xi_o)\left[\frac{B_o}{\xi_o} + 1 - \frac{\lambda_o^2}{3y_o^2} - \frac{2\nu_o^2}{3\,\xi_o}\right]\tau^2 \right\}$$

where $\lambda_o$ is the Jeans' length, $y_o$ is the initial perturbation size, and $\tau$ is the time in units of the unimpeded free fall time. v has been eliminated by another power series expansion.

One can now consider Layzer's model. He takes $B_o \ll 1$, $\xi_o \ll 1$, $\lambda_o = 0$, and $\nu_o^2 = 1$. This corresponds to small initial random velocities, small initial excess densities, zero pressure, and the self-gravitation just balanced by the rotation. One sees that the excess density decreases as he predicts. The case Hunter considers is essentially $B_o \ll 1$, $\lambda_o = 0$, $\xi_o = .1$, and $\nu_o^2 \ll 1$. Thus the excess density increases in time. The reason for the difference between their results is then just a consequence of the difference in initial conditions.

In order to see in more detail how rotation affects the solution, the full equations (26) and (27) were integrated numerically. One set of integrations was done with the pressure rigorously zero. The other was with a finite pressure and $\lambda_J^2/\lambda_p^2 = 1.5$, as in parts III and IV. The initial excess densities and ratios of spin to gravitational forces were varied. The results are plotted in fig(14) and fig(15).

One sees that the pressure term greatly retards growth, in fact, preventing it in some cases where it was possible for the zero pressure case. As is to be expected, increasing the rotation retards the growth. On the other hand, increasing the initial density excess enhances the growth.

The important point to notice is that rotation does not necessarily prevent a fragment from separating out. In setting the rotational forces nearly equal to the gravitational forces for the perturbation, but not for the cloud as a whole, one puts perturbations at a great disadvantage. Regardless, though, of the spin in the fragment, by making the excess density sufficiently large, rotation can always be made a small effect.

One might think that while rotation would impede collapse perpendicular to the axis of rotation, collapse along the axis would still be possible, allowing the density to rise. Hence, just as in the tidal case,although there is a disrupting force, it acts only in one plane. There is a difference however between the two cases. Tidal forces affect the background not only in the perturbation but all around it. Rotational forces, in acting locally, retard not only the collapse of the excess mass, but the background as well. Hence, eventually, even the background density in the perturbation is unable to increase, in contrast to the case of tidal disruption. Thus, since the surrounding density can continue to increase, the excess density is absorbed by the background. Only if collapse along the rotation axis can compensate for both the lowered rate of growth of the background relative to the surroundings as well as the collapse of the excess mass, can the perturbation grow.

D) ROTATION IN ASTROPHYSICAL SYSTEMS

On the basis of this model, it has been shown that rotation can prevent the formation of condensations in a collapsing system if the rotational forces are of the same order of magnitude as the self-gravitational forces. Whether this will affect the formation of stars or galaxies depends on the value of these forces. Since there is no information regarding the motions in the pregalactic medium, only the problem of star formation can be approached here. One is thus led to ask what values of $\nu^2$, the parameter measuring the ratio of spin and gravitational forces, are expected in the present galaxy. Since the Milky Way, in being a flattened system, probably lies in the group of systems with higher than average angular momentum per unit mass, the value of $\nu^2$ found from it should be in the nature of an upper limit.

One should point out that stars do have ways of disposing of angular momentum. Observationally one finds that the sun has only a small percent of the total angular momentum of the system.

If one assumes that multiple stars form from a common envelope, an exceedingly large amount of angular momentum can be deposited in orbital motion of companions. Mention has already been made of the suggestion of Hoyle and Fowler (1963) whereby the magnetic field can couple the contracting star to the medium about it. Limber (1964) has studied the shedding of matter at the equator due to rotational instability. Such a mechanism can carry away angular momentum, though at the expense of mass being lost to the system. All of these ideas, however, begin with the star as an isolated object. The crucial problem from the point of view of fragmentation is whether one can ever achieve this isolated state in the first place. In view of the high percentage of the stars which are multiple, it might be interesting to look at, in more detail, the remark made above that orbital motions are potentially capable of absorbing a larger amount of angular momentum.

Consider, therefore, a cloud of radius $r_o$ and mass $M_o$ that exceeds the Jeans' mass and hence begins collapsing. Assume that it has some angular momentum so that collapse perpendicular to the rotation axis will eventually be halted when the radius is $r_c$. Let $x_o = r_c/r$. Then, by the time collapse stops, the density will have increased by $1/x^3$ and, hence, by eq(2), page 1, the critical mass $M_c$ will be diminished in the fashion $M_c \sim M_o x_o^{3/2}$ . Therefore subunits of the cloud will also be unstable in the Jeans' sense. One should note that the instability proposed here is not one of rotational fission, but of gravitational fragmentation. This point is discussed briefly by Lynden-Bell (1964).

Since the main cloud is rotating, the subunits might also be expected to possess angular momentum. Whether they will be able to collapse, even if they do satisfy the Jeans' condition, thus depends on the ratio of rotational to gravitational forces which has been denoted in the previous section as $\nu^2$.

In order to estimate $\nu^2$, two different approaches are used. One involves the distribution of angular momentum. The other considers in a rough way the mode of break-up that might be expected in a rotating disc.

One can write the orbital angular momentum $L_o$ of a rotating uniform sphere of mass M and radius R as

$$L_o = \frac{2MR^2\Omega}{5} .$$

where $\Omega$ is the angular velocity. Suppose it splits into N pieces of mass $m = M/N$. Further suppose that the angular momentum is distributed equally among the N pieces. Each piece will then possess a spin angular momentum S and an orbital angular momentum L. Thus

$$L_o = N(L+S) .$$

Assume the fragments may be treated as spheres of radius r moving about the original axis of the cloud at a distance $\langle R \rangle$. Then

$$S = \frac{2mr^2\omega_S}{5}$$

$$L = m\langle R\rangle^2\omega_L$$

where $\omega_S$ and $\omega_L$ are the spin and orbital angular velocities respectively. Substituting in now for L and S in the expression for $L_o$,

$$L_o = \frac{2MR^2\Omega}{5} = N\left(\frac{2mr^2\omega_S}{5} + m\langle R\rangle^2\omega_L\right) \quad \text{or}$$

$$\frac{2R^2\Omega}{5} = \frac{2r^2\omega_S}{5} + \langle R\rangle^2\omega_L \ .$$

Now $r \sim R/N^{1/3} << R$ and $\langle R\rangle < R$ since the average distance of a fragment from the cloud center will be less than the total radius of the original cloud. Hence, it is consistent with the conservation of angular momentum to set $\omega_S = \omega_L = \Omega$ as would be the case for uniform rotation.

Suppose now the cloud has contracted by the amount $x_o$ suggested above, conserving angular momentum. Then it will have a new angular velocity given by

$$\Omega = \frac{\Omega_o}{x_o^2} \ .$$

Consider now the ratio of the self-gravitation and the rotation forces, $\nu^2 = \omega^2\tau^2$ as defined in eq(28). It has already been shown that if the cloud contracts, the increased density can make subunits of the cloud unstable in the Jeans' sense. The question is now asked, if the ratio of gravitational to rotational forces in the original cloud is $\nu_o^2$, what is the ratio $\nu^2$ in the subunit? In other words, does fragmentation raise or lower $\nu^2$, corresponding to increasing or decreasing the importance of rotation? If one assumes that $\omega_s = \Omega$, then

$$\nu^2 = \omega^2 \tau^2$$

$$= \Omega^2 \tau^2$$

$$= \frac{\Omega_o^2 \tau^2}{x_o^4} \; .$$

Now

$$\tau^2 = \frac{1}{\frac{4}{3}\pi G \rho}$$

$$= \frac{1}{\frac{4}{3}\pi \frac{G\rho_o}{x_o^3}(1+\varepsilon)}$$

$$= \frac{\tau_o^2 x_o^3}{1+\varepsilon} \; .$$

Therefore

$$\nu^2 = \frac{\Omega_o^2 \tau_o^2}{x_o(1+\varepsilon)}$$

$$= \frac{\nu_o^2}{x_o(1+\varepsilon)} \; .$$

Thus, if the amount of contraction is small before rotation halts the collapse, then $\mathcal{E}$ will not in general be able to obtain a large amplitude and so $\nu^2 > \nu_o^2$ indicating that rotation becomes more important as subunits fragment out.

However, if $x_o$ can become sufficiently small that the nonlinear effects become important, as suggested by Hunter, and found in section C for certain cases, $\mathcal{E}$ can become large. In this case, assuming that $\mathcal{E} \sim \mathcal{E}_o/x^{3\beta}$ where $\beta$ is a factor to allow for the retardation due to rotation and $\beta = .5$ for an isothermal collapse with no rotation,

$$\nu^2 = \nu_o^2 x_o^{3\beta-1} .$$

Hence, in some cases, $\nu^2$ could actually decrease, suggesting that rotation could become less important as subunits fragment. While it would be rash to state that rotation will not affect the growth and subfragmentation of perturbations, the above calculation does suggest that under certain circumstances spin angular momentum can be disposed of through the conversion to orbital angular momentum. This would appear to be possible, though, only if the rotational forces are initially small ($\nu_o^2 << 1$).

A second method for roughly estimating the evolutions of a rotating system is sketched below.

Consider a disc in nonuniform rotation so that $\omega = \omega_o f(r)$, $f(r) = 1$ for uniform rotation and decreases with r if the internal portion of the disc rotates more rapidly than the outer part and in the same sense. If one now cuts a circular piece of radius $\Delta r$ out of the disc at a distance r from center, it is possible to estimate the spin angular velocity. If the disc rotates as a rigid body, the spin and orbital rotations are synchronous. Thus $\omega_S = \omega_L$. However, the differential rotation adds a component to the spin amounting to the difference in angular velocity across the fragment. Thus

$$\omega_S(r) = \omega_L(r) + \frac{d\omega}{dr} \cdot \Delta r = \omega_L(r) + \omega_o \frac{df}{dr} \cdot \Delta r \ .$$

For uniform rotation, df/dr = 0 , and hence $\omega_S = \omega_L$ . If the center is rotating more rapidly than the periphery, then df/dr $< 0$ and hence $\omega_S < \omega_L$ . Hence, in this simple picture the spin angular velocity could actually decrease as fragmentation occurs. This would mean that rotational forces would continuously diminish relative to gravitational forces as long as fragmentation proceeds. One might thus conclude that in a fragmenting and rotating cloud, $\nu_p^2 \leqq \nu_o^2$ .

One now asks what are reasonable values for $\nu_o^2$ in the galaxy? If one assumes that the gross rotation of the cloud is produced by galactic shear forces, one can see that

$$\omega_o^2 = \frac{(\delta v)^2}{r_o^2} \sim A^2$$

where $\delta\nu$ is the change in the rotational velocity per unit distance and A is the Oort constant. $r_o$ is the radius of the cloud. While there is really little reason why this should be representative of the actual velocities encountered, it is used as an order of magnitude estimate.

Consider now what size cloud is unstable. For n = 10 particles / cc and T = 100, $M_{critical} = 2 \times 10^3$ M☉ and $R_{critical}$ = 10 psc. Since one wants at least two subunits to be unstable, M must be about $4 \times 10^3$ M☉ . Taking the Oort constant as 15 km / sec-kpc, one thus has

$$\nu_o^2 \sim \omega_o^2 r_o^2 = \left(\frac{1.5 \times 10^6}{3 \times 10^{21}}\right)^2 \times \left(\frac{1.5 \times 10^{15}}{n}\right)^2 \sim \frac{1}{15}$$

Now the critical condition for growth as deduced from eq(28) is

$$1 + \frac{B_p}{\mathcal{E}_o} - \frac{\lambda_o^2}{3\lambda_p^2} - \frac{2\nu_p^2}{3\mathcal{E}_o} > 0$$

where $B_p$ and $\nu_p$ refer to the perturbation. Now $B_p \sim \omega_L^2 T^2 \sim \omega^2 T^2 \sim \nu_o^2$ and $\nu_p \sim \nu_o$ as was shown above. Thus

$$1 + \frac{\nu_o^2}{3\mathcal{E}_o} - \frac{\lambda_o^2}{3\lambda_p^2} > 0 .$$

Inserting values for $\lambda_o^2 / \lambda_p^2 = 1.5$ and $\mathcal{E}_o = .1$ , one sees that the condition for growth is satisfied. It should be noted, though, that if $B_p < \frac{2}{3}\nu_p^2$ corresponding to the orbital angular velocity being substantially lower than the spin velocity the condition is not met. This could occur if the rotational energy is preferentially transferred to the spin component.

The calculations above are all very rough and hence the conclusions drawn are open to dispute. The reason that any calculations are included at all is to try to make plausible that at least under one set of initial conditions, disposal of angular momentum in the initial stages of fragmentation is not necessarily an insurmountable obstacle.

In summary one can say the following:

1) Rotation can prevent fragmentation. However, it will be expected to play an important role only if the ratio of the rotation period to the free fall time in the absence of rotation is small.
2) Unless the spin of the fragment is considerably larger than that expected if galactic shear determines the rotation, fragments should be able to collapse and achieve large excess densities.
3) The above conclusions have assumed angular momentum is conserved during collapse. Hence, while it would be helpful if angular momentum could be removed, it does not appear to be essential in achieving the formation of isolated objects with densities much larger than the background medium.

## VI - COLLISIONS BETWEEN FRAGMENTS AS A POSSIBLE SOURCE OF DESTRUCTION

### A) CALCULATIONS OF THE MEAN-FREE-PATH OF A FRAGMENT

While fragmentation can occur in collapsing systems, the survival of the pieces may pose difficulties. If, artifically, the cloud is assumed to have rigorously no angular momentum, then the perturbations can readily separate out, as shown in the previous sections. However, since then the main cloud ultimately collapses to a point, all fragments must be destroyed by collisions with one another at the singularity. If one considers cases with nonzero angular momentum, this singularity does not occur, and hence the destruction of fragments is not certain. One is then faced, however, with the crowding of the fragments into a small volume which might be thought to lead to their subsequent destruction. On the other hand, if one increases the orbital angular momentum to limit still more the collapse of the main cloud, another difficulty occurs. If the orbital angular momentum is increased, there is no reason why the spin angular momentum should not also be increased. Greater spin forces, though, will either prevent the growth of fragments outright, or retard the growth. The latter will increase the cross section with the consequence of greater collisional destruction. This is essentially the difficulty raised by Layzer (1963), though in a slightly different guise.

One simple way of estimating the collision probability is to compute $\lambda$, the mean-free-path, as a function of time for a swarm of fragments. The simple kinetic theory of gases gives an expression for the mean-free-path in terms of the density of particles n, and the collision cross section $\phi$. Thus

$$\lambda = \frac{1}{(n\phi)} \tag{29}$$

For a fragment in a collapsing cloud of initial radius $R_o$, one can take the cross section as simply the geometrical size. Thus

$$\phi = \pi r^2$$

where r is the size of the fragment. As collapse proceeds, n increases and $\phi$ decreases since the perturbation is contracting. If one assumes that the perturbation collapses as in free fall, then its density goes asymptotically as $x^{-4.5}$, for isothermal collapse. Hence its radius decreases as $x^{1.5}$. Furthermore, the density of fragments per unit volume goes as $x^{-3}$. Substituting these values into eq(29), one sees that

$$\lambda = \frac{1}{\frac{n_o}{R_o^3 x^3} \pi R_o^2 x^3} = \text{constant} .$$

Since the size of the cloud is actually decreasing all the while, the mean-free-path relative to the cloud size increases in time!

While this expression for r assumes rotation is unimportant and is hence open to question in this application, the result suggests that collisions play only a moderate role in fragment destruction.

In order to make a more refined analysis, the mean-free-path is computed as a function of time. One can then find the number of surviving fragments by integrating over the collapse period. One can thus write

$$31) \qquad \frac{dn}{dt} = -N\phi v_{rel}$$

where $N = n/V$, $V$ is the volume of the cloud, $n$ the total number of fragments, and $v_{rel}$ is the relative velocity between fragments. Substituting $V = 4\pi R^3/3$ and $\phi$ from eq(3), one can integrate eq(31) and obtain

$$n(t) = n(t_o)\, e^{-\frac{3}{4}\int_o^t \frac{v_{rel}\, r^2\, dt}{v^3 R_o^3}}$$

If dimensionless variables are introduced with $y = r/R$ and $R = vR_o$, one obtains

$$32) \qquad n(t) = n_o e^{-\frac{3}{4}\int_o^{\tau} \dot{x}_{rel} \frac{y^2}{v} dt}$$

Some questions might arise about the validity of the mean-free-path theory in this context as it is normally used when one is dealing with isolated particles. The justification for its application to this problem lies in the fact that the linear theory shows that fragments tend toward relatively dense mass centers. Furthermore, the linear theory includes implicitly tidal and other forms of disruption that might occur due to interaction among the fragments. Thus it is only in the nonlinear regime that one need worry about such interactions and it is here that the problem is best represented as an "N-body" one.

In order to make the problem more tractable, a number of simplifying assumptions have been made. The first is that the size of the perturbation can be written as

$$y = y_o v^{\beta}$$

where $\beta = .5$ for a nonrotating, isothermal collapse and $\beta = 0$ if there is no growth. In order to have an analytic expression for the velocity and density in the main cloud, a limited uniform collapse such as described in section B, part V has again been used. One further assumption has been made in that the perturbations are taken to interact only through the potential of the whole cloud. Thus the effect of encounters has been ignored. To justify this assumption, the increase in internal energy of a perturbation due to an encounter is computed on the basis of the theory developed by Spitzer (1958) for studying the disruption of the clusters.

Let $\delta U_T$ be the increase in internal energy of a fragment as the result of a tidal encounter. Let the perturbing object pass the fragment at a distance p with a mass M and a velocity V. Then if m and r are the mass and radius of the fragment,

$$\delta U_T = \frac{4\pi G^2 M^2 r^2 m}{3V^2 p^4} .$$

If one compares this with the energy increase expected from a physical collision, $\delta U_C = mV^2/2$, one obtains

$$\frac{\delta U_T}{\delta U_C} \doteq \frac{G^2 M^2 r^2}{V^4 p^4} .$$

V can be chosen in at least two ways. Following Layzer (1963), $V^2 = V_I^2 = GM/p$ where the subscript "I" stands for interactions among the neighboring particles. On the other hand, V could be chosen as the free-fall velocity, in which case $V^2 = V_{ff}^2 = GM_{cloud}/r_{cloud}$ . If the mass and radius of the cloud are eliminated in terms of the perturbation mass and radius, and if there are N fragments, $V^2 = GMN^{2/3}/r$. Thus

$$\frac{\delta U_T}{\delta U_C} = \frac{r^2}{p^4} \begin{cases} p^2 & \text{if } V = V_I \\ \dfrac{r^2}{N^{4/3}} & \text{if } V = V_{ff} \end{cases}$$

Now even if r is taken as equal to p, certainly an overestimate of r, one sees that the ratio $\delta U_T/\delta U_C$ is less than one regardless of which velocity is chosen. Hence, tidal disruption is likely to be no more important than straight collisions. Thus tidal encounters are ignored.

One can now write down the relevant equations of motion for the system. Letting R be the distance of the fragment from the cloud center and $\theta$ its angular coordinate, one then has

$$\ddot{R} = -\frac{GM(R)}{R^2} + R^2\theta^2$$

$$= -\tfrac{4}{3}\pi G\bar{\rho}R + R^2\theta^2$$

$$\dot{\theta} = \frac{L_o}{mR^2}$$

where $L_o$ is the initial orbital angular momentum of the fragment, m its mass, and $\bar{\rho}$ the mean density of the cloud. For the cloud itself, one can again use eq(28) part V and thus obtain

$$\ddot{r}_c = -\tfrac{4}{3}\pi G\bar{\rho}r_c + r_c^2\theta^2$$

Introducing the dimensionless variables used before, $\bar{\rho} = \rho_o/\nu^3$, $x_c = R_c/r_o$, and assuming conservation of angular momentum, one obtains

33a) $$\ddot{x} = -\frac{x}{\nu^2} + \frac{c}{x^3}$$ for the perturbation

b) $$\dot{\theta} = \frac{c^{1/2}}{x^2}$$

34a) $$\ddot{\nu} = -\frac{1}{\nu^2} + \frac{c}{\nu^3}$$

b) $$\bar{\rho} = \frac{\rho_o}{\nu^3}$$ for the cloud

c) $$c = \frac{L_o^2}{r_o^4 m^2} \cdot \frac{1}{\frac{4}{3}\pi G\rho_o} .$$

To evaluate the relative velocity, one can write

$$\vec{V}_{rel} = \vec{V}_p - \vec{V}_L$$

where $\vec{V}_p$ is the velocity of the perturbation and $\vec{V}_L$ is the velocity of the surroundings measured relative to the center of the cloud. Since the collapse is uniform, $V_L$ is proportional to the radial distance from the cloud center. Thus $V_L = V_o r/r_o$, where $V_o$ is assumed purely radial. If $\theta$ is the angle between the direction of the perturbation velocity vector and that of a cloud radius vector, then

$$35) \quad V_{rel} = \left[(V_p - \chi V_o \cos\theta)^2 + R^2\theta^2\right]^{1/2}$$

The solution of eq(33) through (34) suffices then to determine $V_{rel}$ as a function of C and the position of the perturbation.

One can now return to eq(32) for $n(\tau)$ and since $\dot{x}_{rel}$ and y are known functions of time, a solution is possible.

## B) CALCULATION OF THE NUMBER OF SURVIVING FRAGMENTS

Before proceeding to a more detailed numerical calculation, a rough idea of the results may be obtained if one uses for the relative velocity the free-fall velocity times the average value of the ratio of the peculiar to the free-fall velocity and for the radius the expression introduced before. Since the free-fall velocity is almost certainly an overestimate of the relative velocities, these approximations should tend to overestimate the effects of collisions.

The system will be followed for two free-fall times corresponding to a collapse and a re-expansion. Beyond this point, perturbations that are going to grow should have achieved such a small radius that they will not be influenced by external forces. Furthermore, mixing of the orbits should have begun to have destroyed the radial characteristics of the collapse and the uniformity of the cloud.

If the time is measured in units of the free-fall time, $\mathcal{T}$, then

$$n(\mathcal{T}) = n_o e^{-\frac{3}{4}\int_o^2 y_o^2 v^{2\beta-1} \frac{dv}{dt}\left\langle\frac{v_{rel}}{v_{ff}}\right\rangle dt}$$

$$= n_o e^{-\frac{3}{4} y_o^2 \left\langle\frac{v_{rel}}{v_{ff}}\right\rangle 2\int_o^1 v^{2\beta-1}\, dv}$$

$$36) \qquad = n_o e^{-\frac{3}{4\beta}\cdot\left(\frac{r_p^o}{r_c^o}\right)^2 \cdot \left\langle\frac{v_{rel}}{v_{ff}}\right\rangle}$$

The factor of two in front of the integral sign arises as a result of changing the variable of integration from $\mathcal{T}$ to $\nu$. One must therefore integrate over collapse and re-expansion. One should note how sensitive the resulting expression for n is to the relative initial sizes of the perturbations and the cloud. Even if rotation is important, unless very extreme retardation of growth occurs, ($\beta$ very small) changes in the initial sizes of the perturbation will have much more influence on the number of surviving pieces. It is interesting to note that fragments that are slowly rotating are favored over more rapidly rotating ones because they will have a smaller $\beta$. Another point is that if the relative velocities are decreased, fewer fragments will be destroyed.

It is worth emphasizing that while very small peculiar velocities can prevent the main cloud from collapsing to a point, they seem to have very little effect on the growth of the perturbations. On the other hand, if the peculiar velocities are made very large, while the collapse of the whole cloud is essentially prevented, they also increase enormously the destruction of fragments. Thus lower velocities enhance the ability of a fragment to separate out from the main flow, and higher velocities tend to destroy it by collisions.

In order to test these conclusions, eq(32) was integrated numerically. Having seen above that the growth rate of the perturbation is not as critical as its size in determining the destruction rate, an isothermal collapse was assumed. One is thus left with the initial size of the perturbation and its orbital angular velocity as the parameters. In fig(16) the number of surviving fragments is plotted as a function of time. One sees that destruction is gradual until the last phases of collapse. Furthermore, in no case does the number of fragments drop significantly below .5. This is a result of the fact that as more and more fragments are destroyed, there are fewer ones with which the survivors can collide.

## C) DISCUSSION OF THE EFFECTS OF COLLISION BETWEEN FRAGMENTS

In order to apply these results, it is desirable to know the exact effect of collisions. Kahn (1955) has suggested that at densities typical of interstellar clouds, a collision is more likely to result in heating or break up into only two or three pieces rather than complete destruction of a fragment. If this result is extrapolated to higher densities, one might think that in those cases where collisions are important, the resultant heating would destroy incipient subfragments by raising their critical masses making them stable. On the other hand, the possibility of compression, coupled with the rapid cooling times in metal rich systems, might actually enhance fragmentation as suggested by the calculations of Field and Orzog. However, in metal poor systems, the increased thermal time scale is likely to rule out this last possibility. In those systems, collisions could seriously slow down the growth of fragments.

A rough estimate of the increase in temperature that one expects can be gotten from considerations of the amount of kinetic energy that must be dissipated. Thus $\delta T \sim v^2$, where v is the relative velocity at collision. One sees that $\delta T$ will be much larger in systems with large relative velocities. Such would probably be the case in collapsing galaxies. Furthermore, since in galaxy formation the material is likely to be nearly pure hydrogen, the thermal time scale is nearly the free-fall time scale, as was seen in part IV. Hence, if collisions do occur, perturbation growth could be prevented until the random velocities had decreased or until enough metals were present to allow more efficient cooling. Layzer (1963) has suggested that the time for the decay of turbulence is short compared to the free-fall time. Hence, once collapse of the main cloud is over, heating due to cloud-cloud collisions should rapidly drop.

It was suggested in section B that collisions were more important, other things being equal, in systems where the peculiar velocities were large. Larger peculiar velocities were furthermore associated with systems in which the perturbations had large orbital angular momenta. While the case of a rotating system was not studied, if one assumes that the rotation here too produces large peculiar velocities, one might conclude that rotating systems should be particularly prone to the destruction of fragments. Hence, the major epoch of perturbation growth, associated with star formation, might well be delayed in such a system until the magnitude of the peculiar velocities was reduced. Another possibility might be that star formation would be confined to small regions where the velocities happened to be abnormally low. It is tempting therefore to ascribe some of the differences between elliptical and spiral galaxies to an angular momentum effect which delays star formation in the latter case.

One might summarize as follows:

1) The number of surviving fragments is given roughly by

$$n = n_o e^{-\frac{3}{4}\left(\frac{r_p^o}{r_c^o}\right)^2 \left\langle\frac{v_{rel}}{v_{ff}}\right\rangle}$$

2) One might think that systems with large amounts of angular momentum would be exceptionally vulnerable to the destruction of fragments by collisions as a consequence of the larger peculiar velocities.

3) Even in highly unfavorable cases, the numerical integrations, on the basis of this model, indicate that collisions cannot be expected to destroy many more than half the fragments.

## VII - DESTRUCTION OF FRAGMENTS AS COLLAPSE CEASES

### A) THE OPACITY OF CLOUDS

As long as a collapsing cloud remains optically thin, the analysis of the proceeding six sections should describe in an approximate fashion the growth of perturbations. When the density rises so that the flow of radiation out of the material is impeded, that is the optical depth, $\tau$, becomes one, the character of the collapse is likely to be significantly altered, as pointed out by Hoyle (1953). At least two major effects can now be expected to become important. The first is that the trapping of radiation can start to heat the matter switching the gamma of the gas from 1 to 5/3 with the consequent radical change in behavior of the perturbations found in part III. The second is that the heating is likely to cause pressure gradients to become more important in the main flow as well as in the perturbation with the possibility of bringing the entire cloud into equilibrium. If equilibrium is established, then the virial theorem shows that perturbations will be stable unless they satisfy the local Jeans' condition. Hence, fragmentation is likely to stop. Some qualifications are necessary as will be discussed later. Nevertheless, the point where the cloud becomes optically thick is likely to mark an important dividing line in its evolution, and hence, it is important to know, even if roughly, when this occurs.

Let the optical deptn of gaseous mass M be written as

$$\tau = kR\rho \tag{37}$$

where k is the absorption coefficient in $cm^2/gm$, and $\rho$ is the density in gm/cc. R is a characteristic dimension of the system. R can then be eliminated in terms of M and $\rho$. If this is done, then one can write

$$\tau = \text{constant} \cdot k\rho^{2/3} M^{1/3} \qquad \text{or}$$

$$\frac{\tau}{\tau_o} = \frac{k}{k_o}\left(\frac{\rho}{\rho_o}\right)^{2/3}$$

where the subscript "o" denotes the initial value of the quantity. For the main cloud, $\rho = \rho_o/x^3$, while for the perturbation, $\rho = \frac{\rho_o}{x^3}(1+\mathcal{E})$, where $\mathcal{E}$ is the ratio of the excess density to the background density, and x is the radius of the main cloud in terms of its initial value. For an isothermal collapse $\mathcal{E} \sim 1/x^{3/2}$ and thus $\rho \sim 1/x^{9/2}$ for a perturbation.

With these expressions for $\rho$ it is possible to find $\tau$ as a function of x. Hence, $\frac{\tau}{\tau_o} = \frac{k}{k_o}\cdot\frac{1}{x^2}$ for the main cloud, and $\frac{\tau}{\tau_o} = \frac{k}{k_o}\cdot\frac{1}{x^3}$ for the perturbation. If one assumes that the opacity is constant, then one can find the value of x for which the cloud becomes opaque ( $\tau = 1$). Thus

$$x = \tau_o(\text{cloud})^{1/2}$$

$$x = \tau_o(\text{perturbation})^{1/3}$$

Since initially the perturbation density is nearly that of the cloud and the opacities are equal, the ratio of the initial optical depths is proportional to the ratio of the initial sizes. Thus,

$$\frac{\tau_o(\text{perturbation})}{\tau_o(\text{cloud})} = \frac{r^o_{\text{perturbation}}}{r^o_{\text{cloud}}}$$

Substituting this expression in the equations above for x, one obtains

$$x^3_{\text{perturbation}} = \tau_o(\text{cloud})\left(\frac{r^o_p}{r^o_c}\right)$$

If x is eliminated between the expressions, one can determine for what ratio of perturbation and cloud sizes the cloud will become optically thick before the perturbation. It is important to know if this can occur, because, as was mentioned above, if the main cloud comes into equilibrium, then the growth of perturbations is halted, according to the linear theory, for all masses smaller than the critical Jeans' mass. One finds that this condition occurs if

$$38)\qquad \tau_o(\text{cloud}) > \left(\frac{r^o_p}{r^o_c}\right)^2$$

It is still necessary to evaluate $\tau_o$ in the above expression. If eq(37) is rewritten in terms of the total particle density n and the ratio of the number of absorbing particles to the total particle density, $\beta$, then

$$\tau = n \phi \beta R$$

where $\phi$ is the absorption cross section per absorbing particle. Expressing R in terms of the mass in solar units and the density in particles per cc, one finds

$$39) \quad \tau = 6.6 \times 10^{18} n^{2/3} \phi \beta M^{1/3} .$$

It is thus possible to evaluate $\tau$ and the value of x when the main cloud becomes optically thick. To do this, one must choose a particular environment.

For temperatures much higher than $10^4$ and densities about $10^{-3}$ particles / cc, electron scattering is taken as the main source of opacity. For electron scattering, $\beta = 1$ since the electron density equals the ion density, and $\phi = 6 \times 10^{-25}$. Assuming a mass of $4 \times 10^9$ Mo, $x = 10^{-2}$.

For temperatures near 100, and densities about 10, Gaustad (1963) has shown that the main source of opacity is absorption by grains. Taking $\phi = 10^{-11}$ and $\beta = 10^{-13}$ as the ratio of grains to hydrogen atoms, and a mass of $2 \times 10^3$ Mo, one again finds that $x = 10^{-2}$.

Hence, in both cases, collapse can occur to very small fractions of the initial volume. This in turn implies that very large density excesses can develope.

In order to see whether the perturbations become opaque before the main cloud, it is necessary to compute the initial optical depths. Using expression (37) and the values for the mass, temperature and density above, one sees that $\tau = x^2$ computed above. Hence from eq(38), one sees that only perturbations with radii smaller than 1 percent of the main cloud radius will remain optically thin longer than the cloud.

## B) SUBFRAGMENTATION OF COLLAPSING CLOUDS

It was seen in part I that the critical Jeans' mass for a typical interstellar cloud is about $10^3$ M$_\odot$. The linear and two-point models have shown that in such a cloud perturbations can grow once collapse begins. It is clear that if one is to produce objects whose mass is about 1 M$_\odot$, one must invoke either very small perturbations in the initial cloud or a mechanism which will cause large perturbations to fragment. The latter course is the one suggested by Hoyle (1953), and outlined in part I. The fact that in the linearized theory small perturbations simply oscillate until their mass exceeds the local Jeans' mass suggests that Hoyle's picture is in fact the more reasonable one. In support of this, one could argue that very small oscillatory perturbations will initially be absorbed by the growth of larger ones. Once the density of a larger one has risen so that it satisfies the Jeans' criterion, then it can begin a free-fall collapse essentially independent of the main flow. This is a rigorous result of the asymptotic expressions for the linear theory. One can then apply the linear theory to this fragment and follow perturbations in it. This process continues until opacity or some other effect intervenes either to bring the fragment into hydrostatic equilibrium or to cause its temperature to begin rising. Hoyle has studied this stage of the evolution of a fragment for a gas mixture of essentially pure hydrogen but with a slight admixture of metals. Gaustad (1963) has followed the collapse in a low temperature regime where the opacity is due to grains and gradually switches to hydrogen. Gould (1964) has considered the problem when the opacity is due to $H_2$. The result for all three cases is that while the cloud becomes optically thick, gamma is not 5/3 because it is still able to radiate at a rate sufficient to offset the compressional energy released. Only when the mass is about .1 M⊙ will radiation control the collapse. The question thus arises, since gamma is less than 5/3, and in fact still apparently near 1 for all fragments greater than about .1 M⊙ , why are there any 50 M⊙ stars? Suggestions have been made that angular momentum or some other effect intervenes ( Gaustad (1963)).

Another possibility might be mentioned. Once temperature gradients are built up, pressure gradients are also likely to occur, as was mentioned before. While they may be small, it is possible that these gradients could give rise to bouyant forces acting on the perturbations. One thus has the possibility of a "convective type instability" arising. Schematically, one then has the picture of fragments with positive density excesses sinking toward the cloud center, and those with negative excesses rising toward the cloud border. Since there is no guarantee that the fragments have the same entropy as the background, there is the possibility of "overshooting". A very dense fragment might then tunnel through the cloud center and out the other side. If its self-gravitation were sufficiently large, it might then be able to form a separate condensation. Fragments with small density excesses though might simply move up and down in the main flow, eventually being dissipated by viscous or radiative process.

One might approach the problem quantitatively by assuming that the main flow was that of a polytrope collapsing. As a rough approximation a $\sin kr / kr$ density distribution could be used. One could then consider with the aid of the two-point model, for example, the motion of a perturbation as was done in part III, with the modification that there is now a pressure gradient in the main cloud. The main cloud, in a first approximation, could be taken to be in equilibrium. It should then be possible to follow the density of the perturbation as a function of space and time. The objective would be to see if there was a critical mass or density below which perturbations would fail to grow and separate from the main cloud, and to see what relation this has to the Jeans' mass.

The reason that the problem sketched above is important is that it may clarify the question of what determines whether a gas cloud becomes one star or more. This in turn is clearly what determines the mass function.

One might summarize the suggestions made in this part as follows:

1) For the type of clouds studied, it would appear that the subunits become opaque before the cloud itself.

2) The final mass function is probably determined by the processes that are operative in the subfragmentation of the cloud and hence is more likely to depend on the parameters of the cloud immediately before the end of collapse, rather than on the initial parameters of the medium.

## VIII - SOME CONCLUSIONS REGARDING THE EVOLUTION OF GAS CLOUDS

### A) SUMMARY OF THE TWO-POINT RESULTS

Before considering the application of the results to the formation of astrophysical systems, it might be well to give a general summary of what has been learned so far.

From fig(4) and fig(6), one sees that for the zero-pressure case, a large central concentration in the main cloud and/or the presence of background stars inhibits the formation of large density excesses. For the finite-pressure case with a polytropic relation, smaller gammas cause an increased growth rate as can be seen from fig(7). Note, though, that the inclusion of pressure greatly retards growth. In all cases where the density was found to increase on the basis of the linear theory, the nonlinear effects were found to accelerate the growth.

The inclusion of radiation was found to produce a gamma less than or equal to one, as is seen in fig(12) and fig(13). Thus the collapse is roughly isothermal as suggested by Hoyle (1953) and Mestel and Spitzer (1956).

When rotation is included, growth cannot occur if the rotation period is less than the free-fall time unless the initial density excess is made very large. This agrees with Layzer's finding. For smaller rotation periods, the major effect is a retardation of growth in these early phases.

A study of collisions among fragments showed that unless the initial size of the fragments was large or their growth rate much slower than for an isothermal collapse, destruction was unlikely. Since in a system with large amounts of angular momentum the above conditions are likely to be encountered, it was concluded on the basis of eq(36) that collisions would occur. It was suggested that the consequences of collsions would be unimportant unless the cooling time scale was long, as might be expected in the formation of galaxies or metal poor systems. In such systems therefore, fragmentation would be retarded for periods comparable with the free-fall time of the cloud as a whole.

## B) POSSIBLE EVOLUTION OF COSMIC GAS CLOUDS

It would not yet appear possible on the basis of the theories sketched here to predict the initial mass function of stars. However, reasons have been given for supposing that fragmentation proceeds in a very different fashion from system to system according to certain initial conditions. In particular, the importance of chemical composition through its control on the cooling functions and angular momentum has been mentioned.

In order to summarize these suggestions, consider first the behavior of two large clouds, say of galactic mass. Suppose they differ only in the amount of angular momentum. In the case with low angular momentum, it has been suggested that collisions are not as likely to occur due to the lower random velocities one might expect. Furthermore, it was suggested that where orbital velocities were low, spin velocities might also be small. Hence, fragmentation should proceed efficiently and rapidly in such systems. In addition, Spitzer (1942) has shown that what gas is remaining in such systems will be concentrated toward the center of the system. One would conclude that in such systems virtually all star formation should occur in one single brief epoch at the time of initial collapse of the system and that any residual gas would collect at the center, thus being difficult to observe. A study of spherical systems by Gamow, Belzer, and Keller (1948) showed that in massive systems formed by collapse, the gross synchronous radial motion will be destroyed in times the order of the free-fall time. Hence, any vestiges of the collapse should no longer be evident. (Strictly speaking, this is true only of the interior regions. The radial character of the orbits will still be preserved in the outer regions, but any phase relation will have been destroyed.)

If one considers now systems with large amounts of angular momentum, a very different picture suggests itself. As was found by Spitzer (1963), if a cylinder initially stable against fragmentation collapses along its axis into a disc, it remains stable, Thus if angular momentum restricts the collapse to one direction, it will be very hard for star formation to occur.

If a roughly spherical collapse can occur for a short time, it is possible that the density could rise sufficiently to allow some fragmentation to occur. A collapse to .1 of the initial size will lower the critical mass from $10^{11}$ M☉ to $10^{6}$ M☉ since $\rho = \rho_0/x^{9/2}$. Hence star formation might occur in aggregates of this size. It was further suggested that in systems with high angular momentum, perturbation growth might be retarded due to collisional heating. It is thus plausible that any fragments that do survive will have preferentially low spin angular momentum. Such a picture is consistent with that suggested by Eggen, Lynden-Bell, and Sandage (1962) on the basis of the motion of Pop II stars of large UV excesses. A further consequence of such a collapse is that large amounts of gas would be left that would be unable to accumulate in the central regions due to the rotational forces acting on it.

Since fragmentation in rotating systems seems unlikely to occur on the same sort of scale as in nonrotating ones, it is reasonable to ask how it does occur. The stability of rotating discs has been studied by Bel and Schatzman (1958), Hunter (1963), Toomre (1964), Lin and Shou (1964) and Mestel (1963). It is found in all cases, despite the difference in assumptions regarding relative amounts of stars and gas, that instabilities are likely to occur leading to the formation of regions of higher than average density in the form of rings, bars or spiral arms.

It is natural to suppose that if star formation occurs, it is in these regions. However, with a temperature near $10^4$, it is still necessary to have enormous densities ($10^5$ particles / cc) to achieve instability in the Jeans' sense for a mass anywhere near that of even the largest aggregates of pop I stars. This suggests that to produce stars in such a model an additional source of cooling other than pure hydrogen is required to lower the temperature to the presently observed value of about 100. Whether this cooling can be ascribed to primordial metals, metals produced in stars that were able to form in the initial collapse, or molecular hydrogen is uncertain. If the temperature can be lowered to about $10^2$ by some mechanism, a much lower density (10 particles / cc) is adequate to produce Jeans' type instabilities with the consequence of star formation in groups of about $10^3$ M$_\odot$. The relatively short time of collapse of such a system ($10^7$ years) suggests that in the youngest star complexes the dynamical effects of an original collapse and expansion might still be observable. It is not impossible that the observed expansion of some young associations is a manifestation of the original collapse in which the stars might have formed.

C) THE POSSIBILITY OF SECOND GENERATION FRAGMENTATION

It is of interest to ask whether in systems in which fragmentation has occured once, a second generation of objects can form. This question is suggested by Herbig (1962) with regard to the apparent spread of ages in the Pleiades and the Hyades. In the case of galaxy formation, Weymann (1964) has suggested that if gas accumulates in a system and is unable to fragment a second time, it might cause the formation of a large mass in the nucleus of the galaxy. One can imagine the gas content of a system increasing in time after one generation of objects has formed due to mass ejection, accretion from the sur-surrounding medium, as well as dispersal of those fragments in the first generation that were too small to be stable.

Consider therefore a system in which collapse has produced a system of stars. Unless there is a large amount of uncondensed matter, the gravitational field will be determined by the stars. Hence there will be a tendency for the matter accumulating or remaining uncondensed to concentrate toward the center as shown by Spitzer (1942), who also showed that in non-rotating systems, there is a critical mass for the gas similar to that found by Ebert et al. If the accumulation continues due to mass ejection, say, and the mass eventually exceeds this critical mass, then it may undergo collapse. This time, however, the collapse occurs through a network of stars. Furthermore, there is likely to be a central condensation in the cloud. In part II it was shown that both these factors impede the growth of perturbations.

For the case of the zero-pressure uniform sphere, the density of a perturbation with initial amplitude .1 doubles when the excess density has increased by a factor of ten, or when the main cloud has collapsed to .4 of its original size. If the non-uniformity index, $\eta$, is .5, the growth to a similar density is delayed until the cloud radius is .05. Thus, roughly a factor ten more collapse is required to achieve the same density. If the density of stars is 10 that of gas, an even more extreme state of collapse is required to achieve the same density. If one applies this factor of ten to the isothermal collapse, the main cloud must contract to about 1 percent of its original size to have the density of even a large perturbation icrease by a factor of ten. This is approaching the point where the cloud is likely to be optically thick, as found in section A part VII. It was mentioned there, however, that the work of Gould (1964) and Gaustad (1963) showed that an optically thick HI cloud could still continue collapsing nearly isothermally. Hence, fragmentation could conceivably occur a second time in these systems. For the higher temperature cloud, though, the picture suggests that fragmentation is not as likely to occur once an optically thick state is achieved. (Hoyle (1953)). This could possibly lead to the formation of a massive object in the center of the system. There has been much recent speculation that quasi-stellar radio sources are connected with just such objects (Fowler and Hoyle (1963), and Robinson, Schild and Schucking (1965)).

It is very difficult to draw conclusions about the ultimate evolution of collapsing systems. A number of tentative points have been suggested above. Reasons have been given for supposing that angular momentum affects the ability of a system to fragment, and hence star formation proceeds very differently in elliptical and spiral systems. Firmer conclusions necessitate a clearer understanding of many processes. A number of points are particularly crucial. One very simple point that has been ignored in all the above is the depletion of the background gas that is likely to occur as fragmentation proceeds. One should also ask what modifications are necessary in the linear theory if nonuniformity and rotation are included. A start in this direction has been made by Lynden-Bell (1962, 1964). In general, though, the gravitational stability of a rotating, compressible mass seems to require a great deal more study.

Another problem that has been studied only roughly is the consequences of collisions on moderately dense fragments. Throughout this work there has been no mention of magnetic fields. This must be rectified at some point. While it was suggested that orbital motion could be used to rid a system of rotational energy, just one special solution to see exactly how the density would develope in such a system would be very valuable. Finally, the question of what occurs when pressure gradients are becoming important, as mentioned in part VII, and their influence on the ability of a cloud to fragment must be studied before that interesting question of what determines the stellar mass function can be answered.

APPENDIX I

# COLLAPSE OF A UNIFORM GAS SPHERE

## HYDRODYNAMIC

### Basic Equations

1) $$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\vec{v}$$

2) $$\frac{d\vec{v}}{dt} = \nabla\phi - \frac{\nabla P}{\rho}$$

3) $$\nabla^2\phi = -4\pi G\rho$$

where $\rho$ = density

$\phi$ = gravitational Potential

$p$ = pressure

$\vec{v}$ = velocity

Take $\frac{d}{dt}$ of eq(1)

3) $$\frac{d}{dt}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) =$$

$$-\frac{\nabla\cdot\vec{v}}{t} - (\dot{\vec{v}}\cdot\nabla)\nabla\cdot\dot{\vec{v}}$$

## TWO POINT

### Basic Equations

1) $$\ddot{r}_1 = -\frac{GM_1}{r_1^2} - \frac{Gm_p}{r_p^2} - \frac{1}{\rho}\frac{dP}{dr}\bigg|_1$$

2) $$\ddot{r}_2 = -\frac{GM_2}{r_2^2} - \frac{1}{\rho}\frac{dP}{dr}\bigg|_2$$

3) $$M_i = \frac{4}{3}\pi\bar{\rho}r_i^3 \qquad M_t = \bar{\rho}(1+\xi)V_p$$

$$m_p = \xi\bar{\rho}V_p \qquad P_i = \frac{1}{\rho}\frac{dP}{dr}\bigg|_i$$

$$x = \frac{r}{r_o}$$

For definitions of other quantities, see part II, and fig(1). Assume a spherical perturbation.

$$\ddot{r}_1 - \ddot{r}_2 \equiv \ddot{r}_p = -\frac{4}{3}\pi G\rho r_p - \frac{G\rho\xi V_p}{r_p^2} - P_1 + P_2$$

Substitute divergence of eq(2) in eq(3).

$$\frac{d}{dt}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) - 4\pi G\rho - \nabla\cdot\left(\frac{\nabla P}{\rho}\right) =$$

$$\nabla\cdot\left[(\vec{v}\cdot\nabla)\vec{v}\right] - (\vec{v}\cdot\nabla)\nabla\cdot\vec{v}$$

Let $\vec{v} = \vec{r}f(t) + \vec{v}_p = \vec{r}f + \vec{u}$ where

$$f(t) = \frac{\rho}{3\rho} .$$

Thus $\nabla\cdot\vec{v} = 3f + \nabla\cdot\vec{u}$ .

Let $A = \nabla\cdot\left[(\vec{v}\cdot\nabla)\vec{v}\right] = \frac{\nabla^2 v^2}{2} -$

$$\nabla\cdot\left[\vec{v}\times\mathrm{curl}\,\vec{v}\right]$$

Since the main flow is radial,

$\mathrm{curl}\,\vec{v} = \mathrm{curl}\,\vec{u}$ . Assuming $|u|$ as small,

$$A = \nabla^2\left(\frac{r^2f^2}{2} + \vec{r}\cdot\vec{u}f\right) + f\vec{r}\cdot\mathrm{curl}\,\mathrm{curl}\,\vec{u}$$

Now $r_p^3 = \dfrac{Mt}{\frac{4}{3}\pi\rho(1+\varepsilon)} \equiv \alpha^3(1+\varepsilon)^{-1}$

Thus

$$\ddot{r}_p = \alpha(1+\varepsilon)^{-1/3}\left[\ddot{x} - \frac{2\dot{x}\dot{\varepsilon}}{3(1+\varepsilon)} + \right.$$

$$\left. - \frac{1}{3}\frac{x\ddot{\varepsilon}}{1+\varepsilon} + \frac{4}{9}\frac{x\dot{\varepsilon}^2}{(1+\varepsilon)^2}\right]$$

Thus

$$4)\quad \ddot{x} - \frac{x}{3(1+\varepsilon)}\left[\frac{2\dot{x}\dot{\varepsilon}}{x} + \ddot{\varepsilon} - \frac{4}{3}\frac{\dot{\varepsilon}^2}{1+\varepsilon}\right] =$$

$$- \frac{4}{3}\pi G\rho x(1+\varepsilon) - \frac{(P_1 - P_2)x}{r_p}$$

Since $\ddot{x} = -\frac{4}{3}\pi G\rho x - P_2$ , Now curl curl $\vec{u}\,\big]_r$ =

$$\frac{2}{r}\left(\nabla\cdot\vec{u} - \frac{\partial u_r}{\partial r} - \frac{u_r}{r}\right) + \frac{\partial}{\partial r}\nabla\cdot\vec{u} - \nabla^2 u_r .$$

Also

$$\nabla^2 \hat{r}\cdot\vec{u} = r\nabla^2 u_r + \frac{2u_r}{r} + 2\frac{\partial u_r}{\partial r}$$

Thus

$$A = 3f^2 + 2f\nabla\cdot\vec{u} + rf\frac{\partial}{\partial r}\nabla\cdot\vec{u} .$$

Since $(\vec{v}\cdot\nabla)(\nabla\cdot\vec{v}) = rf\frac{\partial}{\partial r}\nabla\cdot\vec{u}$ ,

4) $\nabla\cdot\left[(\vec{v}\cdot\nabla)\vec{v}\right] - (\vec{v}\cdot\nabla)\nabla\cdot\vec{v} = 3f^2 + 2f\nabla\cdot\vec{u}$

Now $\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\vec{v} = 3f + \nabla\cdot\vec{u}$ .

Thus, $\nabla\cdot\vec{u} = \frac{1}{\rho}\frac{d\rho}{dt} - 3f$ .

Let $\rho \Rightarrow \rho(1+\varepsilon)$ .

5) $$\ddot{\varepsilon} - \frac{4}{3}\frac{\dot{\varepsilon}^2}{1+\varepsilon} + \frac{2\dot{x}\dot{\varepsilon}}{x} = 3(1+\varepsilon)\left[\frac{4}{3}\pi G\rho\varepsilon + \frac{P_1 - P_2}{r_p} - \frac{P_2}{xr_o}\right]$$

This is the basic nonlinear two-point equation.

Recalling now that $P_i = \frac{1}{\rho}\frac{dP}{dr}$ , one can see that since $P_2$ is evaluated at the perturbation center and the main cloud is uniform, $P_2 = 0$.

Also, $$P_1 - P_2 = \frac{1}{\rho}\frac{dP}{dr}\bigg|_1 \approx \frac{1}{\bar{\rho}}\frac{P(r_1) - P(r_2)}{r_p}$$

Then $\frac{1}{\rho}\frac{d\rho}{dt} \sim \frac{\dot{\rho}}{\rho} + \dot{\xi}$

and $\frac{d}{dt}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) = \frac{\ddot{\rho}}{\rho} - \frac{\dot{\rho}^2}{\rho^2} + \ddot{\xi}$ .

Thus

$$5)\quad \frac{\ddot{\rho}}{\rho} - \frac{\dot{\rho}^2}{\rho^2} + -4\pi G\rho(1+\xi) - \nabla\cdot\left(\frac{\nabla P}{\rho(1+\xi)}\right) = 3f^2 + 2f\left(\frac{\dot{\rho}}{\rho} + \dot{\xi} - 3f\right).$$

Substracting the unperturbed solution and eliminating f,

$$6)\quad \ddot{\xi} + \frac{2\dot{x}\dot{\xi}}{x} - 4\pi G\rho\xi - \nabla\cdot\left(\frac{\nabla P}{\rho}\right) = 0.$$

$$= \frac{1}{\bar{\rho}}\frac{P}{r_p}$$

Linearizing now in $\xi$,

$$6)\quad \ddot{\xi} + \frac{2\dot{x}\dot{\xi}}{x} - 4\pi G\rho_o\xi - \frac{3\Delta P}{\bar{\rho} r_p^2} = 0.$$

The two-point equation, as derived in eq(5), is a differential equation in $\xi$. As has been mentioned in part III, it is possible to obtain a more exact form of the equation through the use of scale factors. As can be seen from eq(3), the $\xi$ used here is really a mean $\xi$, $\bar{\xi}$. However, the approximation to the pressure gradient involves the value of $\xi$ at the perturbation center, $\xi_c$. Thus, one wishes to relate

$$\xi = \beta\, \xi_c \quad \text{and} \quad P_1 = \frac{1}{\rho}\frac{dP}{dr}\bigg|_1 = \alpha \frac{1}{\rho}\frac{\Delta P}{\Delta r}\,.$$

Now $\frac{1}{\rho}\frac{dP}{dr} = \frac{\gamma P}{\rho^2}\frac{d\rho}{dr}$. If one assumes

$$\rho = \rho_L + \rho_c(t)\,\frac{\sin kr}{kr}$$

where $\rho_c(t)$ is the central excess density for the perturbation, then

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{\gamma P}{\rho^2}\,\rho_c(t)\,\cdot\left[\frac{\sin kr}{kr^2} + k\,\frac{\cos kr}{kr}\right].$$

The radius of the perturbation is chosen at the first zero of $\sin kr$. Thus

$$7)\qquad P_1 = \frac{1}{\rho}\frac{dP}{dr}\bigg|_{r_p} = \frac{\gamma P}{\rho^2}\bigg|_1 \frac{\rho_c(t)}{r_p} = \frac{\gamma P}{\rho}\bigg|_1 \frac{\xi_c}{r_o xy} = \frac{1}{\rho}\bigg|_1 \frac{\Delta P}{\Delta r}$$

where $\frac{\gamma P}{\rho}$ are evaluated at the perturbation edge and are hence the values in the unperturbed flow. One can see from eq(7) that $\alpha = 1$.

In order to relate $\xi$ and $\xi_c$, it is necessary to find the total excess mass in the perturbation. Thus

$$M(r_p) = \int_0^{r_p} 4\pi\rho_{excess} r^2 \, dr = 4\pi\rho_c(t) \int_0^{r_p} \frac{\sin kr}{kr} r^2 \, dr$$

$$= \frac{4\pi\rho_c}{k^3} \Big[ \sin kr - kr \cos kr \Big]_0^{r_p} = \frac{4\pi\rho_c r_p}{k^2}$$

But $M(r_p) = \frac{4}{3}\pi r_p^3 \bar{\varepsilon}\rho$ and hence

$$\bar{\varepsilon} = \frac{3\varepsilon_c}{r_p k^2} .$$

Now $r_p$ is the physical radius of the perturbation and hence $r_p = r_o xy$. k though must be measured in a Lagrangian frame. Following Hunter (1962, p606) one can thus write

$$k^2 = \frac{1}{\mu^2 x^2} \sim \frac{1}{\lambda_p^2 x^2}$$

where $\mu^2$ is related to the wavelength of the disturbance. Substituting this value of $k^2$ into the expression for $\bar{\varepsilon}$, one finds

8) $$\bar{\varepsilon} = \frac{3\varepsilon_c}{r_p^2 k^2} = \frac{3\varepsilon_c \lambda_p^2 x^2}{r_o^2 x^2 y^2} = \frac{3\varepsilon y \lambda_p^2}{r_o^2 y^2}$$

and thus

$$\beta = \frac{3\lambda_p^2}{r_o^2 y^2} .$$

## APPENDIX II

### EVALUATION OF $\Delta r_o / \Delta r$ AND $\eta$

The equation of continuity of the main sphere is

$$\rho(r_o)\, r_o^2 \Delta r_o = \rho(r)\, r^2 \Delta r$$

if $\beta$ is defined as $\beta = \Delta r_o / \Delta r$, then

$$\beta = \frac{\rho(r)\, r^2}{\rho(r_o)\, r_o^2} .$$

Now $\eta \equiv \bar{\rho}_L / \bar{\rho} = \rho(r)/\bar{\rho}$. Thus

$$\beta = \frac{\eta\, \bar{\rho}\, r^2}{\eta_o \rho_o r_o^2} = \frac{\eta}{\eta_o} \cdot \frac{1}{x} .$$

Furthermore, since $\beta = \Delta r_o / \Delta r$

$$\beta \approx \frac{\dfrac{\partial t}{\partial r}}{\dfrac{\partial t}{\partial r_o}}$$

where

$$t = \int_{r_o}^{r} \frac{dr}{\left[ 2GM\left(\frac{1}{r} - \frac{1}{r_o}\right) + \dot{r}_o^2 \right]^{1/2}} .$$

The integration over r is performed and the derivatives are then taken. Note that M, $r_o$, and $\dot{r}_o$ are functions of $r_o$. Hence $\beta$ depends on their values throughout the cloud.

It is found advantageous to let

$$\Gamma' = \frac{\frac{GM}{r}}{\frac{GM_o}{r_o}} \qquad \text{and}$$

$$\Lambda = \frac{\frac{\dot{r}^2}{2} - \frac{GM}{r}}{\frac{GM_o}{r_o}}$$

Then the following expressions are found for $\beta$ :

For $\Lambda_o = -1$, $\dot{r}_o = 0$

$$\beta = \left\{ \Gamma' + 1 - \frac{3(\Lambda\Gamma)'}{2\Lambda}\left(1 - \frac{x}{3}\right) + \sqrt{\frac{1}{x} - 1}\left[\Gamma' + 1 - \frac{3(\Lambda\Gamma)'}{2\Lambda}\right] \cdot \tan^{-1}\sqrt{\frac{1}{x} - 1} \right\}^{-1} .$$

For $\Lambda_o > 0$,

$$\beta = \frac{1}{\sqrt{1 + \frac{1}{\Lambda_o x}}}\left\{ \frac{-1}{\sqrt{1 + \frac{1}{\Lambda_o}}} + \frac{\Gamma' + 1}{\Lambda_o}\left(\frac{1}{\sqrt{1 + \frac{1}{\Lambda_o x}}} - \frac{1}{\sqrt{1 + \frac{1}{\Lambda_o}}}\right)\right.$$

$$+ \frac{(\Gamma\Lambda)'}{2\Lambda_o}\left(\frac{1 + \frac{3}{\Lambda_o}}{1 + \frac{1}{\Lambda_o}} - \frac{x + \frac{3}{\Lambda_o}}{\sqrt{1 + \frac{1}{\Lambda_o x}}}\right) +$$

$$\frac{1}{2\Lambda_o}\left[\Gamma' + 1 - \frac{3}{2}\frac{(\Gamma\Lambda)'}{\Lambda_o}\right]\ln\left[\frac{\sqrt{1+\frac{1}{\Lambda_o x}}-1}{\sqrt{1+\frac{1}{\Lambda_o x}}+1}\cdot\frac{\sqrt{1+\frac{1}{\Lambda_o}}+1}{\sqrt{1+\frac{1}{\Lambda_o}}-1}\right]\Bigg\}^{-1}$$

For $\Lambda_o < 0$

$$\beta = \sqrt{\frac{(1+\Lambda_o)x}{1+\Lambda_o x}}\ \Bigg\{1 + \frac{1+\Gamma'}{\Lambda_o} - \frac{(\Lambda\Gamma)'}{2\Lambda_o^2}\Bigg[3 + \Lambda_o + x\Lambda_o\sqrt{\frac{(1+\Lambda_o)x}{1+\Lambda_o x}}\Bigg]$$

$$- \left[\sqrt{\frac{(1+\Lambda_o)}{\Lambda_o}}\left(\tan^{-1}\sqrt{\frac{1}{|\Lambda_o| x}-1} - \tan^{-1}\sqrt{\frac{1}{|\Lambda_o|}-1}\right)\right.$$

$$\left. + \sqrt{\frac{(1+\Lambda_o)x}{1+\Lambda_o x}}\right]\left[\frac{1+\Gamma'}{\Lambda_o} + \frac{3(\Lambda\Gamma)'}{2\Lambda_o^2}\right]\Bigg\}^{-1}.$$

$\Lambda_o = 0$

$$\beta = \sqrt{x}\left\{1 + \frac{(\Lambda\Gamma)'}{5}(x^{5/2}-1) + \frac{(\Gamma'+1)(x^{3/2}-1)}{3}\right\}^{-1}$$

APPENDIX III

SURFACE ACCELERATIONS

**Prolate** $\epsilon = 1 - \dfrac{a^2}{c^2}$

$$A_{equator} = \frac{-2\pi G\rho}{\epsilon^3}\left[\epsilon - \frac{1-\epsilon^2}{2}\ln\frac{1+\epsilon}{1-\epsilon}\right]a$$

$$A_{pole} = \frac{-2\pi G\rho(1-\epsilon^2)}{\epsilon^3}\left[-2\epsilon + \ln\frac{1+\epsilon}{1-\epsilon}\right]c$$

**Oblate** $\epsilon = 1 - \dfrac{c^2}{a^2}$

$$A_{equator} = \frac{-2\pi G\rho(1-\epsilon^2)^{1/2}}{\epsilon^3}\left[-\epsilon(1-\epsilon^2)^{1/2} + \sin^{-1}\epsilon\right]a$$

$$A_{pole} = \frac{-4\pi G\rho}{\epsilon^3}\left[\epsilon - (1-\epsilon^2)^{1/2}\sin^{-1}\epsilon\right]c$$

where A is the acceleration, a is the equatorial distance and c is the polar distance.

The expressions for $f(\epsilon)$ and $g(\epsilon)$, the polar and equatorial corrections, thus become

Prolate

$$f(\xi) = \frac{1.5(1-\xi^2)}{\xi^3}\left[\ln\left(\frac{1+\xi}{1-\xi}\right) - 2\xi\right] \approx 1 - .4\xi^2$$

$$g(\xi) = \frac{1.5}{\xi^3}\left[\xi - \frac{1-\xi^2}{2}\ln\left|\frac{1+\xi}{1-\xi}\right|\right] \approx 1 + .2\xi^2$$

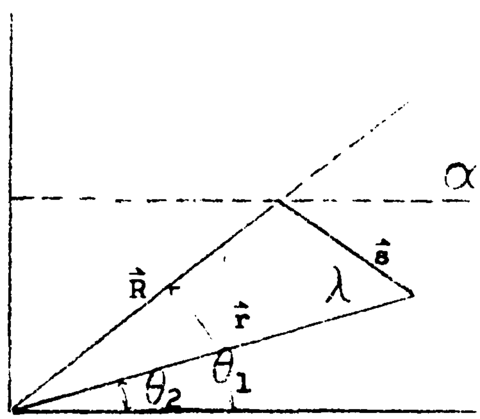
Oblate

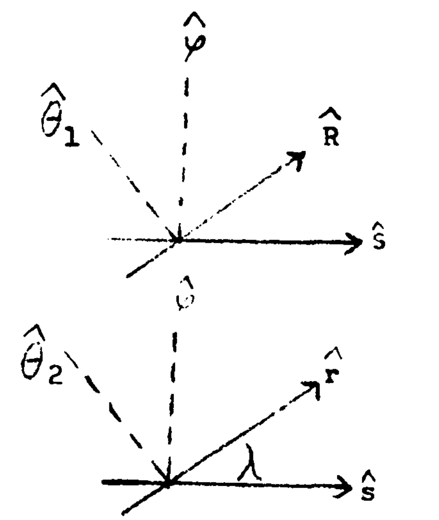
$$f(\xi) = \frac{3}{\xi^3}\left[\xi - (1-\xi^2)^{1/2}\sin^{-1}\xi\right] \approx 1 + .4\xi^2$$

$$g(\xi) = \frac{1.5}{\xi^3}\left[(1-\xi^2)^{1/2}\sin^{-1}\xi - \xi(1-\xi^2)\right] \approx 1 - .2\xi^2$$

## APPENDIX IV

## INFLUENCE OF ROTATION




**Identities**

$$r \cos \theta_2 = R \cos \theta_1 + s \cos \varphi$$

$$r \sin \theta_2 = R \sin \theta_1 - s \sin \varphi$$

$$r \cos \lambda = s + R \cos \alpha$$

$$r \sin \lambda = R \sin \alpha$$

$$r^2 = s^2 + R^2 + 2sR \cos \alpha$$

$$\vec{r} = \vec{s} + \vec{R}$$

**Transformations**

$$\hat{R} = \hat{s} \cos \alpha + \hat{\varphi} \sin \alpha$$

$$\hat{\theta}_1 = -\hat{s} \sin \alpha + \hat{\varphi} \cos \alpha$$

$$\hat{r} = \hat{s} \cos \lambda + \hat{\varphi} \sin \lambda$$

$$\hat{\theta}_2 = -\hat{s} \sin \lambda + \hat{\varphi} \cos \lambda$$

Forces

$$\text{along } R = -\frac{GM(R)}{R^2}$$

$$\text{along } \theta_1 = 0$$

$$\text{along } r = -\frac{GM(r)}{r^2} - \frac{Gm_p \cos\lambda}{s^2} \cdot f$$

$$\text{along } \theta_2 = -\frac{Gm_p \sin\lambda}{s^2} \cdot g$$

f and g are corrections required for asphericity. (See Appendix III.)

Now taking the expression for s and differentiating twice with respect to time,

$$\vec{s} = \vec{r} - \vec{R}$$

$$\ddot{\vec{s}} = -(\ddot{R} - R\dot{\theta}_1^2)\hat{R} - (2\dot{R}\dot{\theta}_1 + R\ddot{\theta}_1)\hat{\theta}_1 + \hat{r}(\ddot{r} - r\dot{\theta}_2^2) + \hat{\theta}_2(2\dot{r}\dot{\theta}_2 + r\ddot{\theta}_2)$$

Substituting for the forces,

$$\ddot{\vec{s}} = \frac{GM(R)}{R^2}\left[\hat{s}\cos\alpha + \hat{\varphi}\sin\alpha\right] - \left[\frac{GM(r)}{r^2} + \frac{Gm_p \cos}{s^2} f\right] \cdot$$

$$(\hat{s}\cos\lambda + \hat{\varphi}\sin\lambda) + \frac{Gm_p \sin}{s^2}\left[\hat{\varphi}\cos\lambda - \hat{s}\sin\lambda\right] \cdot g$$

Now $$\frac{GM(r_i)}{r_i^2} = \frac{4}{3}\pi G \bar{\rho}\, r_i$$

$$\ddot{\vec{s}} = \hat{s}\left[-\tfrac{4}{3}\pi G\bar{\rho}\, s - \frac{Gm_p}{s^2}\,(f\cos^2\lambda + g\sin^2\lambda)\right]$$

$$- \hat{\varphi}\left[\frac{Gm_p \sin\lambda\cos\lambda}{s^2}\,(f - g)\right]$$

If the perturbation starts spherical, then $f = g$. Therefore,

$$\ddot{\vec{s}} = -\tfrac{4}{3}\pi G\bar{\rho}\, s\,\hat{s} - \frac{Gm_p f\hat{s}}{s^2}$$

Note that the properties of the main cloud vanish identically if one goes to the dimensionless time. Hence, apart from the time scale, the growth of a rotating spheroidal fragment in a uniform cloud is completely independent of the cloud itself if the axis of rotation lies along the radius vector of the cloud.

Now, since

$$\ddot{\vec{s}} = \hat{s}(\ddot{s} - s\dot{\varphi}^2) + \hat{\varphi}(2\dot{s}\dot{\varphi} + s\ddot{\varphi}),$$

$$\ddot{s} = s\dot{\varphi}^2 - \tfrac{4}{3}\pi G\bar{\rho}\, s - \frac{Gm_p f}{s^2}$$

Now going to dimensionless variables and letting the unit of length be

$$vr_o = r_o\left(\frac{\rho_o}{\bar{\rho}}\right)^{1/3}$$

and defining $s \rightarrow sv$ and changing the time unit to $\tau_o = (\tfrac{4}{3}\pi G\rho_o)^{-1/2}$, one obtains,

$$\ddot{s} + \frac{2\dot{s}\dot{v}}{v} = -s\left[\frac{\ddot{v}}{v} + \frac{1}{v^3} + \frac{\zeta}{v^3} - \frac{\dot{\varphi}^2\tau_o^2}{v}\right]$$

If one now conserves angular momentum, and assumes the density in the main cloud is determined by,

$$\ddot{v} = -\frac{1}{v^2} + \frac{B_o}{v^3}$$

then since

$$\dot{\varphi}^2 = \frac{\omega_o^2 s_o^4}{v^4 s^4} \equiv \frac{\nu_o^2 s_o^4}{v^4 s^4 \tau_o^2}$$

One finally has, ignoring the pressure terms,

$$\ddot{s} + \frac{2\dot{s}\dot{v}}{v} = -\frac{s}{v^4}\left[B_o - \frac{s_o^4\nu_o^2}{s^4}\right] - \frac{s\zeta}{v^3} .$$

## APPENDIX V

## COOLING CURVES

CASE I - $T \sim 100^{\circ}K$

IONIC

$$F = 7.23 \times 10^{13} \rho \, T^{3.32}$$

[Unno and Simoda (1963)]

MOLECULAR HYDROGEN

$$F = 8.58 \times 10^{-31} \rho \, T^{21.8}$$

[Gould (1964)]

CASE II - $T > 10^{4\,\circ}K$

PURE HYDROGEN

$$F = 3.59 \times 10^{23} \rho \left(\frac{T}{10^6}\right)^{1/2} \left[1.42 + \frac{6.8 \times 10^5}{T} + \frac{7.75 \times 10^{10}}{T^2}\right]$$

[Michie (1963)]

Fig. 1.--Model of Cloud and Perturbation Used for the Two Point Equation

C denotes the central point of the spheroidal perturbation with axes a and b. r is the distance of C from the cloud center.

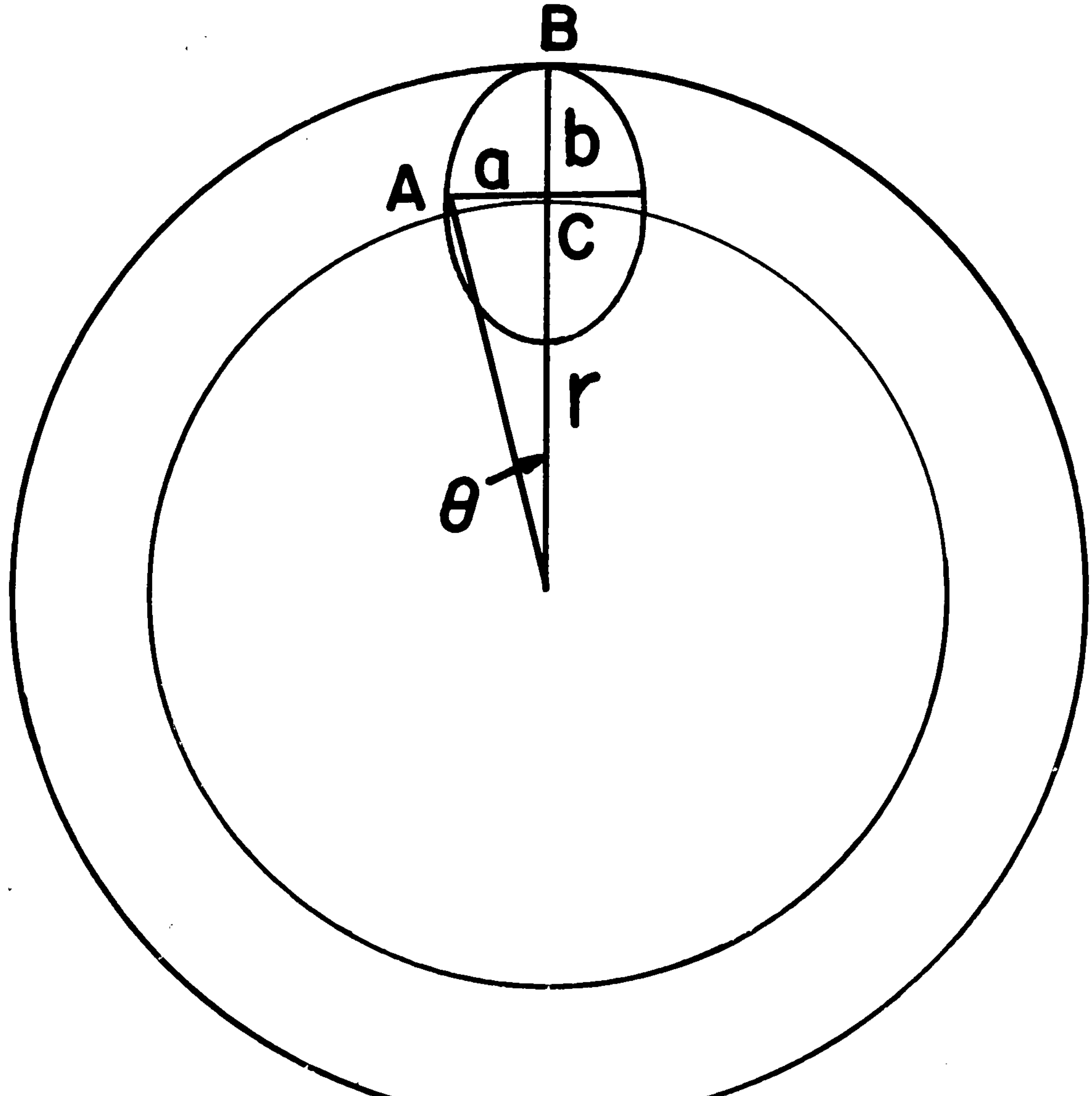

B
b
A
a
c
r
θ

Fig. 2.--Change of Central Concentration, as a Function of the Initial Central Concentration, $N_o$, for a Zero-pressure Collapse

A small $N_o$ implies a more centrally concentrated cloud. N is the ratio of the local density of a shell to the mean density of the matter interior to it.

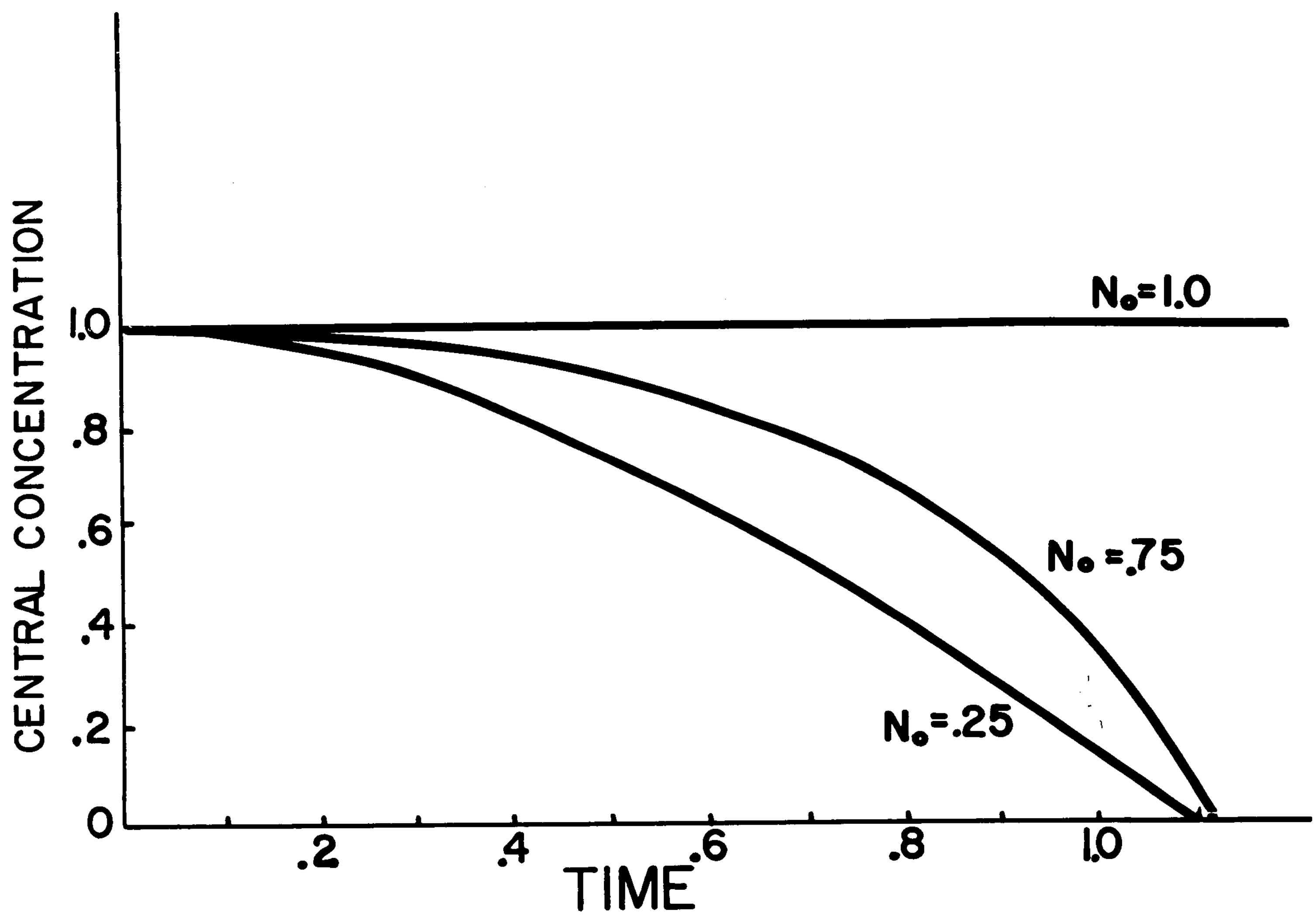
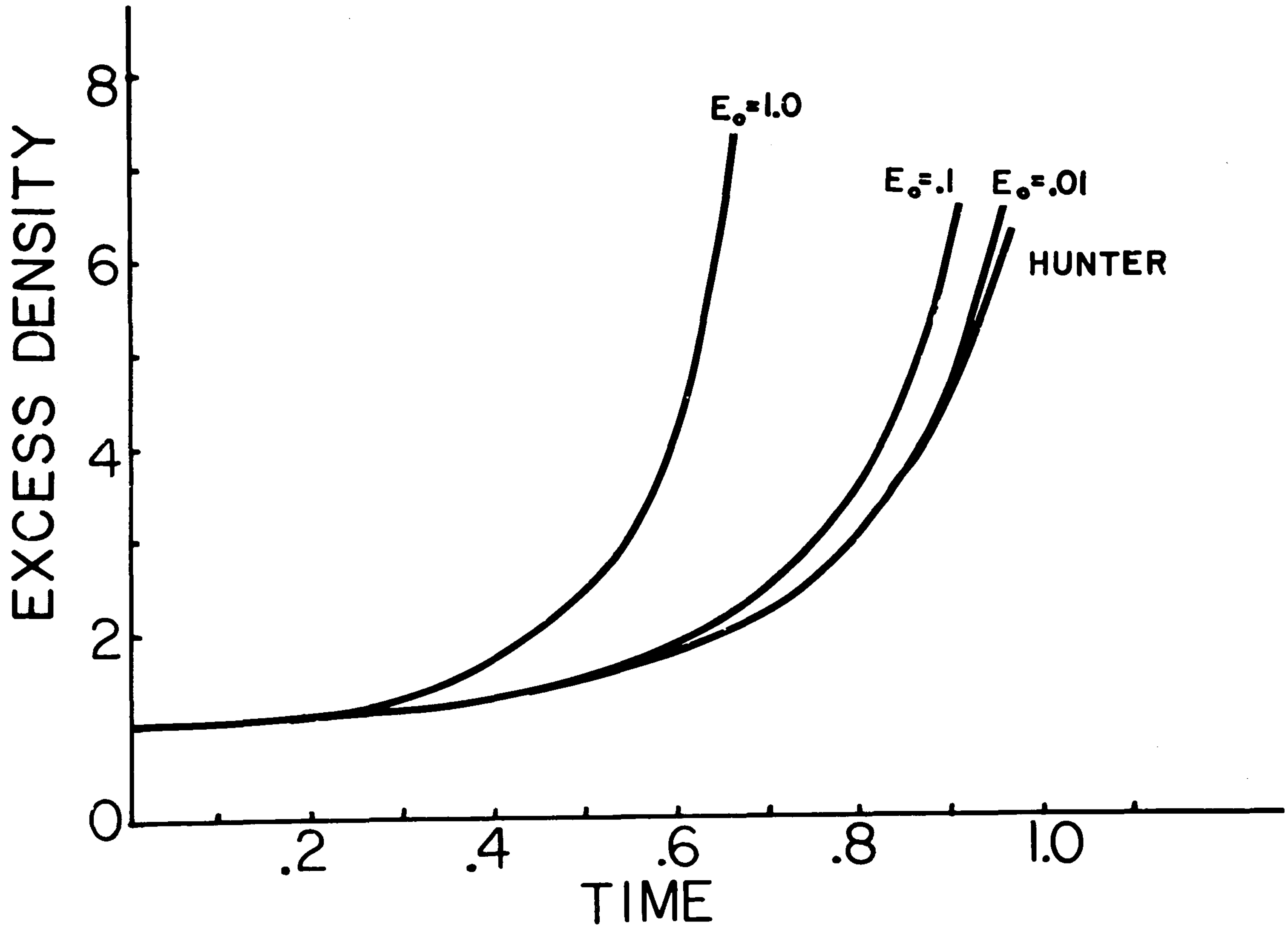

CENTRAL CONCENTRATION
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
TIME
N₀=1.0
N₀ =.75
N₀=.25

Fig. 3.--The Effect of Nonlinearity.

The excess density is plotted as a function of time for different initial excess densities, $E_0$, for a zero-pressure collapse. The curve marked "Hunter" refers to the linear theory. The curves are normalized to unity for the initial excess density.



EXCESS DENSITY
0
2
4
6
8
.2
.4
.6
.8
1.0
TIME
$E_o$=1.0
$E_o$=.1
$E_o$=.01
HUNTER

Fig. 4.--The Effect of Central Concentration

The excess density is shown as a function of time for different initial central concentrations, $N_0$, for a zero-pressure collapse. The curves are normalized to an initial density of 10%. $N_0 = 1.0$ is a uniform cloud.

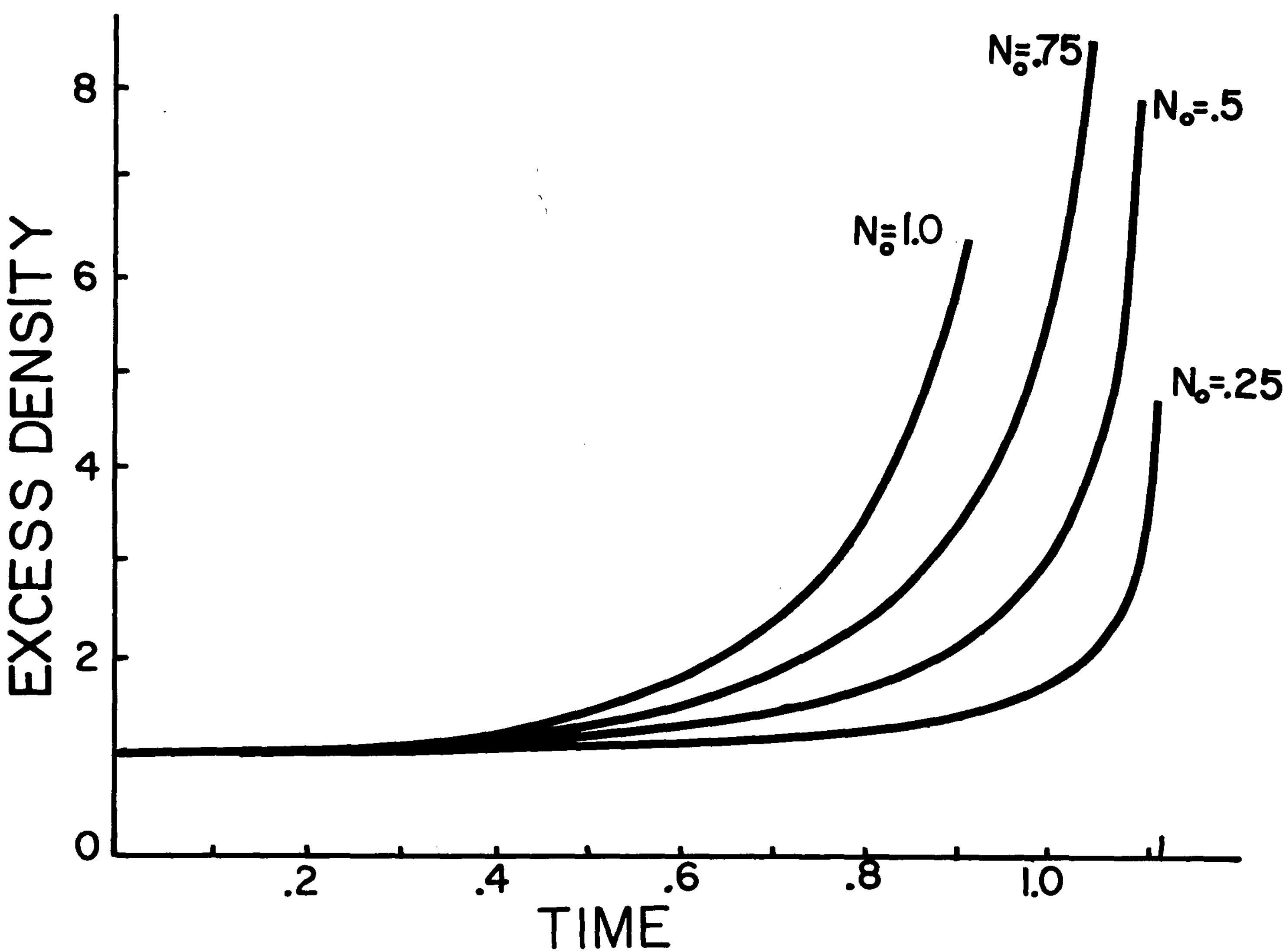

EXCESS DENSITY
0
2
4
6
8
TIME
.2
.4
.6
.8
1.0
N₀=1.0
N₀=.75
N₀=.5
N₀=.25

Fig. 5.--Number of Particles per cc as a Function of Time for a Zero-pressure Collapse for the Main Cloud and the Perturbation

The initial perturbation amplitude is 10%.

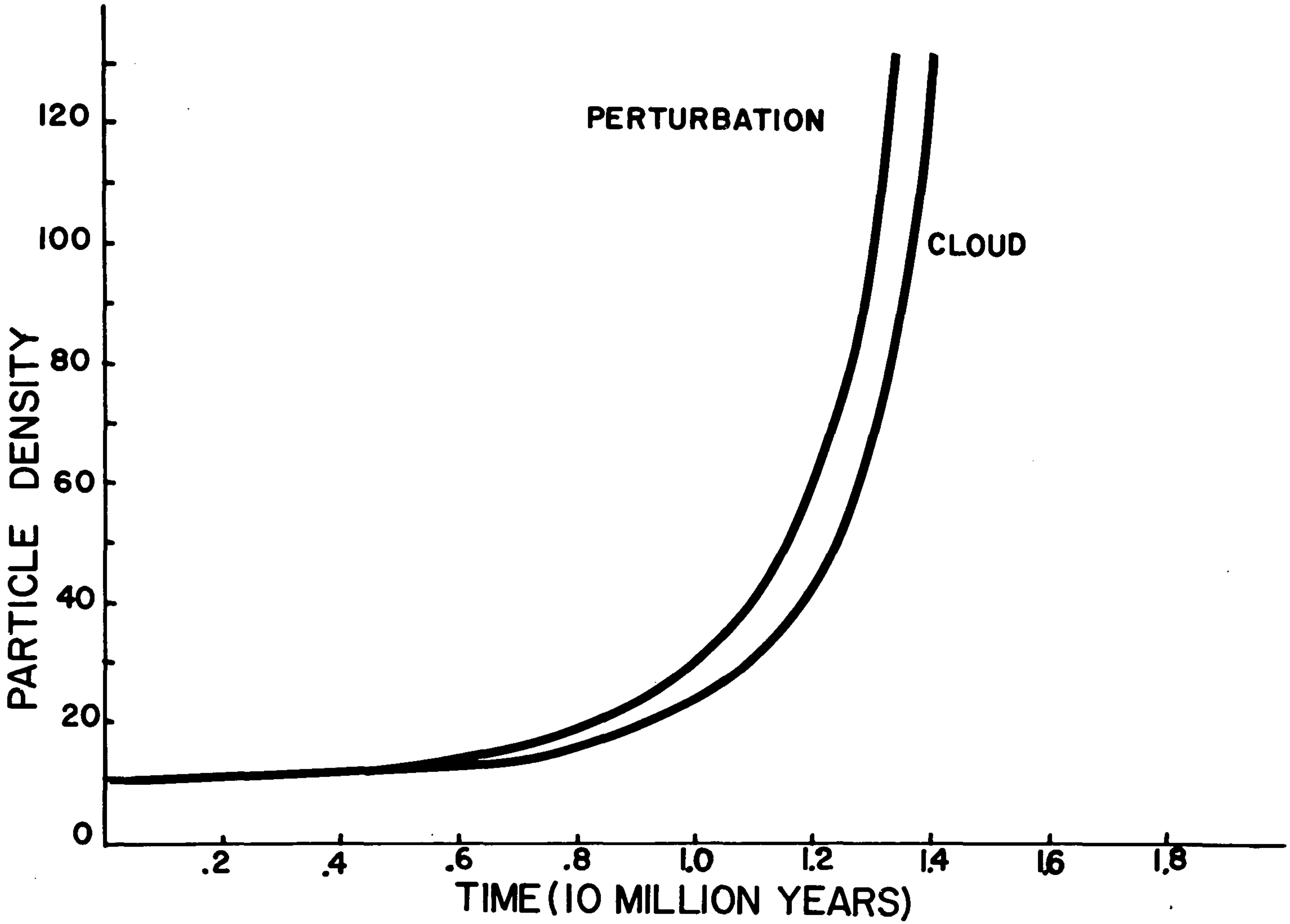

PERTURBATION
CLOUD
PARTICLE DENSITY
0
20
40
60
80
100
120
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
TIME (10 MILLION YEARS)

Fig. 6.--Effect of Background Stars

The excess density as a function of time is shown for a zero-pressure collapse. Q is the ratio of the star density to the gas density. The density is again normalized to an initial amplitude of 10%.

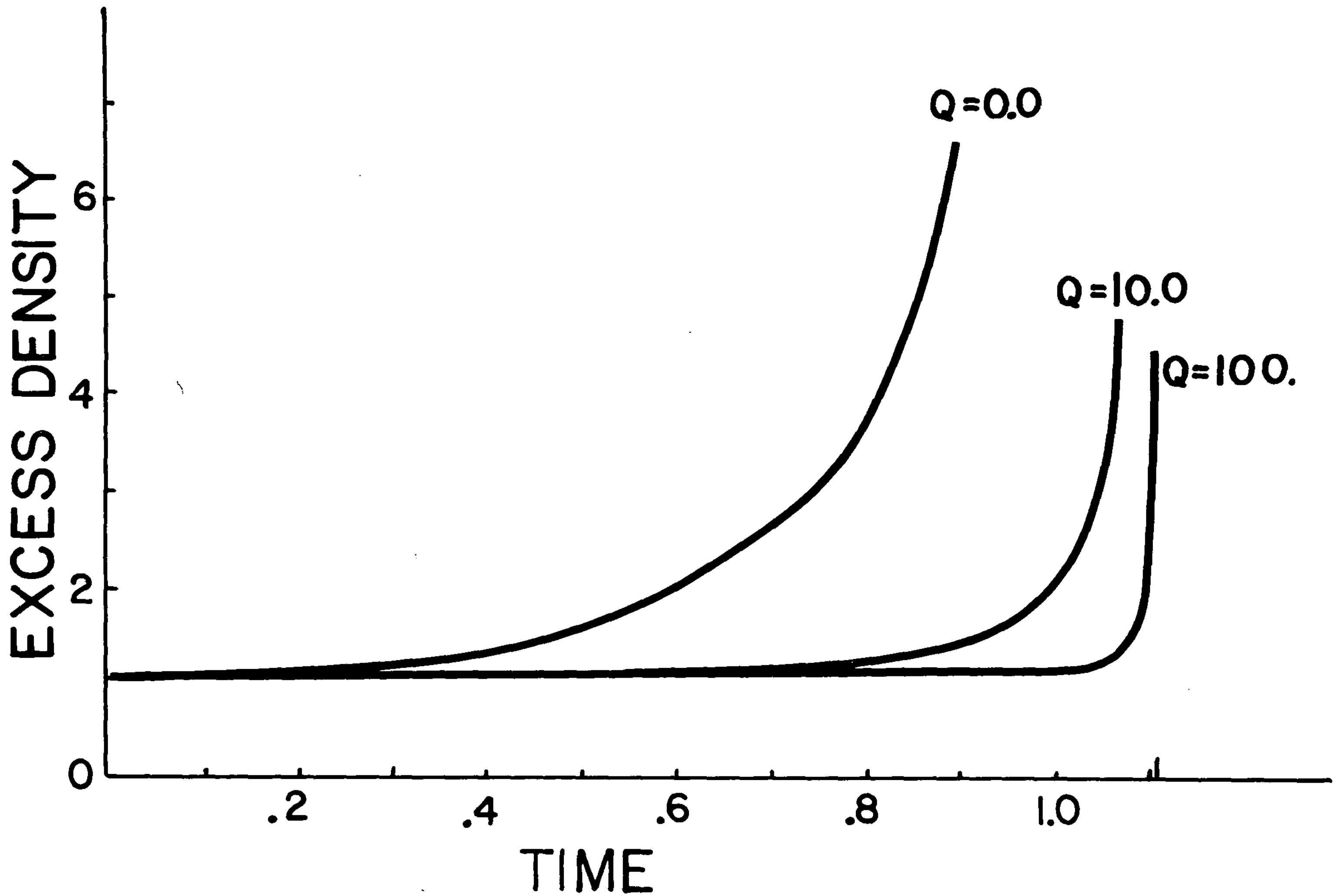

EXCESS DENSITY
0
2
4
6
.2
.4
.6
.8
1.0
TIME
Q=0.0
Q=10.0
Q=100.

Fig. 7.--Effect of Gamma (.5 to 2.0)

The excess density is shown as a function of time for a finite-pressure collapse. The density is again normalized to its initial value of 10%.

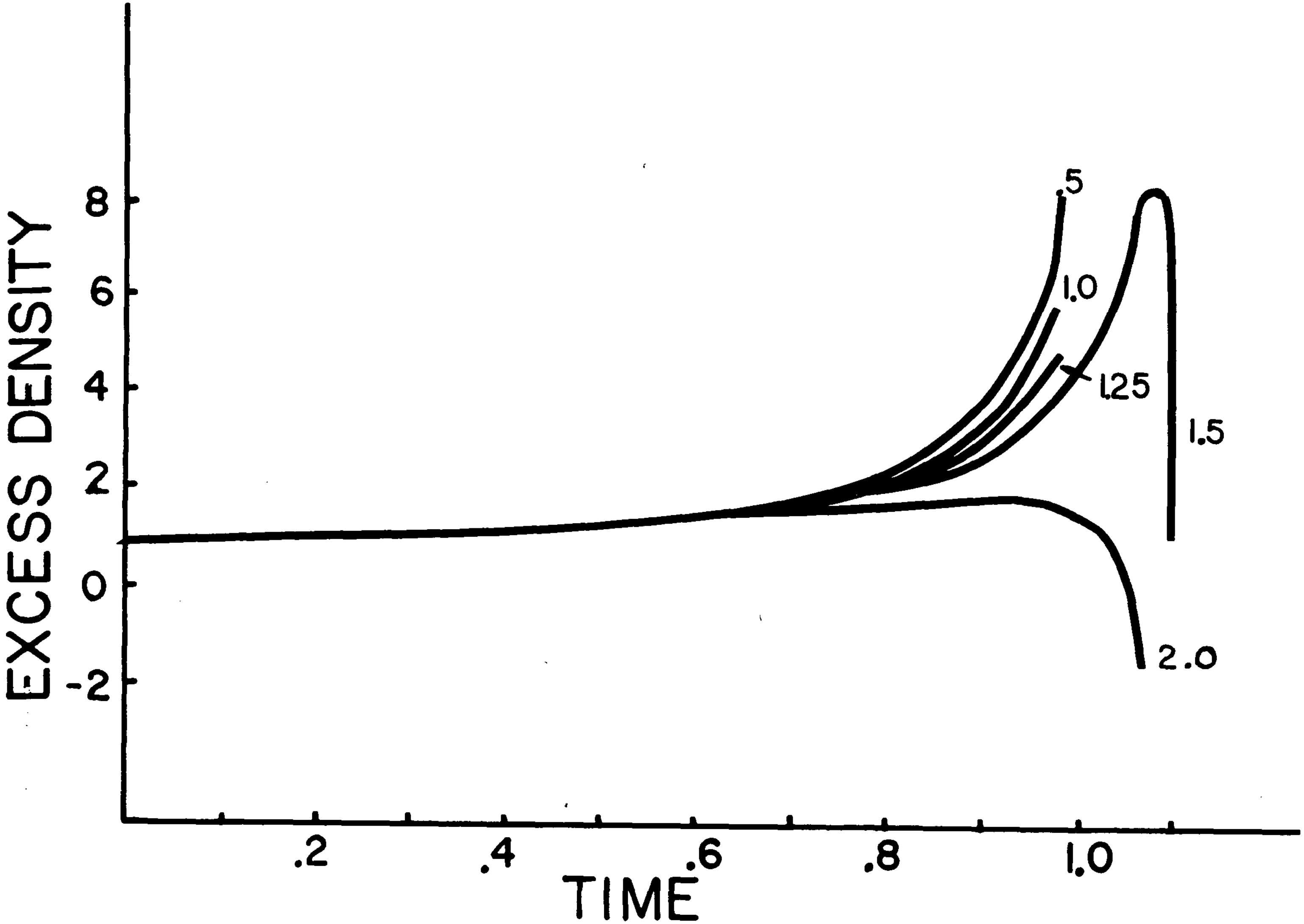

EXCESS DENSITY
8
6
4
2
0
-2
TIME
.2
.4
.6
.8
1.0
.5
1.0
1.25
1.5
2.0

Fig. 8.--Effect of Nonlinearity for a Finite Pressure Collapse

The excess density is shown as a function of time. The results are normalized to the initial excess density, $E_0$.

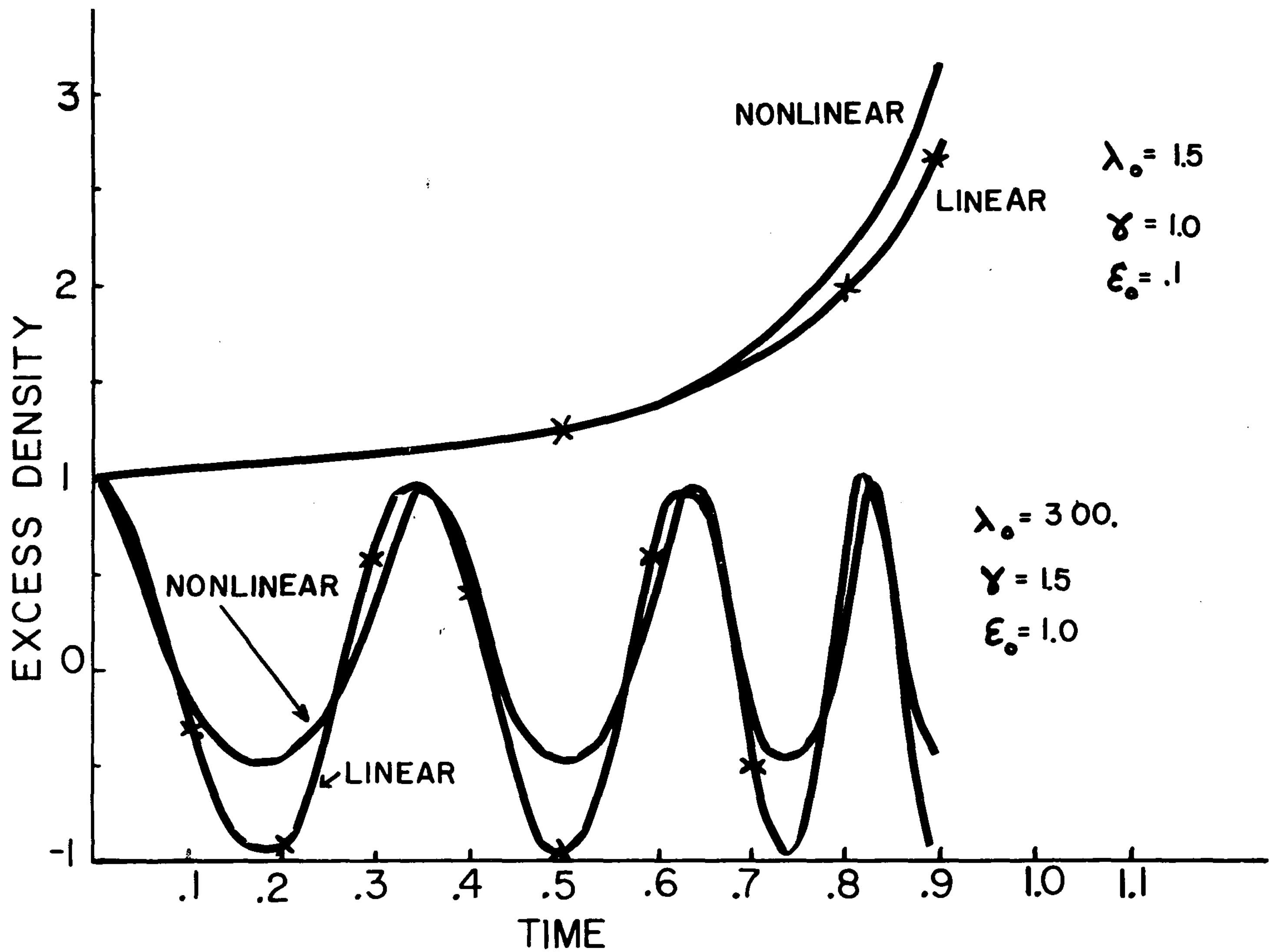

NONLINEAR
LINEAR
$\lambda_0 = 1.5$
$\gamma = 1.0$
$\varepsilon_0 = .1$
NONLINEAR
LINEAR
$\lambda_0 = 300.$
$\gamma = 1.5$
$\varepsilon_0 = 1.0$
EXCESS DENSITY
3
2
1
0
-1
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
TIME

Fig. 9.--Particle Density as a Function of Time for an Isothermal Collapse

The ratio of the Jeans' length to the perturbation scale is .5. The initial amplitude of the perturbation is 10%.

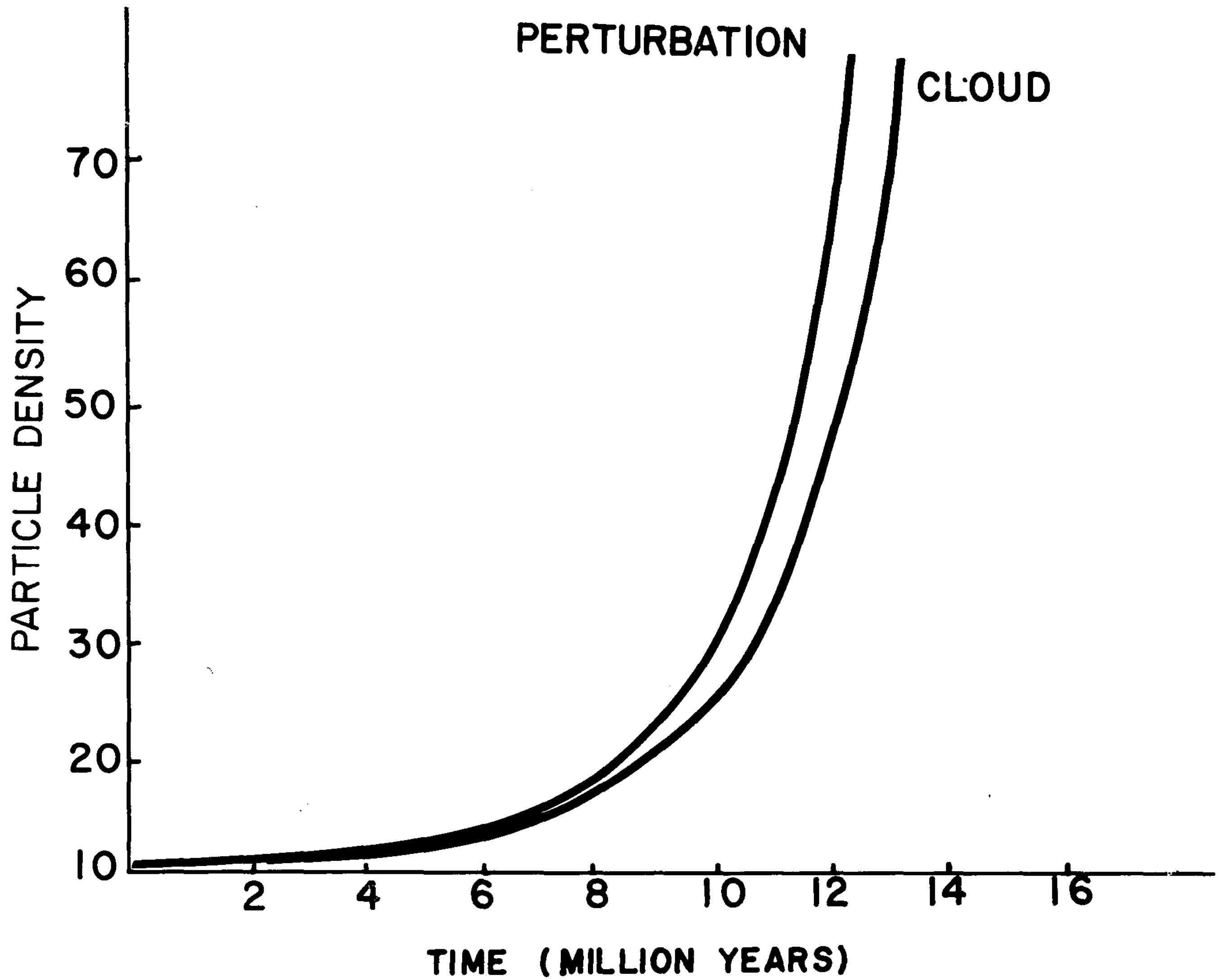

PERTURBATION
CLOUD
PARTICLE DENSITY
70
60
50
40
30
20
10
2
4
6
8
10
12
14
16
TIME (MILLION YEARS)

Fig. 10.--Comparison of the Two-point, Linear and Full Hydrodynamic Solutions

The logarithm (base 10) of the excess density, normalized to its initial value of 10%, is shown as a function of time for an isothermal collapse. The Jeans' length is .5 the perturbation size.

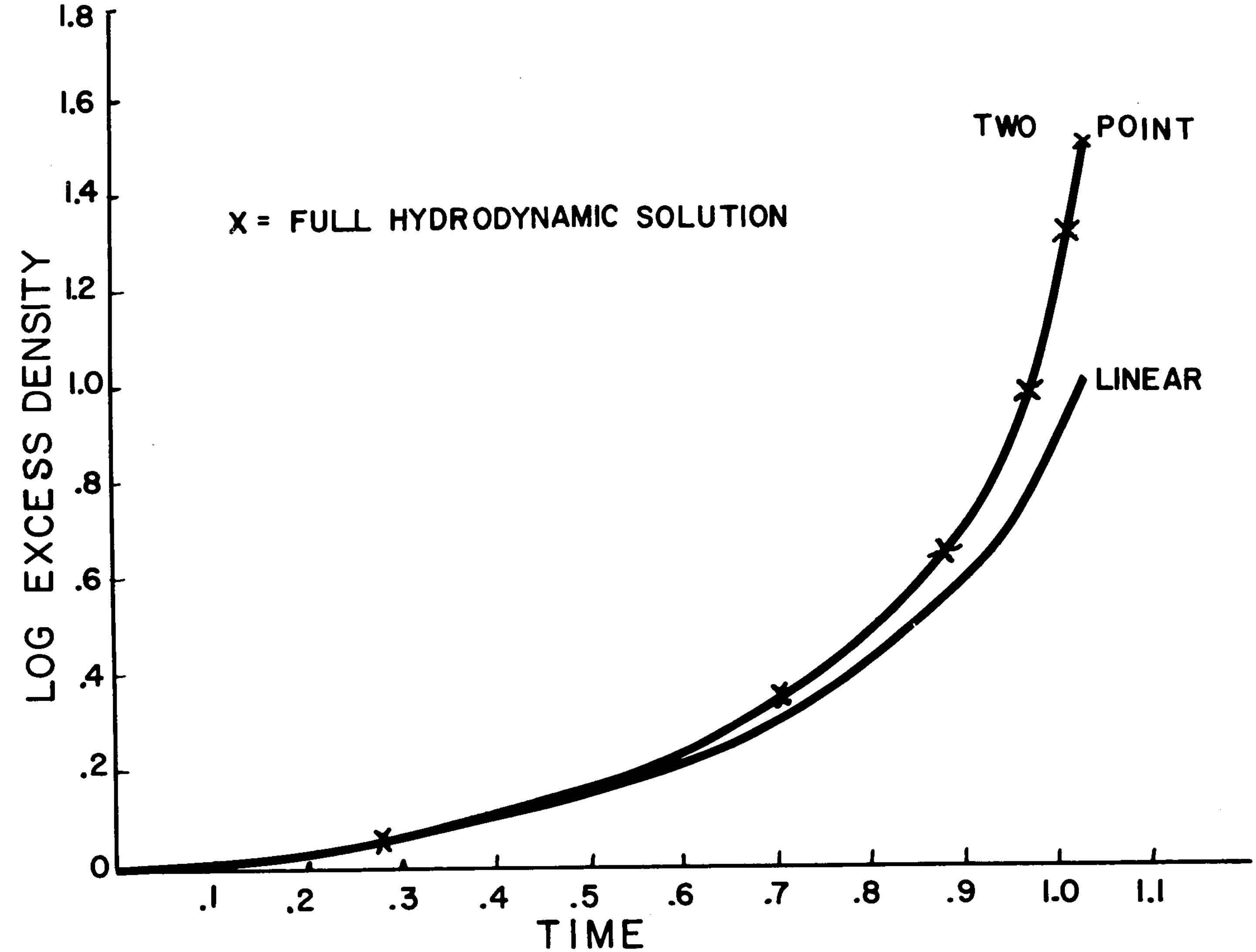

1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
LOG EXCESS DENSITY
X = FULL HYDRODYNAMIC SOLUTION
TWO POINT
LINEAR
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
TIME

Fig. 11.--Schematic Representation of the Change in the Physical Variable During a Collapse Where Oscillations Occur

The top curve represents the ratio of the internal to the external pressure. The middle one is the excess density normalized to its initial value. The bottom one is the ratio of the pressure force to the gravitational force. The abcissa is a time scale.

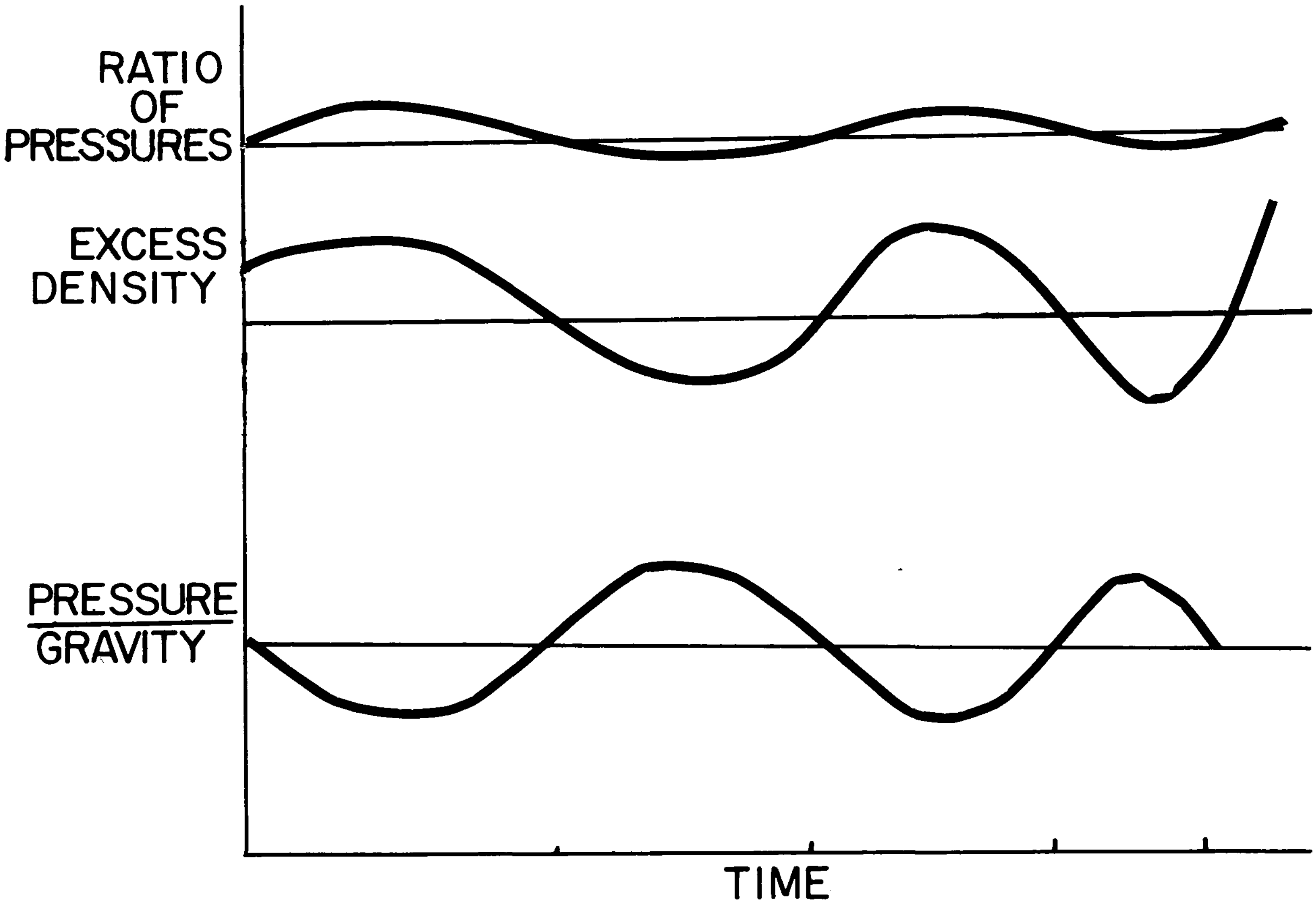

RATIO OF PRESSURES
EXCESS DENSITY
PRESSURE / GRAVITY
TIME

Fig. 12.--Temperature of a Perturbation in an HI Cloud During Collapse

The upper curve is for molecular hydrogen cooling. The lower is for metallic ion cooling. The initial density is 10 particles per cc, and the Jeans' length is .5 the size of the perturbation. The cloud is started in thermal equilibrium. The abcissa is the fractional radius of the main cloud, x.

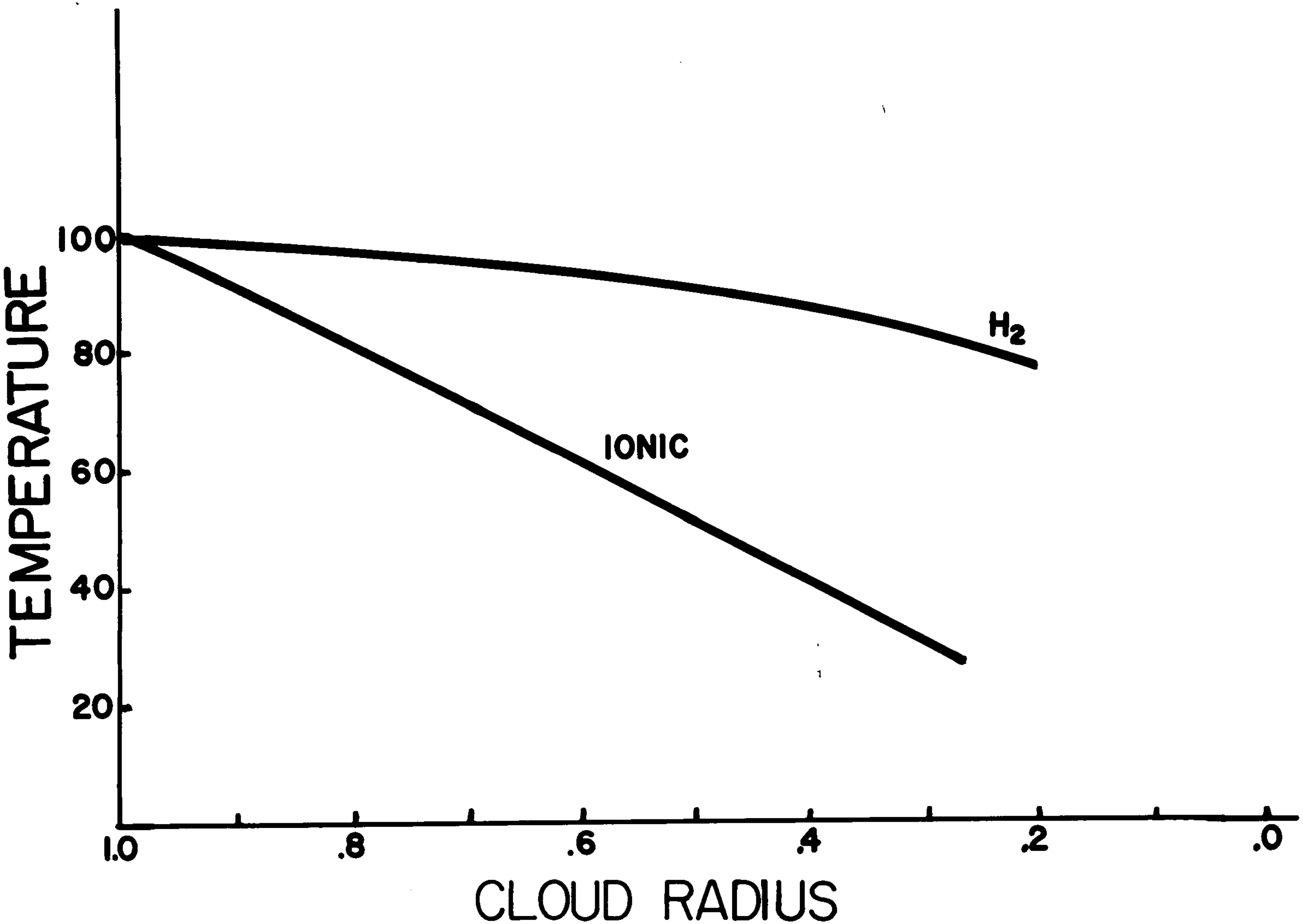

TEMPERATURE
100
80
60
40
20
$H_2$
IONIC
1.0
.8
.6
.4
.2
.0
CLOUD RADIUS

Fig. 13.--Temperature for a Collapsing Cloud of Ionized Hydrogen

The initial density is .001 particles per cc, and the Jeans' length is .5 the perturbation size.

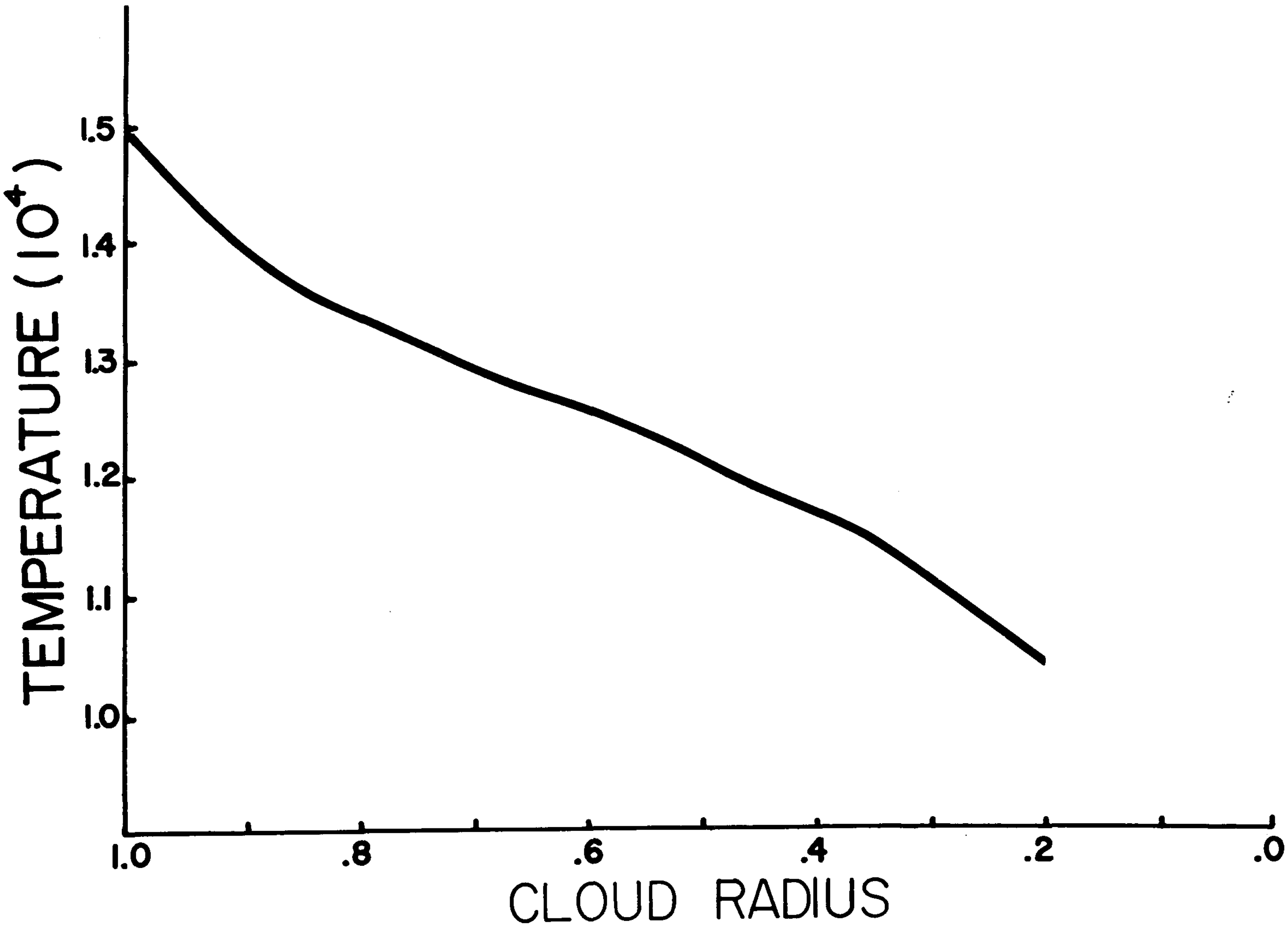

TEMPERATURE ($10^4$)
1.5
1.4
1.3
1.2
1.1
1.0
1.0
.8
.6
.4
.2
.0
CLOUD RADIUS

Fig. 14,--Effect of Rotation on a Perturbation in a Zero-pressure Free-fall Collapse

$\nu_o^2$ is the ratio of the free-fall time to the rotation period, squared. The excess density is normalized to its initial value of 10%. The main cloud is in free-fall.

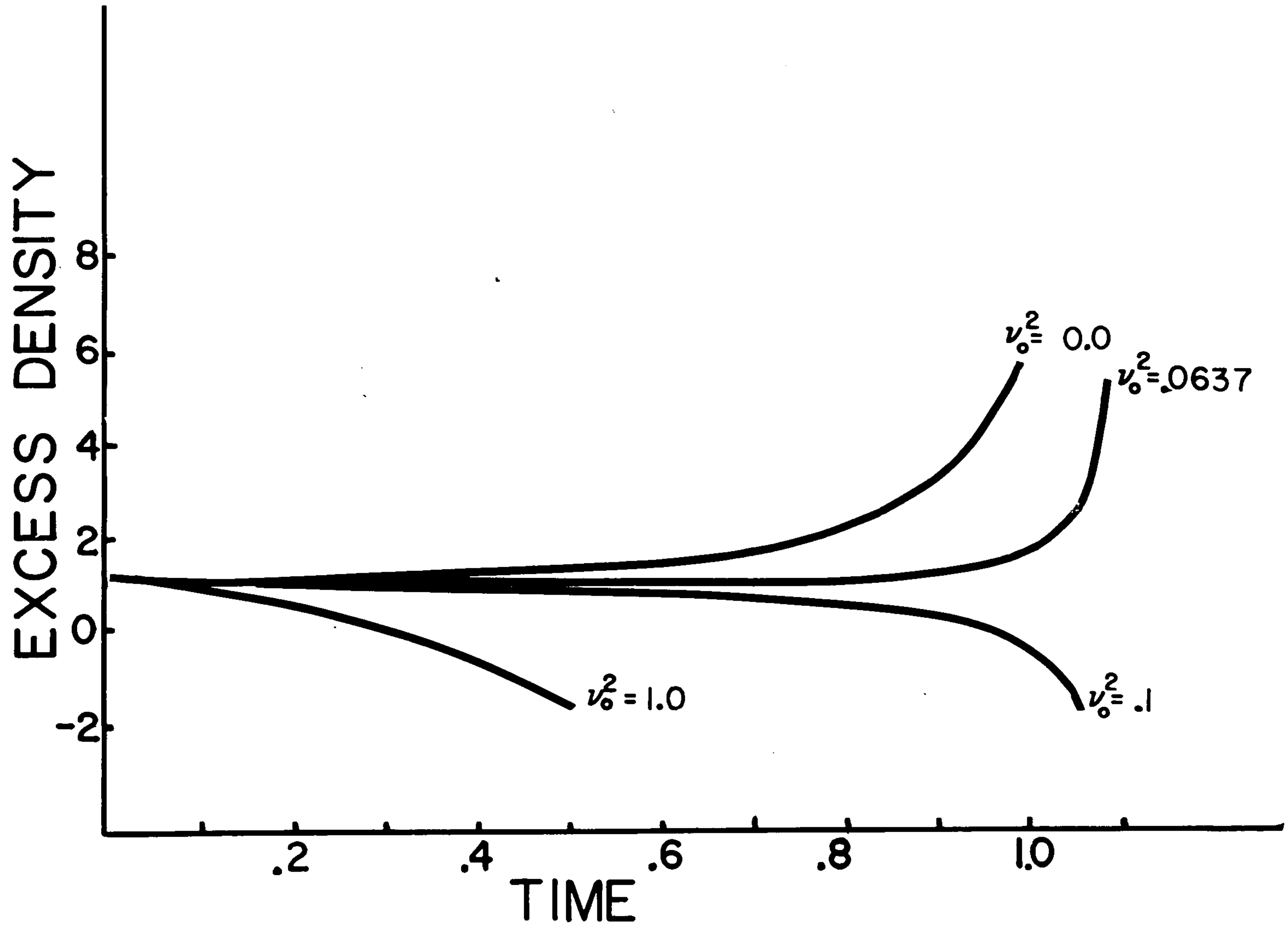

EXCESS DENSITY
8
6
4
2
0
-2
TIME
.2
.4
.6
.8
1.0
$\nu_o^2 = 0.0$
$\nu_o^2 = .0637$
$\nu_o^2 = 1.0$
$\nu_o^2 = .1$

Fig. 15.--Effect of Rotation on a Perturbation for a Finite-pressure Collapse

The curves are normalized to the initial values of the excess density, $E_o$. The main cloud is in free-fall ($B_o = 0$). The collapse is assumed isothermal with the Jeans' length .5 the size of the perturbation.

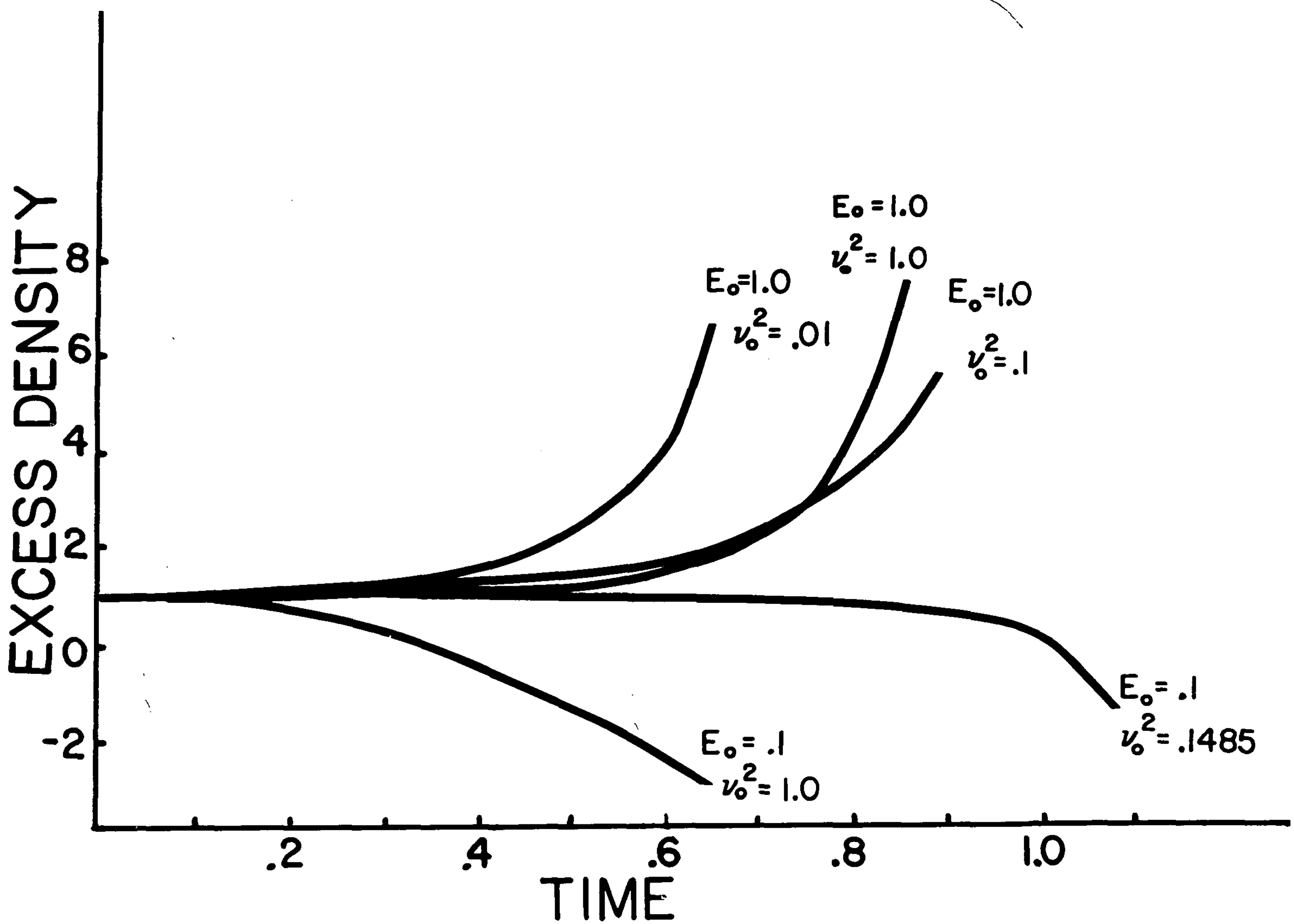
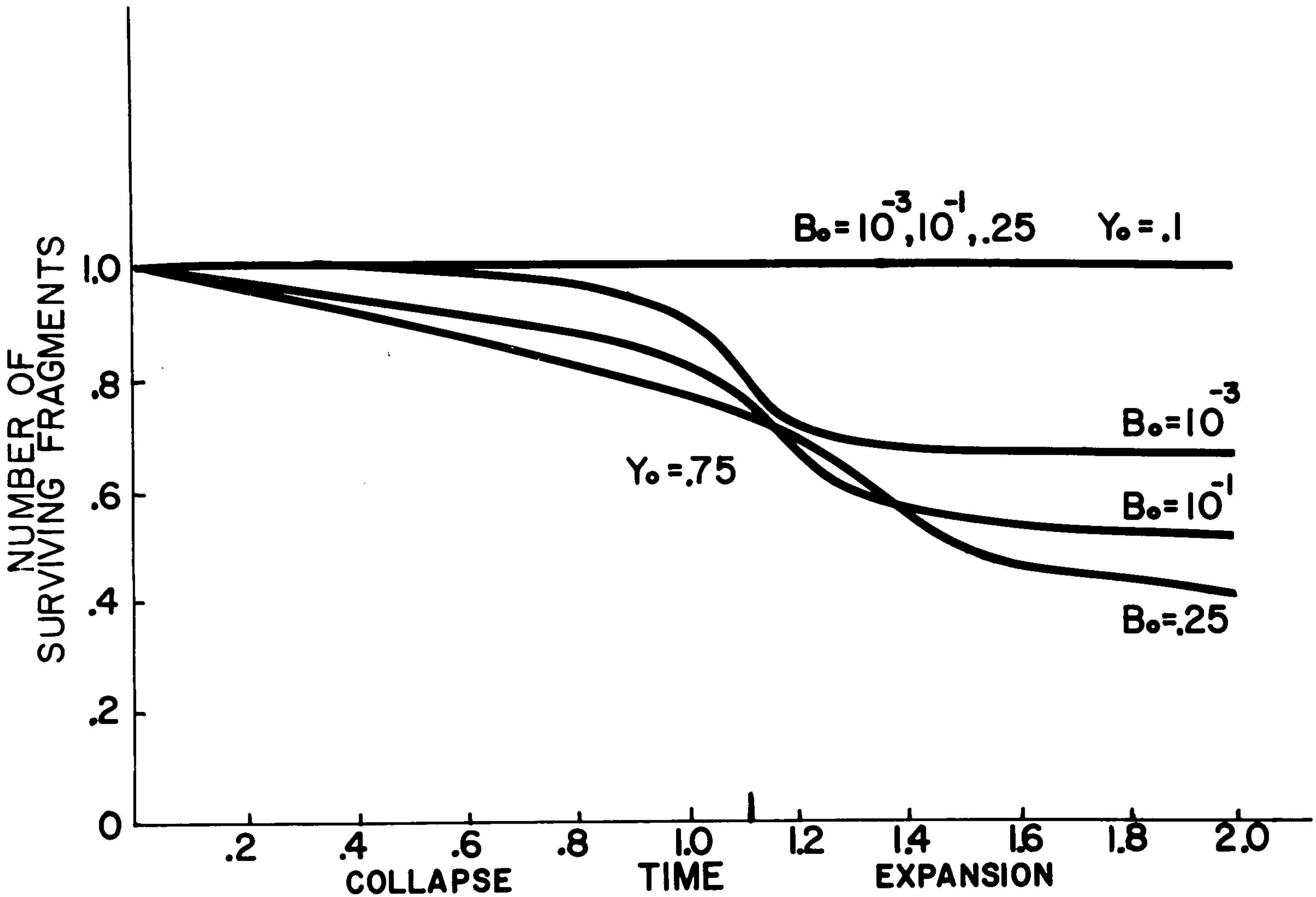

EXCESS DENSITY
8
6
4
2
0
-2
.2
.4
.6
.8
1.0
TIME
$E_o$=1.0 $\nu_o^2$= .01
$E_o$ = 1.0 $\nu_o^2$= 1.0
$E_o$=1.0 $\nu_o^2$= .1
$E_o$= .1 $\nu_o^2$= .1485
$E_o$= .1 $\nu_o^2$= 1.0

Fig. 16.--Fractional Number of Surviving Fragments Illustrating the Effect of Collisions

The different curves show the effect of the ratio of the initial perturbation size to the cloud size ($Y_o$) and the ratio of the effective orbital angular velocity to the free-fall time ($B_o$). $B_o = 0$ is a free-fall collapse.


NUMBER OF SURVIVING FRAGMENTS
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
COLLAPSE
TIME
EXPANSION
Bo = 10^-3, 10^-1, .25
Yo = .1
Yo = .75
Bo = 10^-3
Bo = 10^-1
Bo = .25

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