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ON THE NUMBER OF CRITICAL CONFIGURATIONS OF CHARGES ON AN m-TORUS

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Joseph Bertram Bronder

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I hereby recommend that this dissertation prepared under my direction by Joseph Bertram Bronder entitled On the Number of Critical Configurations of Charges on an m-Torus be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Harvey Cohn — 28 April 1965

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After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

Robert F. De Boer — 4-28-65

Willard D. Pye — 4/28/65

Ronald Jacobowitz — 4/28/65

B. Schweizer — 4/28/65

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ABSTRACT

The problem of enumerating the critical (equilibrium) configurations of n-charges constrained to lie on an m-dimensional torus is investigated. This problem is equivalent to the enumeration of the stationary values of a real-valued function defined on an m(n-1)-dimensional torus.

The law of mutual repulsion is assumed to be a decreasing function of the square of the 2m-dimensional distance between the charges.

The number of critical configurations for the case of two charges is shown to be 2^m. For three charges, the number of critical configurations is between 4^m and 6^m. By appropriately choosing the law of mutual repulsion and the weights of the charges, both the upper and lower bounds may be attained.

The principal tools used in this investigation are some results of M. Morse's topological theory of critical points. A brief development of these results is included.

INTRODUCTION

A problem which arises in connection with an early model of the atom is that of finding the stable configurations of electrons on a conducting sphere (Whyte, 1952). A generalization of this problem will be considered here; namely, the determination of the type and number of critical configurations of N charges constrained to lie on an m-dimensional torus and acted upon by a fairly general law of mutual repulsion. The formal statement of this problem will be deferred until Chapter 3 where it will be formulated in terms of classifying and enumerating the stationary points of a real-valued function defined on a compact, m(N-1)-dimensional manifold. The main tools used to attack this problem are some results of M. Morse's topological theory of critical points.

A method for determining the maxima (minima) of a differentiable, real-valued function, f, defined on an m-dimensional, differentiable manifold, $\mathcal{M}$, is to seek solutions of systems of equations of the form:

$$(*)\qquad \frac{\partial f(T^{-1}(x_1,x_2,\ldots,x_n))}{\partial x_i} = 0, \quad i = 1,2,\ldots,n,$$

where T is a homeomorphism of an open subset of $\mathcal{M}$ onto the open n-dimensional disc:

$$V^n = \left\{ (x_1, x_2, \ldots, x_n) \;\middle|\; x_1^2 + x_2^2 + \ldots + x_n^2 < 1 \right\}.$$

A point, y, of $\mathfrak{M}$ will be called a stationary point of f on $\mathfrak{M}$ if T(y) satisfies a system of the form (*). In general, there will be stationary points which are not extrema of f, since (*) is only a necessary condition for an extremum.

If $\mathfrak{M}$ is compact, there are at least two stationary points of f on $\mathfrak{M}$ since a continuous real-valued function on a compact set has a maximum and a minimum. However, the Morse Theory shows that the total number of stationary points is at least as large as the sum of a set of topological invariants of $\mathfrak{M}$ known as the connectivity numbers (or mod 2 Betti numbers) of $\mathfrak{M}$. In general, this sum will exceed the number two. For example, the sum of the connectivity numbers of the ordinary (2-dimensional) torus is four.

Another consequence of the Morse Theory is that if certain connectivity numbers of $\mathfrak{M}$ do not vanish, then there are necessarily stationary points of f on $\mathfrak{M}$ which do not correspond to extrema of f. Again, the torus provides an example. For, on the torus, there are at least two stationary points of a differentiable, real-valued function which are neither maxima nor minima of that function.

A stationary point, $y = T^{-1}(x_1, x_2, \ldots, x_n)$, of f is said to be non-degenerate if the Hessian of the composite function, $f \circ T^{-1}$, does not vanish at $(x_1, x_2, \ldots, x_n)$. The index of a non-degenerate stationary point is defined to be the number of negative eigenvalues of the matrix:

$$\left[\frac{\partial^2 f(T^{-1}(x_1, x_2, \ldots, x_n))}{\partial x_i \, \partial x_j} \right]$$

If $\mathcal{M}$ is compact and if the stationary points of f are non-degenerate, then the following equation due to M. Morse (Morse, 1925) is valid:

$$\sum_{i=1}^{n} (-1)^i M^i = \sum_{i=1}^{n} (-1)^i R^i ,$$

where M^i denotes the number of stationary points of index i, R^i is the i-th connectivity number of $\mathcal{M}$ and n is the dimension of $\mathcal{M}$. Under certain conditions, this relation may be used to find the exact number of stationary points of a real-valued function on a manifold. An example of this may be found in a paper by H. Cohn (Cohn, 1960). Another example will appear in Chapter 3, below, where the Morse Equation is used to derive an upper bound for the total number of critical configurations of three charges on an m-dimensional torus.

In the following, the first chapter contains a brief review of some of the definitions and results of combinatorial topology.

Chapter 2 is devoted to a proof of the Morse Equation for a compact manifold. The problem of determining the number of critical configurations of charges on an m-dimensional torus is discussed in Chapter 3. In that chapter, the law of mutual repulsion is assumed to be a decreasing function of the square of the 2m-dimensional Euclidean distance between the charges. The number of critical configuration of two charges is shown to be 2^m while, for three charges, the number of critical configuration is shown to be between 4^m and 6^m. By appropriately choosing the law of repulsion and the weights of the charges, both upper and lower bounds may be attained in the three charge case.

CHAPTER 1

TOPOLOGICAL PRELIMINARIES

The Morse approach to the study of stationary points is centered around the topological concepts of absolute and relative cycles and continuous deformations. Thus, by way of introduction, this chapter will be devoted to a discussion of these and other concepts of combinatorial topology. A general and more complete account of the material presented here may be found in the standard texts on combinatorial topology. (See, for example, Seifert and Threlfall, 1934 and Pontryagin, 1952.)

The proofs of theorems presented in this chapter will be given in outline form or omitted completely.

1.1 EUCLIDEAN SIMPLEXES

Let E^n denote Euclidean n-dimensional space, and let $a, b \in E^n$. The set of all points of the form $ta + (1 - t)\, b$, where t is a real number and $0 \leq t \leq 1$, is called the _segment joining a and b_. A set $A \subset E^n$ is _convex_ if, for any two elements $x, y \in A$, each point on the segment joining x and y is also an element of A.

Let A be any subset of E^n. The intersection of all convex sets which contain A (as a subset) is called the <u>convex hull of A</u>. It is easily verified that the intersection of any number of convex sets is also a convex set; i.e., the convex hull of a set is convex. In the sequel, the convex hull of a set, A, will be denoted by $\mathscr{H}A$.

A finite subset $\{a_0, a_1, \ldots, a_q\}$ of E^n is said to be an <u>independent set</u> if $q = 0$ or if $q > 0$ and the vectors $a_1 - a_o$, $a_2 - a_o$, $\ldots$, $a_q - a_o$ are linearly independent; that is, if the vectors $a_1 - a_o$, $a_2 - a_o$, $\ldots$, $a_q - a_o$ generate a linear vector space of dimension q.

Let $A = \{a_0, a_1, \ldots, a_q\} \subset E^n$ be an independent set and let T be the set of all points of the form

$$(1) \qquad x = \sum_{i=0}^{q} \lambda_i a_i$$

where, for $i = 0, 1, \ldots, q$, λ_i is a non-negative real number and

$$\sum_{i=0}^{q} \lambda_i = 1 .$$

By an easy calculation, T may be shown to be convex with $A \subset T$. Therefore, the convex cover, $\mathscr{H}A$, of A has the property that every point $x \in \mathscr{H}A$ may be represented in the form of equation (1). Moreover, the independence of A guarantees that this representation is unique. The numbers $\lambda_0, \lambda_1, \ldots, \lambda_q$ are called the <u>barycentric coordinates</u> of x. On the other hand, every point of E^n which admits a representation in the form of equation (1) is an element of $\mathcal{CL}A$, and hence $\mathcal{CL}A = T$. (see Eggleston, 1963, p. 4)

The convex hull of an independent set uniquely determines that set. For, if A and B are two independent sets and $\mathcal{CL}A = \mathcal{CL}B$, then A = B. (Pontryagin, 1952, p. 10)

<u>Definition 1.1</u>

A set $S \subset E^n$ is called a <u>Euclidean q-simplex</u> or simply a <u>Euclidean simplex</u> if there exists an independent set A of q + 1 points such that $S = \mathcal{CL}A$. The elements of the unique set A are called the <u>vertices</u> of S, and q is called the <u>dimension</u> of S.

The Euclidean 0, 1, 2 and 3 simplexes are, respectively points, straight line segments, triangles, and tetrahedroms.

In the sequel, whenever a symbol of the type $\mathcal{CL}A$ is used to denote a Euclidean simplex, it is to be understood that A is the set of vertices of $\mathcal{CL}A$.

Clearly, any non-empty subset of an independent set is also independent. Hence, if $\mathcal{CL}A$ is a Euclidean simplex and $\emptyset \neq B \subset A$, then $\mathcal{CL}B$ is a Euclidean simplex.

Definition 1.2

A Euclidean q-simplex, $\sigma\!\!\!\mathcal{L}B$, is a q-face or simply a face of a Euclidean simplex, $\sigma\!\!\!\mathcal{L}A$, if B is a subset of A. $\sigma\!\!\!\mathcal{L}B$ is a proper face of $\sigma\!\!\!\mathcal{L}A$ if $\sigma\!\!\!\mathcal{L}B$ is a face of $\sigma\!\!\!\mathcal{L}A$ and $B \neq A$.

When considering collections of Euclidean simplexes, it is desirable that the various Euclidean simplexes "fit together" in a certain prescribed manner. Two Euclidean simplexes, S and T, are said to be properly situated if $S \cap T = \emptyset$ or $S \cap T$ is a face of S and a face of T.

Definition 1.3

A collection, K, of Euclidean simplexes is a complex if

(i) Every two elements of K are properly situated.

(ii) Every face of an element of K is also an element of K.

The set of all faces of a Euclidean simplex, S, (including S itself) is an example of a complex. The set of all proper faces of S is also a complex. These complexes will be denoted by $\dot{S}$ and S^-, respectively; i.e.,

$$\dot{S} = \left\{ t \;\middle|\; t \text{ is a face of } S \right\},$$

and

$$S^- = \left\{ t \;\middle|\; t \text{ is a proper face of } S \right\}.$$

Unless otherwise stated, all complexes will be assumed to be finite. (i.e., contain only a finite number of simplexes.)

The union of all simplexes in a complex, K, is called a polyhedron and will be denoted by $|K|$. K is called a triangulation of $|K|$.

1.2 CONE CONSTRUCTION; NORMAL SUBDIVISION; PRISM CONSTRUCTION

Three methods of constructing a new complex from a given complex will now be introduced. The first of these, the cone construction, is introduced primarily to simplify the other two constructions. The second construction, normal subdivision, provides a method of triangulating a polyhedron into simplexes of arbitrarily small diameter. The third construction, the prism construction, will be used later to relate two functions defined on the same polyhedron and to relate a complex to its normal subdivision.

The Cone Construction

Let $F \subset E^n$ and let $a \in E^n$. The point, a, is said to be in general position with respect to F if $a \in F$ and if for any two distinct points $x \in F$, $y \in F$, the segment joining a and x and the segment joining a and y have exactly one point in common, namely the point a.

Definition 1.4

Let $F \subset E^n$, and let $a \in E^n$ such that a is in general position with respect to F. The set-theoretic union of all segments joining a and points of F is called the <u>cone with vertex a and base F</u>, and is denoted by aF. Thus, aF is the set of all points of the form $at + (1 - t)b$, where $0 \leq t \leq 1$ and b is an element of F.

Let $S = \mathcal{S}\{a_0, a_1, \ldots, a_q\}$ be a Euclidean q-simplex, and let a be in general position with respect to S. Then $\{a, a_0, a_1, \ldots, a_q\}$ is necessarily an independent set, and the cone, aS, is the convex cover of this set; i.e.,

$$aS = \mathcal{S}\left\{a, a_0, \ldots, a_q\right\}.$$

(See Pontryagin, 1952, p. 45) Thus, aS is a Euclidean q+1 simplex.

<u>Definition 1.5</u>

Let K be a complex and let a be in general position with respect to $|K|$. Then the <u>cone complex with vertex a and base K</u>, denoted by aK, is the set of all cones of the form aS where S is a Euclidean simplex of K. In set notation,

$$aK = \left\{ aS \mid S \in K \right\} .$$

It may be shown (Pontryagin, 1952, p. 46) that aK is a complex, K is a subset of aK, and $|aK| = a|K|$.

Normal Subdivision

Let $S = \sigma\mathcal{L}\{a_0, a_1, \ldots, a_q\}$ be a Euclidean q-simplex. The (unique) point in S whose barycentric coordinates are each equal to 1/(q+1) is called the barycenter of S, and will be denoted by b(S). Thus,

$$b(S) = \frac{1}{q+1} \sum_{i=1}^{q} a_i .$$

It may be shown (Pontryagin, 1952, pp. 43-44) that b(S) is in general position with respect to S^- and that $b(S)\left|S^-\right| = S$.

For any complex, K, let K^q denote the set of all simplexes of K whose dimensions are at most q:

$$K^q = \left\{ S \mid S \text{ is an r-simplex and } r \leq q \right\} \cap K.$$

It is easy to show that K^q is a complex; i.e., K^q satisfies (i) and (ii) of Definition 1.3.

The normal or barycentric subdivision of a complex, K, will be denoted by ${}_1K$. Since the purpose of normal subdivision is to obtain a triangulation whose simplexes have arbitrarily small diameters, and since the diameters of the simplexes of K^o are as small as possible already, the normal subdivision of K^o is defined to be K^o. The 1-simplexes of K may be replaced by smaller 1-simplexes by dividing each 1-simplex in half. In other words, each 1-simplex, $S = \sigma\mathcal{L}\{a_0, a_1\} \in K^1$, may be replaced by the two 1-simplexes,

$\mathcal{S}\left\{\frac{1}{2}(a_0 + a_1),\ a_0\right\} = b(S)\ \{a_0\}$ and $\mathcal{S}\left\{\frac{1}{2}(a_0 + a_1),\ a_1\right\} = b(S)\ \{a_1\}$.

Therefore, the normal subdivision of K^1 is defined to be the set of all 1-simplexes of the form $b(S)\{a\}$, together with the faces of these simplexes and all 0-simplexes in K^0, where $S \in K^1$ and a is a vertex of S. (In general, there will be 0-simplexes in K^0 which are not faces of 1-simplexes of K^1. Therefore, K^0 must be included in the subdivision of K^1 to ensure that $|{}_1K^1| = |K^1|$.) To extend this procedure to the complexes K^2, K^3, etc., suppose $q \geq 1$ and that the normal subdivision of K^{q-1} has been carried out. Then the proper faces of each simplex of K^q are already subdivided, and hence, as suggested by the method of subdividing K^1, the procedure for subdividing K^q is to construct cones of the form $b(S)T$, where S is a q-simplex of K^q and T is a simplex in the subdivision of some proper face of S. To ensure that such cones are defined and that no points are added to or deleted from the polyhedron $|K|$ by such a procedure, we require not only that ${}_1K^{q-1}$ be a complex such that $|{}_1K^{q-1}| = |K^{q-1}|$, but also that, for any complex, L, such that $L \subset K^{q-1}$, ${}_1L$ is a complex, ${}_1L \subset {}_1K^{q-1}$, and $|{}_1L| = |L|$. Then, in particular, for any q-simplex, S, of K^q, ${}_1S^-$ is a complex and $|{}_1S^-| = |S^-|$. Hence, the set of all cones of the form b(S)T, together with the faces of these simplexes, is the complex $b(S)({}_1S^-)$, where T is a simplex in the subdivision of a proper face of S. Moreover,

$$|b(S)({}_1S^-)| = b(S)|S^-| = S.$$

The normal subdivision of K^q is now carried out by adding, to ${}_1K^{q-1}$, all complexes of the form $b(S)({}_1S^-)$, where S is a q-simplex of K^q. Thus, for each q-simplex, S, of K^q, a complex whose polyhedron is S is added to ${}_1K^{q-1}$.

This construction is formally described in the following definition.

<u>Definition 1.6</u> (Normal Subdivision of a Complex)

Let K be a complex.

(i) ${}_1K^o = K^o$.

(ii) If $q > 0$, and if, for any complex, L, such that $L \subset K^{q-1}$, ${}_1L$ is a complex, $|{}_1L| = |L|$, and ${}_1L \subset {}_1K^{q-1}$, then

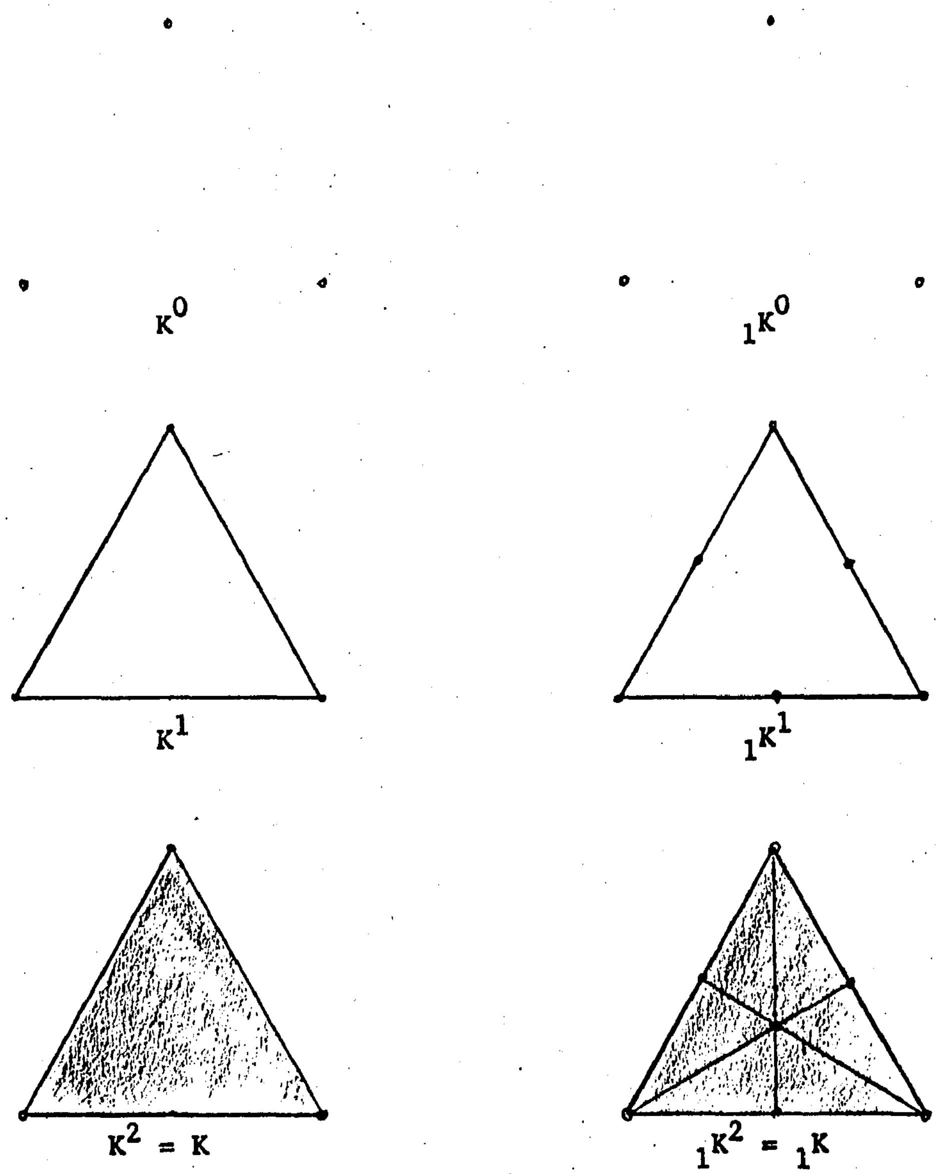
$$ {}_1K^q = ({}_1K^{q-1}) \cup \left(\bigcup_{S \in K^q} b(S)({}_1S^-) \right). $$

It may be shown that, for all $q \geq 0$, ${}_1K^q$ is a complex, $|{}_1K^q| = |K^q|$, and, for every complex L such that $L \subset K^q$, ${}_1L \subset {}_1K^q$. In particular, ${}_1K$ is a complex and $|{}_1K| = |K|$.

The normal subdivision of a complex consisting of a single Euclidean 2-simplex and its faces is shown in Figure 1.

Let ${}_rK$ denote the complex obtained from the complex K by normally subdividing K r-times. To be more precise, let ${}_oK = K$ and, for any $r \geq 0$, let ${}_{(r+1)}K = {}_1({}_rK)$.

For any subset, A, of E^n, let $\rho(A)$ denote the diameter of A.



<u>Figure 1</u>. The normal subdivision of a complex consisting of a 2-simplex and its faces.

The following theorem shows that if $|K|$ is a polyhedron, then the normal subdivision process may be used to obtain triangulation of $|K|$ whose simplexes have arbitrary small diameters.

Theorem 1.1.

Let K be a complex, and let d be the maximum diameter of the simplexes of K. Let q be the maximum dimension of the simplexes of K. Then the diameter of each simplex of ${}_1K$ is at most $qd/(q+1)$; i.e.,

$$\rho(S) \leq \left(\frac{q}{q+1}\right) d$$

for all $S \in {}_1K$. It follows that, for any integer $r \geq 1$,

$$\rho(T) \leq \left(\frac{q}{q+1}\right)^r d$$

for all $T \in {}_rK$.

The proof of this theorem may be found in Pontryagin, (1952, pp. 50-55) or in Seifert and Threlfall (1934, pp. 49-50).

The Prism Construction

Throughout the remainder of this chapter, E^n will be regarded as a subspace of E^{n+1} and the unit vector of E^{n+1} which is normal to every vector of E^n, will be denoted by e.

For any two sets, A and B, such that $A \subset E^n$ and $B \subset E$, the symbol $A \times B$ will be interpreted either as the usual cartesian product of A with B or as the set of all points of the form

$$x + \tau e$$

where $x \in A$ and $\tau \in B$.

Let I denote the closed unit interval; i.e., the set of all $x \in E$ such that $0 \leq x \leq 1$.

Let K be any complex such that $K \subset E^n$. For each $\tau \in I$, let $K \cdot \tau$ be the complex obtained from K by translating each simplex of K a distance τ off the hyperplane E^n in the direction of the vector e. In set notation,

$$K \cdot \tau = \left\{ S \times \{\tau\} \;\middle|\; S \in K \right\}.$$

The prism construction is introduced to provide a method of triangulating the set $|K| \times I$ in such a way that the triangulation contains the complexes $K \cdot 0$ and $({}_rK) \cdot 1$ as subsets. This triangulation will be denoted by $K \cdot I^r$, where the r refers to the index in $({}_rK) \cdot 1$. Some applications of this construction will appear in Section 1.6.

Let S be a Euclidean simplex. The set $S \times I$ is called the <u>prism with base S</u>. (If S is a 2-simplex, then $S \times I$ has the shape of the common optical prism.) The point $c(S) = b(S) + (1/2)e$ is called the <u>center</u> of the prism, $S \times I$. The top and bottom of this prism are, respectively, the sets $S \times \{0\}$ and $S \times \{1\}$, and the sides are prisms and are of the form $T \times I$ where T is a simplex of $S^{\cdot}$. It can be shown (see Pontryagin, 1952, pp. 73-74), that c(S) is in general position with respect to the set of all points which form the top, bottom and sides of this prism, namely, $(|S^-| \times I) \cup (S \times \{1\}) \cup (S \times \{0\})$, and that the cone with vertex $c(S)$ and base $(|S^-| \times I) \cup (S \times \{1\}) \cup (S \times \{0\})$ is the prism $S \times I$.

As in the case of the normal subdivision of a complex, the triangulation of $K \times I$ will be defined by induction, by considering the complexes K^o, K^1, etc.

If S is a 0-simplex, then $S \times I$ is already a 1-simplex and therefore, need not be tirangulated. Therefore, $K^o \cdot I^r$ is defined to be the set of all simplexes of the form $S \times I$, $S \in K^o$, together with the simplexes of $K^o \cdot 0$ and $K^c \cdot 1 = ({}_rK^o) \cdot 1$.

If $S = \sigma\{a_o, a_1\}$ is a 1-simplex, then the simplexes $S \times \{0\}$, $\{a_o\} \times I$, $\{a_1\} \times I$, together with their faces and the simplexes of $({}_r\dot{S}) \cdot 1$ form a subdivision of the top, bottom and sides of $S \times I$. Hence, the set of all cones of the form $c(S)T$ constitute a subdivision of $S \times I$, where T is a simplex of the subdivided top, bottom or sides of $S \times I$. But these cones and their faces form the cone complex:

$$c(S)\left[(S^- \cdot I^r) \cup \left\{S \times \{0\}\right\} \cup (({}_r\dot{S}) \cdot 1)\right].$$

Moreover,

$$\left| c(S)\left[(S^- \cdot I^r) \cup \left\{S \times \{0\}\right\} \cup (({}_rS) \cdot 1)\right] \right|$$

$$= c(S)\left[(|S^-| \times I) \cup (S \times \{0\}) \cup (S \times \{1\})\right]$$

$$= S \times I.$$

Hence, $K^1 \cdot I^r$ is defined to be the union of all such cone complexes for which $S \in (K^1 - K^o)$ and, as in the case of normal subdivision, all simplexes of $K^o \cdot I^r$ must be included to ensure that $|K^1 \cdot I^r| = |K^1| \times I$. Therefore,

$$K^1 \cdot I^r = K^o \cdot I^r \cup \left(\bigcup_{S \in (K^1 - K^o)} c(S) \left[(S^- \cdot I^r) \cup \left(\{S\} \cdot 0 \right) \cup ({}_r\dot{S}) \cdot 1 \right] \right)$$

This procedure is extended to $|K^q| \times I$, for $q > 1$, by induction. Assume $K^{q-1} \cdot I^r$ is defined in such a way that for each q-simplex, S of K^q, $\dot{S} \cdot I^r$ is a complex, $\dot{S} \cdot I^r \subset K^{q-1} \cdot I^r$, and $|S^- \cdot I^r| = |S^-| \times I$. Then the simplexes of $({}_r\dot{S}) \cdot 1$ and $S^- \cdot I^r$, together with the simplexes of $S \times \{0\}$, form a subdivision of the top, bottom and sides of $S \times I$, and the construction of $K^q \cdot I^r$ proceeds as in the case of $q = 1$.

The construction of $K \cdot I^r$ is formally described in the next definition.

Definition 1.7 (Prism Construction)

Let K be a complex.

(i) $K^0 \cdot I^r = (K^0 \cdot 0) \cup (K^0 \cdot 1) \cup \{S \times I \mid S \in K^0\}$

(ii) Let $q \geq 1$ and suppose $K^{q-1}.I^r$ is a complex such that:

(a) $|K^{q-1}.I^r| = |K^{q-1}| \times I$

(b) If M is a complex such that $M \subset K^{q-1}$, then $M.I^r$ is a complex and $M.I^r \subset K^{q-1}.I^r$.

(c) $K^{q-1}.0 \subset K^{q-1}.I^r$ and $({}_rK^{q-1}) \cdot 1 \subset K^{q-1} \cdot I^r$.

Then

$$K^q \cdot I^r = (K^{q-1} \cdot I^r) \cup \left(\bigcup_{S \in K^q - K^{q-1}} c(S) \left[(S^- \cdot I^r) \cup \left| \{S\} \cdot 0 \right| \cup ({}_rS \cdot 1) \right] \right).$$

The proof that $K^q.I^r$ is a complex such that $|K^q.I^r| = |K^q| \times I$, $K^q.0 \subset K^q.I^r$, $({}_1K^q) \cdot 1 \subset K^q \cdot I$ for all $q \geq 0$ may be found (with a slight modification) in Pontryagin, 1952, pp. 74-77.

The triangulation of a prism whose base is a 2-simplex is shown in Figure 2.

1.3 SINGULAR SIMPLEXES AND SINGULAR CHAINS

To extend the ideas of the preceeding sections to curved geometric figures and to more general topological spaces, the concept of a singular simplex will now be introduced.

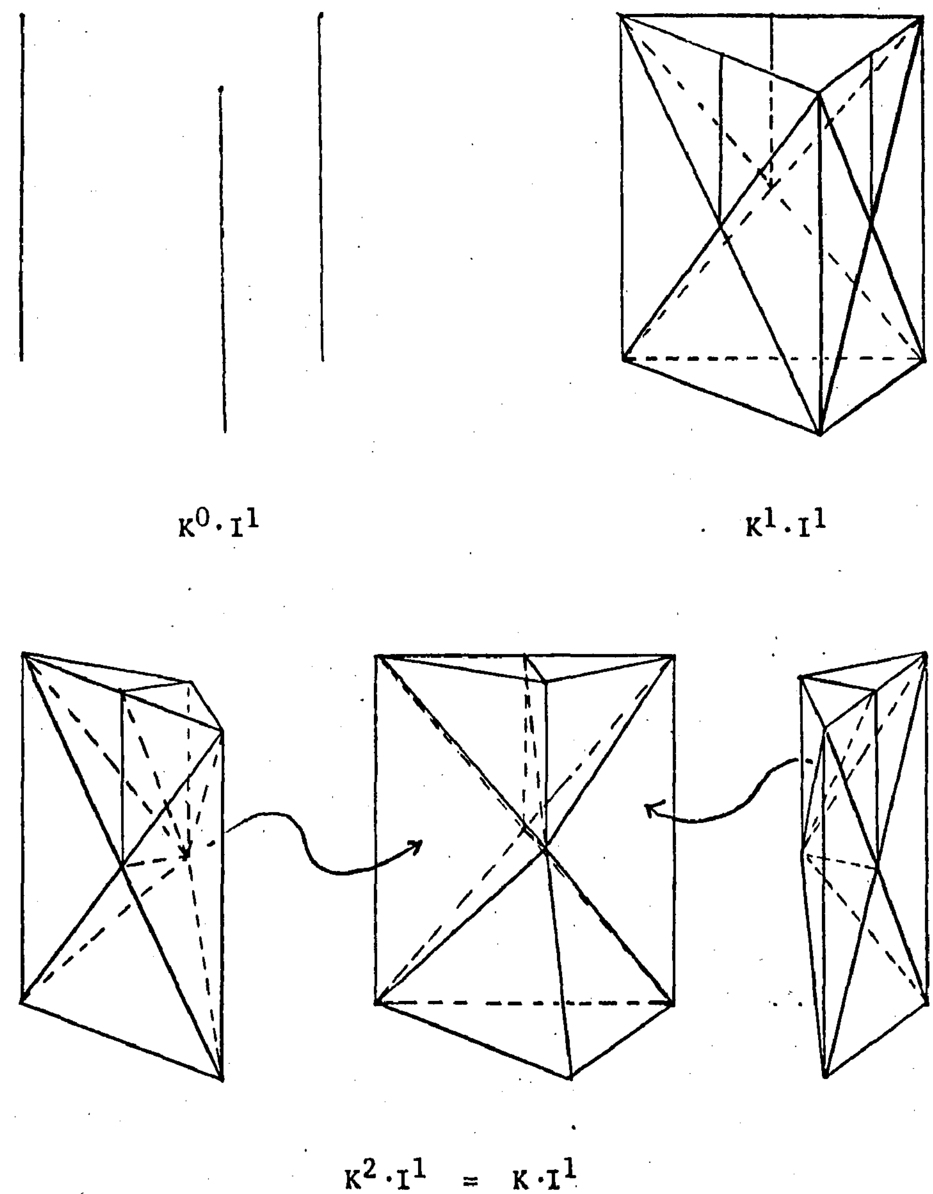


Figure 2 Triangulation of a prism with a 2-simplex for a base.

Heuristically speaking, a singular q-simplex is a continuous function defined on a Euclidean q-simplex of arbitrary size, shape and location in E^n. To present a more precise definition of a singular simplex, let $\mathcal{J}_q$ be the set of all continuous functions whose domains are Euclidean q-simplexes. For each $S \in \mathcal{J}_q$, let $\overline{S}$ denote the domain of S. (By the definition of $\mathcal{J}_q$, $\overline{S}$ is a Euclidean q-simplex.)

Let R be the following relation defined on $\mathcal{J}_q$: for any two elements S and T of $\mathcal{J}_q$, let $(S,T) \in R$ if there exists a non-singular affine transformation f on $\overline{S}$ to $\overline{T}$ such that the composite function, $T \circ f$, is equal to S; that is, if there exists a transformation of the form:

$$f(x)_i = \sum_{i=1}^{q} \alpha_{ij} x_j + b_i, \quad i=1,2,\ldots,n,$$

such that for each $x \in \overline{S}$, $T(f(x)) = S(x)$, where, (i) $f(x)_i$ and x_i denote, respectively, the components of $f(x)$ and x relative to the usual basis of E^n, (ii) the α_{ij}'s and b_i's are real numbers, and, (iii) the α_{ij}'s form a non-singular matrix. It follows from the properties of affine transformations that R is an equivalence relation.

<u>Definition 1.8</u> (Singular Simplex)

The equivalence classes induced on $\mathcal{J}_q$ by R are called <u>singular q-simplexes</u>, or simply <u>singular simplexes</u>.

In the remainder of this discussion, representatives of these classes, rather than the classes themselves, will be called singular q-simplexes. Thus, for our purposes, a singular q-simplex is a continuous function whose domain is a Euclidean q-simplex. Furthermore two singular q-simplexes, S and T, will be regarded as equal if there exists an affine mapping, f, such that f maps $\overline{S}$ onto $\overline{T}$ and $T \circ f = S$. The range of a singular simplex S will be denoted by $|S|$.

Let S be a singular q-simplex and let $\overline{S} = \mathcal{H}\{a_o, a_1, \ldots, a_q\}$. Then, S is a degenerate singular simplex if there exists an affine transformation f of $\overline{S}$ onto itself such that $S(f(x)) = S(x)$ for all $x \in \overline{S}$, and $(f(a_o), f(a_1), \ldots, f(a_q))$ is an odd permutation of the vertices, $(a_o, a_1, \ldots, a_q)$.

Definition 1.9 (Singular Chain)

(i) A singular q-chain is a finite collection of singular q-simplexes. For all q, $\emptyset$ (the empty set) is a singular q-chain.

(ii) If C is a singular chain, then the set-theoretic union of the ranges of all non-degenerate singular simplexes of C will be denoted by $|C|$. Thus, if C' denotes the set of all non-degenerate simplexes in C, then

$$|C| = \bigcup_{S \in C'} |S|.$$

(iii) If C_1 and C_2 are singular q-chains, then the <u>sum</u> of C_1 and C_2, denoted by $C_1 + C_2$, is defined by

$$C_1 + C_2 = (C_1 \cup C_2) - (C_1 \cap C_2).$$

By Definition, if C is a singular chain, $C + C = \emptyset$.

If $C = \{S_1, S_2, \ldots, S_r\}$ is a singular q-chain, then C may be written as the sum

$$C = \sum_{i=1}^{r} \{S_i\}.$$

We shall sometimes omit the brackets and represent C as the "formal" sum:

$$C = \sum_{i=1}^{r} S_i.$$

In particular, the chain $C = \{S\}$ shall sometimes be written as $C = S$.

Let G be any set. A singular q-chain, C, is said to be <u>in G</u> if the range of each simplex of C is a subset of G.

1.4 THE BOUNDARY OF A SINGULAR CHAIN

For any function f, let $f|A$ denote the restriction of f to the set A. In set notation,

$$f\big|A = f \cap \left\{(x,y) \;\middle|\; x \in A\right\}.$$

Clearly, if S is a singular simplex and t is a p-face of $\overline{S}$, then $S\big|t$ is a singular p-simplex. $S\big|t$ will sometimes be referred to as a p-face of the singular simplex S.

For any finite set $A = \left\{a_o, a_1, \ldots, a_q\right\}$, let $\left\{a_o, a_1, \ldots, \hat{a}_i, \ldots, a_q\right\}$ denote the set obtained from A by removing the element a_i. Thus

$$\left\{a_o, a_1, \ldots, \hat{a}_i, \ldots, a_q\right\} = A - \left\{a_i\right\}.$$

<u>Definition 1.10</u> (The Boundary of a Singular Chain)

(i) Let S be a singular q-simplex with $q \geq 1$. The <u>boundary</u> of the singular q-chain, S , denoted by ∂S, is the set of all q-1 faces of S. Thus,

$$\partial S = \sum_{i=0}^{q} S\big| \mathcal{S}\left\{a_o, a_1, \ldots, \hat{a}_i, \ldots, a_q\right\}.$$

If q = 0, then $\partial S = \emptyset$.

(ii) Let $C = \sum_{i=1}^{r} S_i$ be a singular q-chain. Then the boundary of C, denoted by ∂C, is defined by

$$\partial C = \sum_{i=1}^{r} \partial S_i .$$

(iii) $\partial \emptyset = \emptyset$.

Clearly, if C is a singular q-chain, then ∂C is a singular q-1 chain. Also, if C_1 and C_2 are two singular q-chains, then

$$\partial C_1 + \partial C_2 = \partial (C_1 + C_2).$$

An important property of the boundary operator, ∂, is given in the following theorem.

Theorem 1.2

Let C be a singular q-chain. Then

$$\partial \partial C = \emptyset.$$

Proof

It suffices to consider a singular q-chain consisting of the single singular q-simplex S. Let

$$\bar{S} = \mathcal{L}\left\{a_0, a_1, \ldots, a_q\right\}.$$

Then

$$\partial\partial S = \sum_{i=0}^{q} \partial S \;\Big|\; \mathcal{L}\left\{a_0, a_1, \ldots, \hat{a}_i, \ldots, a_q\right\}$$

$$= \sum_{i=1}^{q} \sum_{j=0}^{i-1} S \;\Big|\; \mathcal{L}\left\{a_0, a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_i, \ldots, a_q\right\}$$

$$+ \sum_{i=0}^{q-1} \sum_{j=i+1}^{q} S \;\Big|\; \mathcal{L}\left\{a_0, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_q\right\}.$$

and since each q-2 face of S appears exactly once in each summation,

$$\partial\partial S = \emptyset.$$

Q.E.D.

1.5 CYCLES AND CONNECTIVITY NUMBERS

Two singular q-chains C_1 and C_2 will be regarded as equivalent if $\left|C_1 + C_2\right| = \emptyset$; i.e., if $C_1 + C_2 = \emptyset$ or $C_1 + C_2$ contains only degenerate singular q-simplexes. Equivalence of singular q-chains will be denoted by the symbol $\doteq$.

Definition 1.11 (Absolute Cycle)

A singular q-chain, C, is an (absolute) q-cycle, or simply a cycle, if

$$\partial C \doteq \emptyset.$$

From theorem 1.2, it follows that, if a q-chain, C, is a boundary, (i.e., if there exists a singular q+1 chain D such that $\partial D = C$), then C is a cycle. The subject of combinational topology is centered around the existence of cycles which are not boundaries.

In the remainder of this section, let G be any set.

Definition 1.12 (Bounding Cycles; Homologous Chains)

(i) A q-cycle, C, is said to be a bounding cycle (in G) if there exists a singular q+1 chain, D, in G such that

$$\partial D \doteq C$$

(ii) Two singular q-chains, C_1 and C_2 , are homologous in G, written

$$C_1 \sim C_2 \text{ (in G)},$$

if $C_1 + C_2$ is a bounding cycle in G. In particular, if C is a bounding cycle in G, Then

$$C \sim \emptyset \text{ (in G)}.$$

Definition 1.13 (Connectivity Numbers)

(i) Let $K = \left\{C_1, C_2, \ldots, C_r\right\}$ be a set of q-cycles in G. The set K is said to be homologically independent (in G), if for every non-empty subset, $\left\{C_{i_1}, C_{i_2}, \ldots, C_{i_s}\right\}$, of K, the corresponding chain,

$$C_{i_1} + C_{i_2} + \cdots + C_{i_s},$$

is not a bounding cycle (in G).

(ii) The maximum number of homologically independent q-cycles (in G) is called the q-th connectivity number of G. The q-th connectivity number of G may be infinite.

(The q-th connectivity number of a set G is also known as the q-th Betti-number (mod 2) of G.)

In the remainder of this section, let Z be any subset of G. Two singular q-chains, C_1 and C_2, in G are said to be <u>equal (mod Z)</u>, written

$$C_1 \doteq C_2 \text{ (mod Z)},$$

if $|C_1 + C_2| \subset Z$.

<u>Definition 1.14</u> (Relative Cycle)

A q-chain C in G is a (relative) <u>cycle (mod Z)</u>, if $\partial C \doteq \emptyset$ (mod Z).

<u>Definition 1.15</u> (Relative Boundaries and Relative Homology)

(i) Let C be a q-cycle (mod Z). Then C is a <u>bounding cycle (mod Z in G)</u> if there exists a q+1 chain, D, in G such that

$$\partial D \doteq C \text{ (mod Z)}.$$

(ii) Let C_1 and C_2 be singular q-chains in G. Then <u>C_1 is homologous to C_2, (mod Z in G)</u>, written

$$C_1 \sim C_2 \text{ (mod Z in G)},$$

if $C_1 + C_2$ is a bounding cycle (mod Z in G).

Definition 1.16 (Relative Connectivity Numbers)

(i) Let $K = \left\{C_1, C_2, \ldots, C_r\right\}$ be a set of q-cycles (mod Z) in G. Then K is homologically independent (mod Z in G), if, for every non-empty subset,

$$\left\{C_{i_1}, C_{i_2}, \ldots, C_{i_s}\right\}$$

of K, the corresponding chain,

$$C_{i_1} + C_{i_2} + \ldots + C_{i_s} \quad ,$$

is not a bounding cycle (mod Z in G).

(ii) The maximum number of homologically independent (mod Z in G) q-cycles (mod Z) is called the q-th connectivity number (mod Z) of G.

1.6 DEFORMATIONS; NORMAL SUBDIVISION; SIMPLICIAL APPROXIMATION

The topics of this section form the core of the topological techniques which will be used to prove the theorems in Chapter 2. These topics are presented from an algebraic viewpoint in the modern texts on topology. However, the approach adopted here will be more geometric then algebraic.

Deformations

Let H be a subset of a topological space G and let f be a function on $H \times I$ into G. For each $\tau \in I$, let f_τ be that function on H such that for each $x \in H$,

$$f_\tau(x) = f(x,\tau).$$

The function, f, is called a continuous deformation of H in G if f is continuous on $H \times I$ to G, and f_0 is the identity map of H.

Let $C = \sum_{i=1}^{r} S_i$ be a q-chain in H and let f be a contimuous deformation of H in G. Then, for each i, the composite function $f_1 \circ S_1$ is continuous on $\overline{S}_i$ to G, and is therefore, a singular q-simplex. Hence, $\{f_1 \circ S_1, f_1 \circ S_2, \ldots, f_1 \circ S_r\}$ is a singular q-chain in G. Let D denote this chain. Then we say that f deforms the chain C into the chain D. The next theorem shows that these two chains are related through "connecting" chains in G.

Theorem 1.3

Let $C = \sum_{i=1}^{r} S_i$ be a singular q-chain in $H \subset G$ and let f be a continuous deformation of H in G. Let

$$D = \sum_{i=1}^{r} f_i \circ S_i \quad .$$

Then there exists a q+1 chain, U, in G and a q-chain, V, in $f\left[\left|\partial C\right| \times I\right]$ (the image of $\left|\partial C\right| \times I$ under f) such that

$$\partial U \doteq C + D + V \; .$$

Outline of Proof

For a given i, let A_1, A_2, ..., A_α be the Euclidean q+1 simplexes of $\dot{\bar{S}}_i . I^0$ and let B_1, B_2, ..., B_β be the Euclidean q-simplexes of $\bar{S}^-{}_i . I^0$. Let g^i be that function defined on $\bar{S}_i \times I$ such that

$$g^i(x + \tau e) = f(S_i(x), \tau)$$

for all $x \in \bar{S}_i$ and $\tau \in I$.

Clearly g^i is continuous on $S_i \times I$, and so the functions $g^i \big| A_j$; j = 1, 2, ..., α, are singular q+1 simplexes and the $g^i \big| B_j$'s, j = 1, 2, ..., β, are singular q-simplexes. Let

$$u_i = \sum_{j=1}^{\alpha} g^i|A_j \ ,$$

and let

$$v_i = \sum_{j=1}^{\beta} g^i|B_j .$$

(see Figure 3)

Now, it can be shown that each Euclidean q-simplex of $\overline{S}_i.I^0$ which does not lie on the top, bottom, or a side of $\overline{S}_i \times I$ is a face of an even number of q+1 simplexes of $\overline{S}_i.I^0$. (See for example, Alexandroff and Hoff, 1935, p. 197, eq. 2) Hence, the only q-faces of the singular q+1 simplexes of u_j which are not faces of an <u>even</u> number of the $g^i|A_j$'s are the simplexes

$$g^i|S_i \times \{0\} = g^i_0 = S_i,$$

$$g^i|S_i \times \{1\} = g^i_1 = f_1 \circ S_i,$$

and the simplexes of v_i. Therefore,

$$\partial u_i = S_i + f_1 \circ S_i + v_i .$$

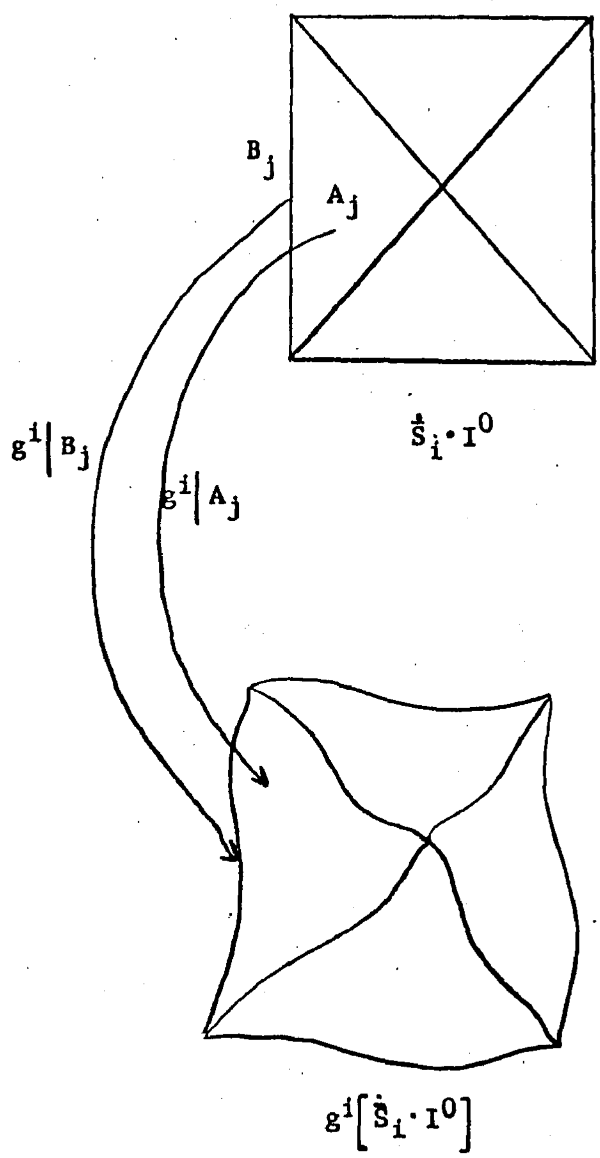


Figure 3 Construction of connecting chains.

The conclusions of the theorem are now satisfied by setting

$$U = \sum_{i=1}^{\alpha} u_i$$

and

$$V = \sum_{i=1}^{\beta} v_i.$$

(For a detailed proof see Seifert and Threlfall, 1934, p. 31)

If C is a cycle, then $|\partial C| = \emptyset$ and so $v \doteq \emptyset$. Therefore, an important consequence of this theorem is the following corollary.

<u>Corollary 1</u>

If C is a cycle, then

$$C \sim D \text{ (in } G).$$

A similar result holds for relative cycles:

<u>Corollary 2</u>

If C is a cycle (mod Z) and if $f\left[|\partial C| \times I\right] \subset Z$ then

$$C \sim D \text{ (mod } Z \text{ in } G).$$

Normal Subdivision of a Chain

The normal subdivision of a complex will now be used to subdivide a singular chain.

Definition 1.17

(i) Let S be a singular simplex and let $A_1, A_2, \ldots, A_\alpha$ be the Euclidean q-simplexes of ${}_k\dot{\bar{S}}$. Then

$$_kS = \sum_{i=1}^{\alpha} S|A_i,$$

(ii) Let $C = \sum_{i=1}^{n} S_i$ be a singular q-chain. Then

$$_kC = \sum_{i=1}^{n} {}_kS_i.$$

It may be shown that if S is a degenerate singular simplex, then ${}_kS = \emptyset$. Hence, by Theorem 1.1, every singular chain may be replaced by a singular chain whose simplexes are "arbitrarily small" in some sense.

The requirement of definition 1.6 that ${}_1L$ be a subset of ${}_1K$ whenever L is a complex and a subset of K suggests the following theorem.

Theorem 1.4

If C is a singular chain, then

$$\partial {}_kC = {}_k\partial C.$$

The proof of this theorem may be found in Seifert and Threlfall, (1934, p. 30).

In the modern approach to algebraic topology, the invariance of certain groups (i.e., the Betti groups) of a polyhedron under normal subdivision is demonstrated. The next theorem and its corollaries may be regarded as the geometric analog to these results.

Theorem 1.5

Let C be a singular chain. Then there exist chains U and V such that,

$$\partial U \doteq C + {}_kC + V,$$

where U is a chain in $|C|$ and V is a chain in $|\partial C|$.

The proof of this theorem parallels that of Theorem 1.3 with $\dot{\bar{S}}_i \cdot I^0$ and $\bar{S}^-{}_i \cdot I^0$ replaced by $\dot{\bar{S}}_i \cdot I^k$ and $\bar{S}_i^- \cdot I^k$, and $g^i(x + \tau e) = f(x,\tau)$ replaced by $g^i(x + \tau e) = S_i(x)$. (See Seifert and Threlfall, 1934, p. 30)

Corollary 1

If C is a cycle, then

$$D \sim C \text{ (in } |C|).$$

Corollary 2

If C is a cycle (mod Z) then

$$D \sim C \text{ (mod } Z \text{ in } |C|).$$

Simplicial Approximation

If C is a singular chain such that $|C|$ is a subset of a polyhedron $|L|$, where L is a complex, then C may be "approximated" by a singular chain D with the property that each singular simplex of D is a function onto a Euclidean simplex of L. The method for obtaining an approximating chain will now be outlined and chains "connecting" the chains C and D will be derived. The 1-dimensional analog of this process is the approximation of a curve by polygonal segments. The connecting chains are obtained by first subdividing the curve into arcs which are small enough to be approximated by straight lines lying in the polyhedron, and then deforming each arc into its straight line approximation.

Let K and L be two complexes. A continuous function, f, on K to L is called a simplicial mapping of the complex K into the complex L if, for each Euclidean simplex $S = \mathcal{H}\{a_0, \ldots, a_q\} \in K$, there exists a Euclidean simplex $T \in L$ such that f maps the vertices of S onto the vertices of T and maps each point $x = \sum_{i=0}^{q} \lambda_i a_i$ of S onto the point $f(x) = \sum_{i=0}^{q} \lambda_i f(a_i)$ of T, where the λ_i's are the barycentric coordinates of x.

Let $\{a_0, a_1, \ldots, a_q\}$ be an independent set. The set of all points of the form

$$x = \sum_{i=1}^{q} \lambda_i a_i,$$

where $\sum_{i=1}^{q} \lambda_j = 1$ and the λ_j's are <u>strictly positive</u>, is called an <u>open simplex</u>. If S is a Euclidean simplex, the open simplex corresponding to S will be denoted by S'.

Let K be a complex and let a be a vertex of one of the simplexes of K. The <u>star of a</u>, denoted by St(a), is defined by

$$St(a) = \bigcup_{S \in M} S',$$

where

$$M = \left\{ S \,\middle|\, S \in K \text{ and } a \text{ is a vertex of } S \right\}.$$

Definition 1.18

Let K and L be complexes and let g be a continuous function on $|K|$ into $|L|$. Then a simplicial mapping, f, is a simplicial approximation to g if, for each vertex a of a simplex of K, St(a) is mapped by g into a subset of St(f(a)).

Theorem 1.6

Let K be a finite complex and let g be a continuous function on $|K|$ to $|L|$, where L is a complex. Then there exists a function f and an integer k such that f is a simplicial mapping of the k-fold normal subdivision of K into L and f is a simplicial approximation of g.

The proof of this theorem may be found in Hilton and Wylie (1962, p. 37).

Theorem 1.7

Let $C = \sum_{i=1}^{r} S_i$ be a singular q-chain such that for some complex L, C is in $|L|$.

Then there exists a singular chain $D = \sum_{i=1}^{s} S_i$ and connecting chains u in $|L|$ and v in $|L|$ such that for each singular simplex T_i in D, T_i is a Euclidean simplex in L and

$$\partial u = C + D + v.$$

To prove this theorem, the domains of the simplexes of C are first chosen in such a way that the collection of the $\bar{S}_i$'s together with all their faces form a complex and $g = \bigcup_{i=1}^{r} S_i$ is a continuous function. This is always possible since r is finite and hence, the $\bar{S}_i$'s may be chosen to be disjoint. Let K denote this complex. Next, g is approximated by a simplicial mapping f of the k-fold normal subdivision of K into L, which is possible by Theorem 1.6.

Let $A_1, A_2, \ldots, A_s$ be the Euclidean q-simplexes of ${}_kK$. Then

$${}_kC = \sum_{j=1}^{s} g|A_j,$$

and so, by Theorem 1.5, there exist singular chains u_1 and v_1 such that

$$\partial u_1 = C + {}_kC + v_1.$$

Let $f|A_j = T_j$, $j = 1, 2, \ldots, s$, and let

$$D = \sum_{j=1}^{s} T_j.$$

Then, since f is a simplicial mapping into L, $|T_j| = f[A_j]$ is an Euclidean simplex of L. The proof is now completed by "deforming" $\sum_{j=1}^{s} g|A_j$ into D:

Let

$$h(x,\tau) = \tau f(x) + (1-\tau)g(x).$$

Then $h_o(x) = g(x)$ and $h_1(x) = f(x)$, and, by the construction used in Theorems 1.3 and 1.5, there exist connecting chains u_2 and v_2 such that

$$\partial u_2 = \sum_{j=1}^{s} g|A_j + D + v_2.$$

Let $u = u_1 + u_2$ and $v = v_1 + v_2$. Then

$$\partial u \doteq C + D + v.$$

Now, u_1 and v_1 are in $|L|$, by Theorem 1.5. For any $x \in |C|$, $f(x)$ and $g(x)$ are elements of the same simplex of $|L|$, since St(a) is mapped by g into St(f(a)) for each vertex $a \in K^0$. (See Seifert and Threlfall, 1934, pp. 107-108) Therefore, by convexity, $h(x,\tau)$ is an element of $|L|$, and so, u_2 and v_2 are in $|L|$. Thus, u and v are in $|L|$.

<u>Corollary 1</u>

If C is a cycle, then

$$D \sim C \text{ (in } |L|).$$

(See Seifert and Threlfall, 1934, p. 111)

This corollary states that every singular cycle C in a polyhedron $|L|$ may be approximated by a "straight line" cycle of L which is homologous in $|L|$ to C.

CHAPTER 2

THE MORSE EQUATION ON A COMPACT MANIFOLD

The main theorem which we wish to prove in this chapter states that the Morse Equation holds for a real-valued function defined on a compact differentiable manifold, $\mathfrak{M}$, provided the second partial derivatives of the function (in terms of the local coordinates of $\mathfrak{M}$) are continuous and the stationary points of the function are non-degenerate.

Since an n-dimensional manifold is "locally homeomorphic" to an open subset of E^n, the first two sections of this chapter are devoted to the behavior of a real-valued function defined on the n-dimensional open disc, V^n. The local results of sections 2.1, and 2.2, are combined and extended to a differentiable manifold in section 2.3. The main result appears in section 2.4.

The material presented in this chapter may be found in Seifert and Threlfall (1938).

2.1 STATIONARY POINTS

Let f be a real-valued function whose second partial derivatives exist and are continuous in V^n. For any $\gamma \in E$, let $\{f < \gamma\}$ be the set of all x such that $f(x) < \gamma$. The sets $\{f \leq \gamma\}$, $\{f = \gamma\}$, $\{f > \gamma\}$ and $\{f \geq \gamma\}$ are defined in a similar way.

Definition 2.1

A point $x = (x_1, x_2, \ldots, x_n) \in V^n$ is a stationary point of f if each of the first partial derivatives, $\partial f/\partial x_1$, $\partial f/\partial x_2$, ..., $\partial f/\partial x_n$, vanish at x. A real number, μ, is a stationary value of f if $f^{-1}(\mu)$ contains a stationary point of f.

Let $\bar{x}$ be a non-stationary point of f. By a well-known theorem of ordinary differential equations (Coddington and Levinson, 1955, pg. 22, Theorem 7.1), there exists an open subset Q of E^n, a real number $b > 0$, and a function $h(x,t)$ such that $\bar{x} \in Q$, $h(x,t)$ is continuous on $Q \times [0,b]$ and, for each $x \in Q$ and $t \in [0,b]$, $h(x,t)$ satisfies the system of ordinary equations:

$$\text{(1-a)} \qquad \frac{dh_i(t)}{dt} = -\frac{\partial f}{\partial x_i}\Big(h_1(t), \ldots, h_n(t)\Big), \; i=1,2,\ldots,n,$$

together with the initial conditions

$$\text{(1-b)} \qquad h_i(0) = x_i, \; i=1,2,\ldots,n.$$

Clearly h deforms Q into some subset of E^n.

Since the partial derivatives of f are continuous and one of these partial derivatives is non-zero at $\bar{x}$, there exists an open neighborhood of $\bar{x}$ such that for each x in that neighborhood, one of the partial derivatives does not vanish at x. Let G denote the intersection of Q with that neighborhood. Then, for each $x \in G$,

$$\frac{df(h(x,t))}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dh_i}{dt}$$

$$= -\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2$$

$$< 0.$$

Thus, for each $x \in G$, as t increases, $f(h(x,t))$ decreases. Therefore, h deforms G into some subset of E^n such that $\{f \leq f(\bar{x})\} \cap G$ is deformed (in $\{f \leq f(\bar{x})\}$) into a subset of $\{f < f(\bar{x})\}$, and $\{f < f(\bar{x})\} \cap G$ is deformed (in $\{f < f(\bar{x})\}$) into a subset of $\{f < f(\bar{x})\}$. Therefore, by corollary 2 of theorem 1.3, the connectivity numbers (mod $\{f < f(\bar{x})\}$) of $\{f \leq f(x)\} \cap G$ are zero.

Now, suppose $\bar{x}$ is a stationary point rather than a non-stationary point. Then the above argument breaks down if we try to use the solution of (1-a) and (1-b) to show that the connectivity numbers (mod $\{f < f(\bar{x})\}$) of $\{f < f(\bar{x})\} \cap G$ are zero, where G is any open neighborhood of $\bar{x}$. For, $h(\bar{x},t) = \bar{x}$ is the unique solution of (1-a) subject to the initial conditions $h_i(0) = \bar{x}_i$, $i = 1, 2, \ldots, n$. Therefore, for any open set G such that $x \in G$, $h(\bar{x};t)$ does not deform $\{f < f(\bar{x})\} \cap G$ into $\{f < f(\bar{x})\}$. This suggests the possibility that the local topological behavior of a stationary point is different from that of a non-stationary point. To a certain extent, this is the case.

A stationary point, x, of f is <u>isolated</u> if there exists an open set G which contains x and contains no other stationary points of f.

<u>Definition 2.2</u> (Type Number of an Isolated Stationary Point)

Let x be an isolated stationary point of f, and let G be an open neighborhood of x which contains no other stationary point of f. Then the <u>q-th type number</u> of x, denoted by $m^q(x)$, is defined to be the q-th connectivity number (mod $\{f < f(x)\}$) of $\left(\{f < f(x)\} \cap G\right) \cup \{x\}$.

It may be shown (Seifert and Threlfall, 1938, pg. 30-31) that the q-th type number of a stationary point is actually a property of the point and the function f, and is independent of the set G.

An isolated maximum and an isolated minimum (if they exist in V^n) provide two examples of isolated stationary points of a differentiable, real-valued function defined on V^n. They also provide examples of stationary points which are topologically different from non-stationary points, as is demonstrated by the next two theorems.

<u>Theorem 2.1</u>

Let x be an isolated minimum of f.

Then

$$m^0(x) = 1,$$

and

$$m^q(x) = 0, \text{ for } q > 0.$$

Proof

Let G be an open neighborhood of x which contains no other stationary points of f. Then

$$\left(\left\{f < f(x)\right\} \cap G\right) \cup \{x\} = \{x\},$$

and

$$\left\{f < f(x)\right\} \cap G = \emptyset.$$

Clearly, the only cycle (either absolute or relative) in $\{x\}$ is the 0-cycle $C = \left\{(b,x)\right\}$, where b is an arbitrary point in E^m. Moreover, $\left\{(b,x)\right\}$ is independent (mod $\emptyset$ in $\{x\}$).

Q.E.D.

Theorem 2.2

Let x be an isolated maximum of f.

Then

$$m^n(x) = 1,$$

and
$$m^q(x) = 0, \text{ for } q \neq n.$$

<u>Proof</u>

Let G be an open neighborhood of x which contains no other stationary points of f. Then, since x is an isolated maximum,

$$\{f < f(x)\} \cap G = G - \{x\} \text{ and } \{f < f(x)\} \cap G \cup \{x\} = G.$$

To complete the proof, it suffices to show that the q-th connectivity number (mod $(G - \{x\})$) of G is 0, for $q \neq n$, and 1, for $q = n$.

Let C be a q-cycle (mod $(G - \{x\})$) in G. Let K be a complex such that $|K| = G$. K may be chosen in such a way that for some Euclidean simplex $A \in K$, $x \in A'$ (A' is the open simplex corresponding to A) and A has no points in common with the compact subset, $|\partial C|$, of $G - \{x\}$, (see Alexandroff and Hopf, 1935, p. 143). Let D, u and v be chains in G such that, for each singular simplex $t \in D$, t is a simplicial mapping into K, and $\partial u = C + D + v$. (See Theorem 1.7)

Now, the simplexes of v are constructed by connecting those points of $|\partial C|$ and $|\partial D|$ which lie in a common Euclidean simplex of K. Since none of the points of $|\partial C|$ are in A, it follows that v is in $G - A' \subset G - \{x\}$. Therefore,

$$C \sim D \ (\mathrm{mod}\ (G - \{x\})\ \text{in}\ G)$$

For each singular simplex $S \in D$, S is a simplicial mapping of the Euclidean q-simplex $\bar{S}$ onto the Euclidean p-simplex $|S| \in K$. Since the vertices of $\bar{S}$ are mapped onto the vertices of S, it follows that $p \leq q$. Hence, if $q < n$, then S is degenerate or, since the Euclidean n-simplex A is the only simplex of K which contains x, $|S| \subset G - \{x\}$. Thus, for $q < D$, D is in G which implies:

$$m^q(x) = 0, \text{ for } q < n.$$

On the other hand, if $q > n$, then S is ncessarily degenerate (K contains no Euclidean simplexes of dimension greater than n) and it follows that $D \doteq \emptyset \subset G - \{x\}$. Thus,

$$m^q(x) = 0, \text{ for } q > n.$$

Now, let T denote the identity map of A onto itself. Then, since $x \notin (A - A')$, $\{T\}$ is a cycle (mod $(G - \{x\})$) in G. Moreover, $\{T\}$ is independent (mod $(G - \{x\})$ in G) since there are no non-degenerate simplexes in G. Thus, $m^n(x) \geq 1$.

Suppose D is not in $G - \{x\}$. Then, there exists a simplex $S \in D$ such that $|S| = A$. Since T is the identity on A and S is a simplicial mapping, $T \circ S = S$ and S is an affine transformation on $\bar{S}$ onto $A = \bar{T}$. Thus, S and T are the same singular simplex. (see Definition 1.8). Since $S = T$ is the only simplex of D which is not in $G - \{x\}$, it follows that

$$D = \{T\} \ (\mathrm{mod}\ (G - \{x\})).$$

Thus,

$$m^n(x) = 1.$$

Q.E.D.

As is well known, a stationary point, $\bar{x}$, of f is an isolated minimum if the matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \right] \tag{2}$$

is positive definite and is an isolated maximum if this matrix is negative definite. One of the methods used to determine if (2) is positive or negative definite is to reduce the quadratic form

$$Q(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \ (x_i - \bar{x}_i)(x_j - \bar{x}_j)$$

to the normal form:

$$(3) \qquad \sum_{i=1}^{n} \nu_i \, \xi_i^2$$

by an affine transformation $\xi = \xi(x - \bar{x})$, where $\nu_i^2 = 1$, $i = 1,2,\ldots,n$. Then (2) is positive definite if $\nu_i = 1$, $i = 1, 2, \ldots, n$ and is negative definite if $\nu_i = -1$, $i = 1, 2, \ldots, n$. The number of negative coefficients of the normal form (3) is called the <u>index</u> of the matrix (2).

<u>Definition 2.3</u>

(i) A stationary point $\bar{x}$ of f is <u>non-degenerate</u> if the Hessian

$$\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]$$

does not vanish at $\bar{x}$.

(ii) The index of a non-degenerate stationary point $\bar{x}$ of f is the index of the matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})\right]$$

Theorems 2.1 and 2.2 show that the following equation is valid if $\bar{x}$ is an isolated maximum or minimum of f:

$$m^q(\bar{x}) = \delta_i^q ,$$

where i is the index of $\bar{x}$, and δ_i^q is the "Kronecker delta". The next theorem shows that this equation holds for all non-degenerate stationary points.

Theorem 2.3

Let $\bar{x}$ be a non-degenerate stationary point of f. Then $\bar{x}$ is an isolated stationary point of f and

$$m^q(\bar{x}) = \delta_i^q,$$

where i is the index of $\bar{x}$.

Proof

Suppose $\bar{x}$ is not isolated. Then there exists a sequence $\{u^1, u^2, \ldots\}$ of stationary points converging to $\bar{x}$ with $u^j \neq \bar{x}$ for any $k \geq 1$. By the Mean Value Theorem (Apostol, 1957, p. 117), for each $k \geq 1$, there exists $v^k \in E^n$ lying on the segment joining u^k and $\bar{x}$ such that

$$0 = \frac{\partial f}{\partial x_\alpha}(u^k) = \sum_{\beta=1}^{n} \frac{\partial f(v^k)}{\partial x_\beta \partial x_\alpha}(u^k_\beta - x_\beta), \quad \alpha=1,2,\ldots,n.$$

Hence, since $u^k \neq \bar{x}$,

$$\det \left[\frac{\partial^2 f(v^k)}{\partial x_\beta \partial x_\alpha} \right] = 0,$$

for all $k \geq 1$. Clearly, the sequence $\{v^1, v^2, \ldots\}$ converges to $\bar{x}$, and so by continuity,

$$\det \left[\frac{\partial^2 f(\bar{x})}{\partial x_\beta \partial x_\alpha} \right] = 0,$$

which is impossible since $\bar{x}$ is non-degenerate. Thus, $\bar{x}$ is isolated.

Now by Taylor's theorem, there exists a function $R(x)$ such that for all x in some neighborhood, N, of $\bar{x}$,

$$(4) \qquad f(x) - f(\bar{x}) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f}{\partial x_\beta \partial x_\alpha}(\bar{x})(x_\alpha - \bar{x}_\alpha)(x_\beta - \bar{x}_\beta) + R(x)$$

where the second derivatives of $R(x)$ are continuous in N,

$$\text{(5a)} \qquad \lim_{|x - \bar{x}| \to 0} \left\{ \frac{R(x)}{|x - \bar{x}|^2} \right\} = 0,$$

and

$$\text{(5b)} \qquad \lim_{|x - \bar{x}| \to 0} \left\{ \frac{\partial R/\partial x_\nu (x)}{|x - \bar{x}|} \right\} = 0, \quad \nu = 1,2,\ldots,n.$$

We may assume that the quadratic form in (4) is in the normal form,

$$\sum_{r=1}^{n} \upsilon_r x_r^2 ,$$

where $\upsilon_r = -1$, $r = 1, 2, \ldots, i$, $\upsilon_r = 1$, $r = i + 1, \ldots, n$. (Since an affine transformation is a special case of a Homeomorphism, this assumption will have no effect on the connectivity numbers involved.) We may further assume that $f(\bar{x}) = 0$.

Let A be the set of all $x \in E^n$ such that

$$x_{i+1} = x_{i+2} = \ldots = x_n = 0,$$

and let B be the set of all $x \in E^n$ such that

$$x_1 = x_2 = \ldots = x_i = 0.$$

To complete the proof, a deformation F will be constructed which, for some neighborhood, U, of $\bar{x}$, deforms $(U \cap \{f < 0\}) \cup \{\bar{x}\}$ into A in such a way that $U \cap \{f<0\}$ remains in $\{f<0\}$ during the deformation.

Let

$$Q(x) = \sum_{r=1}^{n} \nu_r x_r, \text{ and } T(x) = \frac{\sum_{r=1}^{n} \nu_r x_r^2}{|x|^2}$$

Clearly for any $x \in E$, there exist unit vectors a and b such that $x = \alpha a + \beta b$ where $a \in A$, $b \in B$ and $\alpha, \beta \equiv E$. Hence x lies on the circle

$$\varphi(t) = |x| \left[\frac{1-t}{2} a + \frac{1+t}{2} b \right] . \tag{6}$$

As t varies from 1 to -1, a point on this curve travels from the hyperplane B to the hyperplane A. Also, for any point $\varphi(t)$ on this curve,

$$T(\varphi(t)) = \frac{|x|^2 \left[-\left(\frac{1-t}{2}\right)|a|^2 + \left(\frac{1+t}{2}\right)|b|^2 \right]}{|x|^2 \left[\left(\frac{1-t}{2}\right)|a|^2 + \left(\frac{1+t}{2}\right)|b|^2 \right]} = t.$$

(since a and b are unit vectors).

We now show that, if $\varphi(t)$ is in a sufficiently small neighborhood of $\bar{x}$ and $\varphi(t) \in \{f<0\}$, then, as t decreases, $f(\varphi(t))$ remains in $\{f<0\}$.

Now along $\varphi(t)$, $|\varphi(t)|$ is a constant. Therefore,

$$\frac{1}{|\varphi(t)|^2}\frac{d}{dt}f(\varphi(t)) = \frac{d}{dt}\left[\frac{f(\varphi(t))}{|\varphi(t)|^2}\right]$$

$$= \frac{d}{dt}T(\varphi(t)) + \frac{d}{dt}\left[\frac{R(\varphi(t))}{|\varphi(t)|^2}\right]$$

$$= \text{grad } T \cdot \frac{d\varphi(t)}{dt} + \frac{1}{|\varphi(t)|^2}\text{grad } R \cdot \frac{d\varphi}{dt},$$

where "." denotes the scalar product. But,

$$\varphi_\alpha(t) = \frac{|x|}{\sqrt{2}}(1 - t)^{\frac{1}{2}} a_\alpha, \quad \alpha = 1,2,\ldots,i,$$

and

$$\varphi_\alpha(t) = \frac{|x|}{\sqrt{2}}(1 + t)^{\frac{1}{2}} a_\alpha, \quad \alpha = i+1, \ldots, n.$$

Hence, for $t \in [-1,1]$,

$$\frac{d\varphi_\alpha}{dt} = \frac{\nu_\alpha \varphi_\alpha(t)}{2(1 - t)}, \quad \alpha = 1,2,\ldots,i,$$

and

$$\frac{d\varphi_\alpha}{dt} = \frac{\nu_\alpha \varphi_\alpha(t)}{2(1 + t)}, \quad \alpha = i+1,\ldots,n.$$

Also

$$\frac{\partial T}{\partial x_a}(\varphi(t)) = \left[\frac{v_a|\varphi(t)|^2 - Q(\varphi(t))}{|\varphi(t)|^4}\right] 2\,\varphi_a(t)$$

$$= \frac{2\,\varphi_a(t)}{|\varphi(t)|^2}\left[v_a - T(\varphi(t))\right]$$

$$= \frac{2\,\varphi_a(t)}{|\varphi_a(t)|^2}\left[v_a - t\right], \quad a = 1,\ldots,n.$$

Hence, for $t \in [-1,1]$,

$$\operatorname{grad} T \cdot \frac{d\varphi}{dt} = \sum_{a=1}^{i} \frac{\varphi_a^2(t)}{|\varphi_a(t)|^2}\left(\frac{1+t}{1-t}\right)$$

$$+ \sum_{a=i+1}^{n} \frac{\varphi_a^2(t)}{|\varphi_a(t)|^2}\left(\frac{1-t}{1+t}\right)$$

$$= 2\left(\frac{1+t^2}{1-t^2}\right)\sum_{a=1}^{n}\frac{\varphi_a^2(t)}{|\varphi(t)|^2}$$

$$- 2\left(\frac{2t}{1-t^2}\right)\sum_{a=1}^{n}\frac{v_a\varphi_a^2(t)}{|\varphi(t)|^2}$$

$$= \left(\frac{2}{1-t^2}\right)\left[1 + t^2 - 2t\,\frac{Q(\varphi(t))}{|\varphi(t)|^2}\right]$$

$$= 2.$$

Also,

$$\text{grad } R(\varphi(t)).\frac{d\varphi}{dt} = \sum_{\alpha=1}^{i} \frac{\partial R}{\partial x_\alpha} \frac{\upsilon_\alpha \varphi_\alpha(t)}{2(1 - t)}$$

$$+ \sum_{\alpha=i+1}^{n} \frac{\partial R}{\partial x_\alpha} \frac{\upsilon_\alpha \varphi_\alpha(t)}{2(1 + t)}$$

$$= \frac{1}{(1 - t^2)} \sum_{\alpha=1}^{n} \frac{\partial R}{\partial x_\alpha} \varphi_\alpha(t)(\upsilon_\alpha - t).$$

Let U' be a neighborhood of $\bar{x}$ such that for all $x \in U'$,

$$\left| \frac{1}{|x|} \frac{\partial R}{\partial x_\alpha}(x) \right| < \frac{1}{2n}$$

Then there exists a Δ neighborhood of $t = 0$ with $\Delta \leq 1/2$ such that, for all t in this neighborhood, $\varphi(t) \in U'$. Hence, for $-\Delta < t < \Delta$.

$$\left| \frac{1}{|\varphi(t)|^2} \text{Grad } R(\varphi(t)) \cdot \frac{d\varphi}{dt} \right| < \frac{1}{1 - t^2} \sum_{\alpha=1}^{n} \left| \frac{1}{|\varphi(t)|} \frac{\partial R}{\partial x_\alpha} \right| \left| \frac{\varphi_\alpha(t)}{\varphi(t)} \right| \left| \upsilon_\alpha - t \right|$$

$$< \frac{4}{3} \sum_{=1}^{n} \frac{1}{2n} \cdot 2$$

$$< 2.$$

Thus,

$$\frac{df(\varphi(t))}{dt} > 0$$

for all $t \in (-\Delta, \Delta)$, which implies

(7) $$f(\varphi(t)) < f(\varphi(t_o)),$$

whenever $-\Delta < t < t_o < \Delta$.

Let U be a spherical neighborhood of $\bar{x}$ such that $U \subset U'$ and, for each $x \in U$,

$$\left|\frac{R(x)}{|x|^2}\right| < \Delta$$

(See 5a) Let $U^- = U \cap \{f < 0\}$. Then for each $x \in U^-$,

$$T(x) = \frac{f(x)}{|x|^2} + \frac{R(x)}{|x|^2} < \Delta .$$

Thus, $U^- \subset \{T < \Delta\}$.

Now, we show that if $\varphi(t) \in U^-$, then as t decreases, $\varphi(t)$ remains in $\{f < 0\}$. Let $\varphi(t_o) \in U^-$ and let $t < t_o$. Then either $t \leq -\Delta$ or $t > -\Delta$. If $t > -\Delta$, then by (7),

$$f(\varphi(t)) < f(\varphi(t_o)) < 0$$

If $t \leq -\Delta$, then

$$\frac{f(\varphi(t))}{|x|^2} = T(\varphi(t)) + \frac{R(\varphi(t))}{|x|^2}$$

$$= t + \frac{R(\varphi(t))}{|x|^2}$$

$$\leq -\Delta + \frac{R(\varphi(t))}{|x|^2}$$

But since U is a spherical neighborhood of $\bar{x}$, and φ is a circle whose center is $\bar{x}$, $\varphi(t) \in U$. Therefore,

$$\frac{f(\varphi(t))}{|x|^2} < -\Delta + \Delta = 0.$$

The required deformation may now be constructed using the circles described by equation (6). For each $x \in U^-$, let $\Phi(t;x)$ be the circle through x. Let

$$F(x;\tau) = \Phi((1-\tau)T(x) - \tau ; x)$$

for $x \in U^-$, $\tau \in I$, and let $F(\bar{x},\tau) = \bar{x}$, $\tau \in I$. Clearly, F is continuous on $(U^- \cup \{\bar{x}\}) \times I$ into $U^- \cup \{\bar{x}\}$. Also,

$$F(x,0) = \Phi(T(x),x) = x,$$

$$F(x,1) = \Phi(-1,x) \in A.$$

Furthermore, $F(x,\tau) \in \{f<0\}$ for $x \in U^-$, $\tau \in I$ since $(1-\tau)T(x)-\tau < T(x)$.

Thus, F is a continuous deformation which deforms $(\{f<0\} \cap U) \cup \{\bar{x}\}$ into A in such a way that image of $\{f<0\} \cap U$ is a subset of $\{f<0\}$.

It follows from theorem 1.3 that each cycle (mod $\{f<0\} \cap U$) in $(\{f<0\} \cap U) \cup \{\bar{x}\}$ is homologolous (mod $\{f<0\}$ in $\{f<0\} \cup \{\bar{x}\}$) to a cycle (mod $(A - \{\bar{x}\})$) in A. Hence, the q-th connectivity number (mod $\{f<0\} \cap U$) of $(\{f<0\} \cap U) \cup \{\bar{x}\}$ is the same as the q-th connectivity number (mod$(A - \{\bar{x}\})$) of A. But by theorem 2.2, this number is δ_q^i, since A is homeomorphic to E^i.

Q.E.D.

2.2 CYLINDRICAL NEIGHBORHOODS AND f-DEFORMATIONS

Let $\bar{x}$ be an isolated stationary point of f and let G be an open neighborhood of $\bar{x}$ which contains no other stationary points of f.

Let $y \in G$ be a non-stationary point of f. Now, the system

$$(8a) \qquad \frac{d\Phi_i}{dt} = \frac{\frac{\partial f}{\partial x_i}(\Phi_1,\Phi_2,\ldots,\Phi_n)}{\sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}(\Phi_1,\Phi_2,\ldots,\Phi_n)\right]^2}$$

$$(8b) \qquad \Phi_i(f(y)) = y_i, \quad i = 1,2,\ldots,n,$$

has a unique solution defined for all t in some Δ-neighborhood of f(y). Let $\Phi(t)$ denote the vector-valued function whose components satisfy (8a) and (8b).

Then along this integral curve

$$\frac{d}{dt} f(\Phi(t)) = \frac{d}{dt} f(\Phi_1(t), \ldots, \Phi_n(t))$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{d\Phi_j}{dt} = 1.$$

Therefore, for each $t \in \left[f(y) - \Delta, f(y) + \Delta\right]$, $f(\Phi(t)) = t$.

The curve defined by Φ is called the <u>line of descent</u> <u>(Fallinien)</u> of f through y. This curve will also be denoted by the symbol Φ.

Suppose, now that the initial condition (8b) is replaced by

$$(8b') \qquad \Phi_i(f(x)) = x_i, \quad i = 1,2,\ldots,n.$$

Then there exists an ε-neighborhood of y and a Δ neighborhood of f(y) such that for each x in this ε neighborhood of y, the system (8a) together with the initial condition (8b'), has a unique solution defined for all t in the Δ-neighborhood of f(y). For each $x \in \{f = f(y)\}$, let $\Phi_x(t)$ be the line of descent through x. Let Z be the set of all points $\Phi_x(t)$, where $x \in \{f = f(y)\}$, $|x - y| < \varepsilon$, and $|t - f(y)| \leq \Delta$. ($|x-y|$ denotes the Euclidean norm of x-y.) Then Z is a neighborhood of y; i.e., there exists an open neighborhood G' of y such that $G' \subset Z$. (See Seifert and Threlfall, 1938, p. 94, Note 14.) Z is called a <u>cylindrical neighborhood</u> of y of height 2Δ.

The method of constructing a cylindrical neighborhood of the stationary point $\bar{x}$ will now be described.

A line of descent, Φ, through a point $x \in G - \{\bar{x}\}$ is said to <u>empty (einmünden) into $\bar{x}$</u> if $\Phi(t)$ is defined for $f(x) \geq t > f(\bar{x})$ and as t approaches $f(\bar{x})$ from above, $\Phi(t)$ approaches $\bar{x}$. The line of descent is said to <u>recede (ausgehen) from $\bar{x}$</u> if $\Phi(-t)$ empties into $\bar{x}$.

Now, let U be a bounded open neighborhood of $\bar{x}$ whose closure is a subset of G. Let Φ be a line of descent through a point $x \in U - \{\bar{x}\}$. It may be shown (Seifert and Threlfall, 1938, p.g 37) that as t decreases, $\Phi(t)$ either empties into $\bar{x}$ or approaches a point on the boundary of U. A similar result holds as t increases.

Let C be a bounded open subset of the space $\{f = f(\bar{x})\}$ and suppose that the closure of C is a subset of G. Let K be the set of all lines of descent which empty into $\bar{x}$, recede from $\bar{x}$ or pass through a point of $C - \{\bar{x}\}$. Then there exists $\Delta > 0$ such that for each $\Phi \in K$ and $t \in [f(\bar{x}) - \Delta, f(\bar{x}) + \Delta]$, $\Phi(t)$ is an element of G. (See Seifert and Threlfall, 1938, pg. 95, Note 15) Let Z' be the set of all points of the form $\Phi(t)$ such that $\Phi \in K$ and

(i) $f(\bar{x}) < t \leq f(\bar{x}) + \Delta$, if Φ empties into $\bar{x}$

(ii) $f(\bar{x}) - \Delta \leq t < f(\bar{x})$, if Φ recedes from $\bar{x}$

(iii) $f(\bar{x}) - \Delta \leq t \leq f(\bar{x}) + \Delta$, if Φ passes through a point of $C - \{\bar{x}\}$.

Then $Z' \cup \{\bar{x}\}$ is a neighborhood of $\bar{x}$ (Seifert and Threlfall, 1938, pg. 96, Note 16) and is called a <u>cylindrical neighborhood of $\bar{x}$ of height 2Δ.</u>

<u>Lemma 2.1</u>

Let Z be a cylindrical neighborhood of $\bar{x}$ of height 2Δ. Then there exists a continuous function F on $Z \times I$ to Z such that

(i) $F_1(x) \in \{f = f(\bar{x})\}$, for all $x \in Z \cap \{f \geq f(\bar{x})\}$

(ii) $F(x,\tau) = x$, for all $x \in Z \cap \{f \leq f(\bar{x})\}$ and $\tau \in I$.

<u>Proof</u>

For each $x \in Z \cap \{f \geq f(\bar{x})\}$, let Φ_x be the line of descent through x. For each $x \in Z \cap \{f > f(\bar{x})\}$ and $\tau \in I$, let

$$F(x,\tau) = \Phi_x(\ (1-\tau)f(x) + \tau f(\bar{x})\ )$$

(If Φ_x empties into $\bar{x}$, let $F(x,1)$ be the limit of $F(x,\tau)$ as τ approaches 1). For each $x \in Z \cap \{f \leq f(\bar{x})\}$ and $\tau \in I$, let $F(x,\tau) = x$. Clearly, F is on $Z \times I$ to Z and

$$F_1(x) \in \{f \leq f(\bar{x})\} .$$

Since each line of descent is a solution of (8a) and (8b), and since f is continuous, F is continuous.

Q.E.D.

Lemma 2.2

Let Z be a cylindrical neighborhood of $\bar{x}$ of height 2Δ. Let $W = \{f \leq f(\bar{x})\} \cap Z$ and let $W^- = \{f < f(\bar{x})\} \cap Z$. Then there exists a continuous function H on $W \times I$ to W such that

(i) $H_1(x) \in W^-$ for each $x \in W - \{\bar{x}\}$

(ii) $H(\bar{x}, r) = \bar{x}$ for all $r \in I$.

Proof

Let $\Psi = (\Psi_1, \ldots, \Psi_n)$ be a solution of the system of differential equations:

(9a) $$\frac{d\Psi_i}{dt} = - f_i(\Psi_1, \ldots, \Psi_n) \left[f(\Psi_1, \ldots, \Psi_n) + \Delta - f(\bar{x}) \right] ,$$

(9b) $$\Psi_i(0) = x_i, \quad i = 1, 2, \ldots, n.$$

For each $x \in W$, let Ψ_x be the solution of (9a) and (9b) subject to the initial condition $\Psi(0) = x$. Let H be on $W \times I$ such that for each $x \in W$ and $\tau \in I$,

$$H(x,\tau) = \Psi_x(\tau).$$

Now, the only solution of (9a) and (9b) for $x \in (W \cap \{f = f(\bar{x}) - \Delta\}) \cup \{\bar{x}\}$ is the solution $\Psi_x(t) = x$. Therefore,

$$H(\bar{x},\tau) = \bar{x} \quad \text{for all } \tau \in I,$$

and

$$H(x,\tau) \in W^- \quad \text{for all } x \in (W \cap \{f = f(\bar{x}) - \Delta\}).$$

Suppose $x \in (W \cap \{f(\bar{\bar{x}}) - \Delta < f\}) - \{\bar{\bar{x}}\}$. Then

$$\frac{d}{dt} f(\Psi_x(t)) = -\sum_{i=1}^{n} \left[\frac{\partial}{\partial x_i} f(\Psi(t))\right] \left[f(\Psi(t)) + \Delta - f(\bar{x})\right],$$

which is negative for $t = 0$. Therefore, as t increases, $\Psi_x(t)$ moves from x into the set $\{f < f(\bar{x})\} \subset \{f < f(\bar{\bar{x}})\}$. Moreover, $\Psi_x(t)$ cannot cross the "boundary" set $\{f = f(\bar{\bar{x}}) - \Delta\}$ at some "time" $t_o > 0$. For, if $\Psi_x(t_o) = y \in \{f = f(\bar{\bar{x}}) - \Delta\}$, for some $t_o > 0$, then $\Psi_x(t) = \Psi_y(t - t_o) = y$ for all $t > t_o$ by the uniqueness of the solutions of (9a) and (9b).

Thus,

$$H(x,\tau) \in W^- \quad \text{for } \tau > 0 \text{ and } x \in (W \cap \{f(\bar{x}) - \Delta < f\} - \{\bar{x}\}$$

Q.E.D.

Combining lemmas 2.1 and 2.2, we have the following result:

Theorem 2.4

Let Z be a cylindrical neighborhood of $\bar{x}$. Then there exists a continuous function, F, on $Z \times I$ to Z such that

(i) $F_1(x) \in \{f < f(\bar{x})\} \cup \{\bar{x}\}$ for each $x \in Z \cap \{f \geq f(\bar{x})\}$

(ii) $F(x,\tau) \in \{f < f(\bar{x})\}$ for each $x \in Z \cap \{f < f(\bar{x})\}$ and $\tau \in I$

and

(iii) $F(\bar{x},\tau) = \bar{x}$ for all $\tau \in I$.

In other words F continuously deforms the cylindrical neighborhood Z into the set $\{f < f(\bar{x})\} \cup \{\bar{x}\}$ in such a way that the point $\bar{x}$ remains fixed and the points of $\{f < f(\bar{x})\}$ remain in $\{f < f(\bar{x})\}$. The function, F, is called an f-deformation.

2.3 COMPACT n-DIMENSIONAL MANIFOLDS

An n-dimensional manifold is a topological space $\mathcal{M}$ with the property that for each $x \in \mathcal{M}$ there exists an open subset, G, of $\mathcal{M}$ such that $x \in G$ and G is homeomorphic to the open n-dimensional disc:

$$V^n = \left\{ x \;\middle|\; |x| < 1 \right\} \subset E^n.$$

Let $\mathcal{M}$ be a compact n-dimensional manifold. Let $\left\{G_1, G_2, \ldots, G_r\right\}$ be an open covering of $\mathcal{M}$ such that each G_i, $i = 1,2,\ldots,r$, is homeomorphic to V^n. Let $T_1, T_2, \ldots, T_r$ be the homeomorphisms corresponding to $G_1, G_2, \ldots, G_r$. Then for any $i,j = 1,2,\ldots,r$, the composite function $T_i \circ T_j^{-1}$ maps a subset of V^n onto itself. ($T_i \circ T_j^{-1}$ may be the empty function, $\emptyset$.) The manifold $\mathcal{M}$ is said to be differentiable of class C^m, if $T_1, T_2, \ldots, T_r$ may be chosen in such a way that for $i,j = 1,2,\ldots,r$, $T_i \circ T_j^{-1}$ possesses continuous m-th order partial derivatives and has a non-vanishing Jacobian at every point of the domain of definition of $T_i \circ T_j^{-1}$.

A real-valued function, f, on a differentiable manifold $\mathcal{M}$ is differentiable at $y \in \mathcal{M}$ if, for some homeomorphism, T, which maps a neighborhood of y onto V^n, the composite function $f \circ T^{-1}$ is differentiable at the point $T^{-1}(y)$.

In the remainder of this section, and in the next section, let $\mathcal{M}$ be a compact, differentiable, n-dimensional manifold of class C^3, and let f be a real-valued function whose third order partial derivatives are continuous on $\mathcal{M}$. Under these conditions, the preceding theory may be applied to $\mathcal{M}$ and f. For example, lines of descent in $\mathcal{M}$ may be constructed by replacing the function f in equations (8a) and (8b) by the composite function $F = f \circ T^{-1}$. The lines of descent on $\mathcal{M}$ are then given by $T^{-1}(\varphi(t))$ and since $F(\varphi(t)) = t$, we have

$$f(T^{-1}(\varphi(t))) = F(\varphi(t)) = t.$$

We make the further assumption that the total number of stationary points of f on $\mathcal{M}$ is <u>finite</u>.

<u>Lemma 2.3</u>

Let γ_1 and γ_2 be stationary values of f such that $\gamma_1 < \gamma_2$ and no stationary values of f lie between γ_1 and γ_2. Let g be the set of all stationary points in $\{f = \gamma_1\}$, and let ϵ be a positive number such that $\gamma_1 < \gamma_2 - \epsilon$.

Then, there exists a continuous deformation, F, in $\{f \leq \gamma_2 - \epsilon\}$ such that

(i) $F_1(x) \in \{f \leq \gamma_1\}$ for each $x \in \{\gamma_1 < f \leq \gamma_2 - \epsilon\}$,

and (ii) $F(x,\tau) = x$, for each $x \in \{f \leq \gamma_1\}$ and $\tau \in I$.

(In other words, $\{f \leq \gamma_2 - \epsilon\}$ may be continuously deformed into $\{f \leq \gamma_1\}$ in such a way that $\{f \leq \gamma_1\}$ remains fixed.)

Proof

It sufficies to show that the line of descent through any $x \in \{\gamma_1 \leq f \leq \gamma_2 - \epsilon\}$ empties into a point of g or intersects the set $\{f = \gamma_1\}$.

We note first that $\{\gamma_1 \leq f \leq \gamma_2 - \epsilon\}$ is a closed subset of the compact set $\mathfrak{M}$ and hence is also compact. Also, at each point of this set, a cylindrical neighborhood may be constructed. The set of interiors of these cylindrical neighborhoods forms an open covering of $\{\gamma_1 \leq f \leq \gamma_2 - \epsilon\}$ and hence contains a finite subcovering. Let $\{G_1, G_2, \ldots, G_r\}$ denote the finite set of interiors of cylindrical neighborhoods which covers $\{\gamma_1 \leq f \leq \gamma_2 - \epsilon\}$. Let $Z_1, Z_2, \ldots, Z_r$ be the corresponding cylindrical neighborhoods. Let $2\Delta_1, 2\Delta_2, \ldots, 2\Delta_r$ be the heights and $x_1, x_2, \ldots, x_r$ be the centers of these neighborhoods, (i.e., the points about which the neighborhoods are constructed).

Let $x \in \{\gamma_1 < f \leq \gamma_2 - \epsilon\}$. Then x is an element of one of the G_i's, say G_1, and hence lies on a line of descent, $\varphi^1(t)$, which intersects $\{f = f(x_1)\}$ or empties into a point of g. If $f(x_1) - \Delta_1 \leq \gamma_1$, then we are through; for then, $\varphi^1(t)$ intersects $\{f = \gamma_1\}$ or empties into an element of g. If $f(x_1) - \Delta_1 > \gamma_1$, then $\varphi^1(t)$ intersects $\{f = f(x_1) - \Delta_1\}$ at the point

$$\varphi^{1}(f(x_1) - \Delta_1) \in \left\{ f = f(x_1) - \Delta_1 \right\} \cap Z_1 .$$

Since $\gamma_1 < f(x_1) - \Delta_1 < f(x) \leq \gamma_2 - \epsilon$,

$$\varphi^{1}(f(x_1) - \Delta_1) \in \left\{ \gamma_1 < f \leq \gamma_2 - \epsilon \right\} .$$

Therefore, $\varphi^1(f(x_1) - \Delta_1)$ is an element of one of the G_i's. Now, $\varphi^1(f(x_1) - \Delta_1) \notin G_1$, since every neighborhood of a point of $\left\{f = f(x_1) - \Delta_1\right\} \cap Z_1$ contains points of $\left\{f < f(x_1) - \Delta_1\right\}$ and hence contains points which do not belong to Z_1. Therefore $\varphi^1(f(x) - \Delta_1)$ is an element of some G_i different from G_1, say G_2. Hence, $\varphi^1(f(x) - \Delta_1)$ lies on a line of descent $\varphi^2(t)$ which intersects $\left\{f = f(x_2)\right\}$ or empties into a point of g. Since $\varphi^1(t)$ and $\varphi^2(t)$ are defined in some open set containing $\varphi^1(f(x) - \Delta_1)$ and $\varphi^1(f(x) - \Delta_1) = \varphi^2(f(x) - \Delta_2)$, by the uniqueness of solutions of (8a) and (8b) of section 2.2, $\varphi^1 \cup \varphi^2$ is a (single valued) function, and hence is a line of descent through x. Continuing this process, we arrive at a sequence $\left\{\varphi^1, \varphi^2, \ldots, \varphi^j\right\}$ of lines of descent such that $j \leq r$, $\varphi^1 \cup \varphi^2 \cup \cdots \cup \varphi^j$ is a line of descent, and φ^j intersects $\left\{f = f(x_j) - \Delta_j\right\}$ or empties into an element of g. This process must terminate for some $k \leq r$. Then φ^k intersects $\left\{f = f(x_k) - \Delta_k\right\}$ or empties into a point of g. If φ^k does not empty into g, then $f(x_k) - A_k < \gamma_1$ (otherwise the process would not terminate at k). But $f(x_k) \geq \gamma_1$. Therefore φ^k intersects $\{f = \gamma_1\}$ or empties into an element of g. Thus

$$\varphi = \varphi^1 \cup \varphi^2 \cup \ldots \cup \varphi^k .$$

is a line of descent through x which empties into an element of g or intersects $\{f = \gamma_1\}$.

The remainder of the proof parallels the proof of lemma 2.1, section 2.2.

Q.E.D.

Lemma 2.4

Let γ_1 and γ_2 be stationary values of f such that $\gamma_2 < \gamma_1$ and no stationary values of f lie between γ_1 and γ_2. Let g be the set of stationary points of f. Then there exists $\epsilon > 0$ and a continuous deformation F in $\{f \leq \gamma_1\}$ such that F deforms $\{\gamma_1 - \epsilon < f \leq \gamma_1\} - g$ into a subset of $\{f < \gamma_1\}$ and $F(x,r) = x$ for all $x \in \{f = \gamma_1 - \epsilon\} \cup g$.

Proof

As in the proof of Lemma 5, the compact set $\{f = \gamma_1\}$ may be covered by a collection of open sets, $G_1, G_2, \ldots, G_r$, which are the interiors of cylindrical neighborhoods, $Z_1, Z_2, \ldots, Z_r$, whose centers are points of $\{f = \gamma_1\}$. Let $x_1, x_2, \ldots, x_r$ and $2\Delta_1, 2\Delta_2, \ldots, 2\Delta_r$, be, respectively, the centers and heights of $Z_1, Z_2, \ldots, Z_r$.

Since $\gamma_1 - \gamma_2 > 0$, the neighborhoods may be chosen in such a way that none of the neighborhoods $Z_1, \ldots, Z_r$ intersect $\{f = \gamma_2\}$. Let ϵ be the minimum of $\{\Delta_1, \ldots, \Delta_r\}$. Then, for each $x \in \{\gamma_1 - \epsilon \leq f \leq \gamma_1\}$, there exists a solution Ψ_x of (9a) and (9b) (section 2.2) such that $\Psi_x(0) = x$, and, moreover, as t increases, $f(\Psi_x(t))$ decreases except for $x \in \{f = \gamma_1 - \epsilon\} \bigcup g$.

Let $F(x,\tau) = \Psi_x(\tau)$, $\tau \in I$, $x \in \{\gamma_1 - \epsilon \leq f \leq \gamma_1\}$. Then, since the solutions of (9a) and (9b) vary continuously with the initial conditions, (9b), F is continuous for each $x \in \{\gamma_1 - \epsilon \leq f \leq \gamma_1\}$.

If $x \in \{\gamma_1 - \epsilon < f \leq \gamma_1\} - g$, then for each $\tau > 0$,

$$f(\Psi_x(x)) = f(F(x,\tau)) < f(x) \leq \gamma_1 .$$

Therefore, $F(x,\tau) \in \{\gamma_1 - \epsilon \leq f < \gamma_1\}$.

For $x \in \{f = \gamma_1 - \epsilon\} \bigcup g$, the (unique) solution of (9a) and (9b) is

$$\Psi_x(\tau) = x.$$

Therefore, $F(x,\tau) = x$, $\tau \in I$.

Thus, F satisfies the conditions of the Lemma.

Q.E.D.

Combining Lemmas 2.4 and 2.5 we have the analog (on $\mathfrak{M}$) of Theorem 2.4 of section 2.2.

Theorem 2.5

Let γ be a stationary value of f and let γ_1 be the smallest (higher) critical value such that $\gamma < \gamma_1$. Let g be the set of stationary points of γ. Then for any $\epsilon \geq 0$ such that $\gamma + \epsilon < \gamma_1$, there exists a continuous deformation, F, in $\{f \leq \gamma + \epsilon\}$ such that

(i) If $x \in \{f < \gamma\}$, then $F(x,\tau) \in \{f < \gamma\}$ for each $\tau \in I$,

(ii) If $x \in \{\gamma \leq f \leq \gamma + \epsilon\}$, then $F_1(x) \in \{f < \gamma\} \cup g$.

(iii) If $x \in g$, then $F(x,\tau) = x$.

The function F in Theorem 2.5 deforms the set $\{f \leq \gamma + \epsilon\}$ into the set $\{f < \gamma\} \cup g$ for any $\epsilon > 0$ which is such that no stationary points lie between γ and $\gamma + \epsilon$. In particular, F deforms a cycle, C (mod $\{f < \gamma\}$) in $\{f \leq \gamma + \epsilon\}$ into an homologous cycle (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$ in such a way that ∂C remains in $\{f < \gamma\}$. The points of g form what may be regarded as a barrier beyond which C cannot be deformed continuously (e.g., without tearing) unless ∂C leaves $\{f < \gamma\}$, in which case, C is not deformed into an homologous (mod $\{f < \gamma\}$ in $\{f < \gamma + \varepsilon\}$) cycle. Note that even though a cycle cannot be deformed beyond a stationary point, $\bar{x}$, by an f deformation, it may still be homologous in $\{f \leq f(\bar{x})\}$ to a chain in $\{f < f(\bar{x})\}$. For example, 0 is a stationary point of the function $f(x) = x^3$. However, the cycle $C = \{(0,0)\}$ is homologous (mod $\{f < 0\}$ in $\{f \leq 0\}$) to the cycle $\{(-1,-1)\}$ even though $\{(0,0)\}$ cannot be deformed into $\{(-1,-1)\}$ by an f-deformation.

Let γ be a non-stationary value of f and let γ_1 be the largest stationary value below γ. Let g be the set of all stationary points in $\{f = \gamma_1\}$. Then, from theorem 2.5 and corollary 2 of theorem 1.3, it follows that the q-th connectivity number (mod $\{f < \gamma_1\}$) of $\{f \leq \gamma\}$ is equal to 0, while the q-th connectivity number (mod $\{f < \gamma_1\}$) of $\{f \leq \gamma\}$ is equal to the q-th connectivity number (mod $\{f < \gamma_1\}$) of $\{f < \gamma_1\} \cup g$. The next theorem relates the latter connectivity numbers to the type numbers of the stationary points in g.

Theorem 2.6

Let γ be a stationary value of f and let $g = \{x_1, x_2, \ldots, x_r\}$ be the set of all stationary points in $\{f = \gamma\}$. Then the q-th connectivity number (mod $\{f < \gamma\}$) of $\{f < \gamma\} \cup g$ is equal to

$$\sum_{i=1}^{r} m^q(x_i) \;.$$

<u>Proof</u>:

Let m^q denote the q-th connectivity number (mod $\{f<\gamma\}$) of $\{f<\gamma\} \cup g$.

We begin by establishing two propositions:

(i) Each q-cycle, C, (mod $\{f<\gamma\}$) in $\{f<\gamma\} \cup g$ is homologous (mod $\{f<\gamma\}$ in $\{f<\gamma\} \cup g$) to a sum, $C_1 + C_2 + \ldots + C_r$, of q-chains, where, for i = 1,...,r, C_i is a cycle (mod $\{f<\gamma\}$) in $\{f<\gamma\} \cup \{x_i\}$.

(ii) If, for i = 1,2,...,r, C_i is a cycle (mod $\{f<\gamma\}$) in $\{f<\gamma\} \cup \{x_i\}$, and

$$C_1 + C_2 + \ldots + C_r \sim \emptyset \text{ (mod } \{f<\gamma\} \text{ in } \{f<\gamma\} \cup g),$$

then, for i = 1,2,...,r,

$$C_i \sim \emptyset \text{ (mod } \{f<\gamma\} \text{ in } \{f<\gamma\} \cup \{x_i\}).$$

Proof of (i)

Let C be a q-cycle (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$. Then, since each x_i is isolated, there exists an integer k such that no two elements of g lie in the same simplex of the k-fold subdivision, ${}_kC$, of C. For each i, let C_i be the set of all simplexes of ${}_kC$ which contain x_i:

$$C_i = \left\{ S \mid S \in {}_kC \text{ and } x_i \in |S| \right\}.$$

Then, since the only simplexes of k which do not appear in $\sum_{i=1}^{r} C_i$ are in $\{f < \gamma\}$,

$$\sum_{i=1}^{r} C_i = {}_kC \quad (\text{mod } \{f < \gamma\}).$$

Also, by Theorem 2.4 of section 1.6,

$${}_kC \sim C \quad (\text{mod } \{f < \gamma\} \text{ in } \{f < \gamma\} \cup g).$$

Thus,

$$\sum_{i=1}^{r} C_i \sim C \quad (\text{mod } \{f < \gamma\} \text{ in } \{f < \gamma\} \cup g).$$

Clearly, $\sum_{i=1}^{r} C_i$ satisfies the requirements of (i).

Proof of (ii)

For $i = 1, 2, \ldots, r$, let C_i be a cycle (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup \{x_i\}$, and suppose

$$C_1 + C_2 + \ldots + C_r \sim \emptyset \quad (\text{mod } \{f < \gamma\} \text{ in } \{f < \gamma\} \cup g).$$

Then there exists a q + 1 cycle, D, in $\{f < \gamma\} \cup g$ such that

$$\partial D = C_1 + C_2 + \ldots + C_r \quad (\text{mod } \{f < \gamma\}).$$

Let k be an integer such that no two elements of g belong to the same simplex of the k-fold normal subdivision, ${}_kD$, of D. For $i = 1, 2, \ldots, r$, let D_i be the set of simplexes of ${}_kD$ which contain x_i:

$$D_i = \{ S \mid S \in {}_kD \text{ and } x_i \in |S| \},$$

Then

$${}_kD = \sum_{i=1}^{r} D_i \quad (\text{mod } \{f < \gamma\}).$$

Now,

$$\partial_k D = {}_k(\partial D) = \sum_{i=1}^{r} {}_kC_i .$$

Therefore,

$$\sum_{i=1}^{r} \partial D_i = \sum_{i=1}^{r} {}_kC_i \pmod{\{f < \gamma\}}$$

This last equation may be written

$$\sum_{i=1}^{r} \left[\partial D_i + {}_kC_i\right] = \emptyset \pmod{\{f < \gamma\}},$$

which implies, for each i,

$$x_i \notin |\partial D_i + {}_kC_i|.$$

But by construction,

$$x_j \notin |\partial D_i + {}_kC_i|$$

for $i \neq j$. Therefore, for $i = 1, 2, \ldots, r$, $\partial D_i + {}_kC_i$ is in $\{f < \gamma\}$.

Hence,

$$\partial D_i = {}_kC_i \quad (\text{mod } \{f<\gamma\}).$$

Also, by construction, D_i is in $\{f<\gamma\}\bigcup\{x_i\}$. Therefore,

$${}_kC_i \sim \emptyset \ (\text{mod } \{f<\gamma\} \text{ in } \{f<\gamma\}\bigcup\{x_i\}),$$

$i = 1, 2, \ldots, r$.

Now, by Theorem 1.4 of section 1.6,

$$C_i \sim {}_kC_i \quad (\text{mod } \{f<\gamma\} \quad \text{in } \{f<\gamma\}\bigcup\{x_i\}),$$

and so

$$C_i \sim \emptyset \quad (\text{mod } \{f<\gamma\} \text{ in } \{f<\gamma\}\bigcup\{x_i\}),\ i = 1, 2, \ldots, r.$$

Thus, (ii) is established.

From the definition of $m^q(x_i)$, there exists a maximal set, $\left\{z_i^1, z_i^2, \ldots, z_i^{m^q(x_i)}\right\}$ of cycles (mod $\{f<\gamma\}$) in $\{f<\gamma\}\bigcup\{x_i\}$ which are independent (mod $\{f<\gamma\}$ in $\{f<\gamma\}\bigcup\{x_i\}$). By (ii), no sum of chains formed from the set

$$K = \left\{ z_1^1, \ldots, z_1^{m^q(x_1)},\ z_2^1, \ldots, z_2^{m^q(x_2)},\ \ldots,\ z_r^1, \ldots, z_r^{m^q(x_r)} \right\}$$

is bounding (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup g$). Therefore, there are at least $m^q(x_1) + m^q(x_2) + \ldots + m^q(x_r)$ independent cycles (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$.

On the other hand, by (i), a q-cycle, C, (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$ is homologous to a chain, $\sum_{i=1}^{r} C_i$, where, for $i = 1, 2, \ldots, r$, C_i is a cycle (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup \{x_i\}$.

Since $\left\{ z_i^1, z_i^2, \ldots, z_i^{m^q(x_i)} \right\}$ is a maximal independent set (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup \{x_i\}$), $i = 1, 2, \ldots, r$, each C_i is homologous (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup \{x_i\}$) to a chain formed from a subset of $\left\{ z_i^1, \ldots, z_i^{m^q(x_i)} \right\}$. Hence, C is homologous (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup g$ to a chain formed from a subset of K. Therefore, unless C is one of the z_i^j's, $K \cup \{C\}$ is not independent (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup g$). Thus, K is a maximal homologically independent set (mod $\{f < \gamma\}$ in $\{f < \gamma\} \cup g$).

Q.E.D.

2.4 THE MORSE EQUATION ON $\mathcal{M}$

Let $\mathcal{M}$ be a compact, differentiable, n-dimensional manifold of class C^3, and f be a real-valued function such that the third order partial derivatives of f are continuous on $\mathcal{M}$ and the stationary points of f on $\mathcal{M}$ are non-degenerate. Let R^q denote the q-th connectivity number of $\mathcal{M}$.

Let $\gamma \in E$. We say that a cycle, C, (mod $\{f < \gamma\}$) may be completed to an absolute cycle in $\{f < \gamma\}$ if there exists a chain, D, in $\{f < \gamma\}$ such that $\partial C \doteq \partial D$.

Let Q denote the set of all stationary values of f, and let $\gamma \in Q$. Let $t^q(\gamma)$ denote the number of stationary points of index q in $\{f < \gamma\}$. Then there exists an independent set,

$$\left\{A_1, A_2, \ldots, A_{t^q(\gamma)}\right\},$$

of q-cycles (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$, where g is the set of stationary points in $\{f = \gamma\}$. By a simple inductive argument, this set may be replaced by an homologous, independent set

$$\left\{B_1, B_2, \ldots, B_{r^q(\gamma)};\ C_1, C_2, \ldots, C_{s^q(\gamma)}\right\}$$

of q-cycles (mod $\{f < \gamma\}$) in $\{f < \gamma\} \cup g$, where $r^q(\gamma) + s^q(\gamma) = t^q(\gamma)$, each B_i may be completed to an absolute cycle and no combination of the C_i's may be completed to an absolute cycle. (If, say $C_1 + C_2 + \ldots + C_s$, $s \leq s^q(\gamma)$, may be completed to an absolute cycle, we may set $B_{r^{q}(\gamma)+1} = C_1 + C_2 + \dots + C_r$, and replace C_1 by $B_{r^{q}(\gamma)+1}$. This process may be repeated as often as necessary until the required set is constructed.)

Let

$$M^q = \sum_{\gamma \in Q} t^q(\gamma)$$

$$M_+^q = \sum_{\gamma \in Q} r^q(\gamma)$$

and

$$M_-^q = \sum_{\gamma \in Q} s^q(\gamma).$$

Then M^q is the total number of stationary points of index q and $M^q = M_+^q + M_-^q$. Since $M_-^{n+1} \leq M^{n+1} = 0$, the Morse Equation is an immediate consequence of the following theorem.

Theorem 2.7

$$\sum_{i=1}^{q} (-1)^{q-i} (M^i - R^i) = M_+^{q+1} , \quad q = 0,1,\dots.$$

Proof:

For all $\gamma \in E$, let $R^q(\gamma)$ be the q-th connectivity number of $\left\{ f < \gamma \right\}$. For any γ less than the minimum of $f\left[\mathfrak{M}\right]$, $R^q(\gamma) = 0$, while for γ greater than the maximum of $f\left[\mathfrak{M}\right]$, $R^q = R^q(\gamma)$.

We first note that $R^q(\gamma)$ is a constant between any two consecutive stationary values (by Theorem 2.5).

Let $\bar{\gamma} \in Q$. We will now compute the saltus (jump) of $R^q(\gamma)$ at the stationary value $\bar{\gamma}$.

The saltus of $R^q(\gamma)$ at $\bar{\gamma}$ is given by

$$\lim_{\varepsilon \to 0} R^q(\bar{\gamma}+\varepsilon) - \lim_{\varepsilon \to 0} R^q(\bar{\gamma}-\varepsilon).$$

Now, for ε sufficiently small, the q-th connectivity number of $\{f < \bar{\gamma} + \varepsilon\}$ is equal to the q-th connectivity number of $\{f < \bar{\gamma}\} \cup g$, where g is the set of stationary points in $\{f = \bar{\gamma}\}$. Therefore, $\lim_{\varepsilon \to 0} R^q(\bar{\gamma}+\varepsilon)$ is the q-th connectivity number of $\{f < \bar{\gamma}\} \cup g$. On the other hand, if γ' is the largest stationary value of f such that $\gamma' < \bar{\gamma}$, then every chain in $\{f < \bar{\gamma}\}$ may be continuously deformed into a chain in $\{f < \gamma'\} \cup g'$ where g' is the set of all stationary points of f in $\{f = \gamma'\}$. Hence, the q-th connectivity number of $\{f < \bar{\gamma}\}$ is equal to that of $\{f < \gamma'\} \cup g'$, and, moreover, for every sufficiently small $\varepsilon > 0$, the q-th connectivity number of $\{f < \gamma' - \varepsilon\}$ is equal to that of $\{f < \gamma'\} \cup g'$. Therefore, the saltus of f at $\bar{\gamma}$ is equal to the q-th connectivity number of $\{f < \bar{\gamma}\} \cup g$ minus the q-th connectivity number of $\{f < \bar{\gamma}\}$. In other words, as γ increases from $\bar{\gamma} - \varepsilon$ to $\bar{\gamma} + \varepsilon$, (for ε small), $R^q(\gamma)$ is increased or decreased by the number of independent cycles which are added to or removed from $\{f < \bar{\gamma}\}$ by adding the points of g to $\{f < \bar{\gamma}\}$.

We will next show that the saltus of $R^q(\gamma)$ at $\bar{\gamma}$ is given by $r^q(\bar{\gamma}) - s^{q+1}(\bar{\gamma})$. To shorten the notation, let $r^q(\bar{\gamma}) = r^q$,

$s^q(\bar{\gamma}) = s^q$ and $t^q(\bar{\gamma}) = t^q$.

For each $q \geq 0$, let

(10) $$\left\{ \xi_1^q, \xi_2^q, \ldots, \xi_{r^q}^q; \eta_1^q, \eta_2^q, \ldots, \eta_{s^q}^q \right\}$$

be an independent set of q-cycles (mod $\{f < \bar{\gamma}\}$) in $\{f \leq \bar{\gamma}\} \cup g$ such that the ξ_i^q's may be completed to absolute q-cycles, and no combination of the η_i^q's may be completed to absolute cycles.

Now the set of absolute q-cycles,

(11) $$\left\{ \partial\eta_1^{q+1}, \partial\eta_2^{q+1}, \ldots, \partial\eta_{s^{q+1}}^{q+1} \right\}$$

are independent in $\{f < \gamma\}$, since no combination of the η_i^{q+1}'s may be completed to an absolute cycle. Therefore, this set may be expanded to a maximal homologically independent set by adding u^q absolute q-cycles,

(12) $$\Gamma_1^q, \Gamma_2^q, \ldots, \Gamma_{u^q}^q, \qquad (\text{in } \{f < \gamma\})$$

to the set $\left\{ \partial\eta_1^{q+1}, \ldots, \partial\eta_{s^{q+1}}^{q+1} \right\}$. Then the q-th connectivity number of $\{f < \bar{\gamma}\}$ is

(13) $$u^q + s^{q+1}.$$

We now show that

$$(14) \qquad \left\{ \Gamma_1^q, \ldots, \Gamma_{u^q}^q; \quad \zeta_1^q, \ldots, \zeta_{r^q}^q \right\}$$

is a maximal homologically independent set of q-cycles in $\{f < \bar{\gamma}\} \cup g$, where ζ_i^q is the cycle obtained by completing ξ_i^q to an absolute cycle, $i = 1, 2, \ldots, r^q$.

To show that (14) is an independent set, suppose some combination of chains of (14) is bounding in $\{f < \bar{\gamma}\} \cup g$, say

$$\sum_{i=1}^{\alpha} \Gamma_i^q + \sum_{i=1}^{\beta} \zeta_i^q \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\} \cup g).$$

Then, since Γ_i^q is in $\{f < \bar{\gamma}\}$ for $i = 1, 2, \ldots, u^q$,

$$(15) \qquad \sum_{i=1}^{\beta} \zeta_i^q \sim \emptyset \quad (\text{mod } \{f < \bar{\gamma}\} \text{ in } \{f < \bar{\gamma}\} \cup g).$$

But $\displaystyle\sum_{i=1}^{\beta} \zeta_i^q = \sum_{i=1}^{\beta} \xi_i^q \quad (\text{mod } \{f < \bar{\gamma}\})$, and so (15) implies

$$\sum_{i=1}^{\beta} \xi_i^q \sim \emptyset \quad (\text{mod } \{f < \bar{\gamma}\} \text{ in } \{f < \bar{\gamma}\} \cup g),$$

which contradicts the choice of $\xi_1^q, \ldots, \xi_{r^q}^q$. Thus, no combination of the chains of (14) which contains one or more of the ζ_i^q's is bounding in $\{f < \bar{\gamma}\} \cup g$.

Suppose

$$\sum_{i=1}^{\alpha} \Gamma_i^q \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\} \cup g).$$

Then there exists a $(q + 1)$-chain, D, in $\{f < \bar{\gamma}\} \cup g$ such that

$$\partial D = \sum_{i=1}^{\alpha} \Gamma_i^q.$$

Since D is a $(q + 1)$-cycle (mod $\{f < \gamma\}$), D is homologous (mod $\{f < \bar{\gamma}\}$ in $\{f < \bar{\gamma}\} \cup g$) to a combination of

$$\xi_1^{q+1}, \ldots, \xi_{r^{q+1}}^{q+1};\ \eta_1^{q+1}, \ldots, \eta_{s^{q+1}}^{q+1},$$

that is

$$D \sim \sum_{i=1}^{\mu} \xi_i^{q+1} + \sum_{i=1}^{\nu} \eta_i^{q+1} \quad (\text{mod } \{f < \bar{\gamma}\} \text{ in } \{f < \bar{\gamma}\} \cup g),$$

$\mu \le r^{q+1}$, $\nu \le s^{q+1}$. Then, there exists a $(q + 2)$-chain, P, in $\{f < \bar{\gamma}\} \cup g$ and a $(q + 1)$-chain, Q, in $\{f < \bar{\gamma}\}$ such that

$$\partial P = D + \sum_{i=1}^{\mu} \xi_i^{q+1} + \sum_{i=1}^{\nu} \eta_i^{q+1} + Q.$$

Hence, since $\partial\partial P = \emptyset$,

$$\partial D = \sum_{i=1}^{\alpha} \Gamma_i^q = \sum_{i=1}^{\mu} \partial \xi_i^{q+1} + \sum_{i=1}^{\nu} \partial \eta_i^{q+1} + \partial Q.$$

But since, each ξ_i^{q+1} can be completed to an absolute cycle in $\{f < \bar{\gamma}\}$,

$$\sum_{i=1} \partial \xi_i^{q+1} \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\}).$$

Therefore,

$$\sum_{i=1}^{\alpha} \Gamma_i^q + \sum_{i=1}^{\nu} \partial \eta_i^{q+1} + \partial Q \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\}),$$

and since Q is in $\{f < \bar{\gamma}\}$,

$$(16) \qquad \sum_{i=1}^{\alpha} \Gamma_i^q + \sum_{i=1}^{\nu} \partial \eta_i^{q+1} \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\}).$$

But (16) contradicts the independence of

$$\Gamma_1^q, \ldots, \Gamma_{t^q}^q \; ; \; \partial \eta_1^{q+1}, \ldots, \partial \eta_{s^{q+1}}^{q+1}.$$

Thus, (14) is an independent set in $\{f < \bar{\gamma}\} \cup g$.

To show that (14) is maximally independent, let T be an absolute q-cycle in $\{f < \bar{\gamma}\} \cup g$. Then, T is also a cycle (mod $\{f < \bar{\gamma}\}$), and hence, since (10) is maximally independent (mod $\{f<\bar{\gamma}\}$ in $\{f<\bar{\gamma}\}\bigcup g$), T is homologous (mod $\{f<\bar{\gamma}\}$ in $\{f<\bar{\gamma}\}\bigcup g$) to a combination of chains of (10); that is

$$T \sim \sum_{i=1}^{\sigma} \xi_i^q + \sum_{i=1}^{\rho} \eta_i^q \quad (\text{mod } \{f<\bar{\gamma}\} \text{ in } \{f<\bar{\gamma}\}\bigcup g).$$

Then, there exists a (q + 1)-chain, W, in $\{f<\bar{\gamma}\}\bigcup g$ and a q-chain U in $\{f<\bar{\gamma}\}$ such that

$$\partial W = T + \sum_{i=1}^{\sigma} \xi_i^q + \sum_{i=1}^{\rho} \eta_i^q + U. \tag{17}$$

Since each ξ_i^q can be completed to an absolute cycle in $\{f<\bar{\gamma}\}$, there exists a q-chain V in $\{f<\bar{\gamma}\}$ such that

$$\partial V = \partial T + \sum_{i=1}^{\sigma} \partial \xi_i^q.$$

Hence by (17),

$$\partial V = \sum_{i=1}^{\rho} \partial \eta_i^q + \partial U,$$

and since U is in $\{f<\bar{\gamma}\}$, this implies that a combination of the η_i^q's may be completed to an absolute cycle. Thus, no combination of chains of (10) which contains one or more of the η_i^q's is homologous (mod $\{f<\bar{\gamma}\}$ in $\{f<\bar{\gamma}\}\bigcup g$) to T.

Suppose

$$T \sim \sum_{i=1}^{\sigma} \xi_i^q \quad (\text{mod } \{f < \bar{\gamma}\} \text{ in } \{f < \bar{\gamma}\} \cup g).$$

Then,

$$T + \sum_{i=1}^{\sigma} \zeta_i^q \sim \emptyset \quad (\text{mod } \{f < \bar{\gamma}\} \text{ in } \{f < \bar{\gamma}\} \cup g),$$

where, as before, ζ_i^q is the cycle obtained by completing ξ_i^q to an absolute cycle. Then there exists a $(q + 1)$-chain F in $\{f < \bar{\gamma}\} \cup g$ and a q-chain G in $\{f < \bar{\gamma}\}$ such that

$$\partial F = T + \sum_{i=1}^{\sigma} \zeta_i^q + G.$$

But since G is an absolute cycle in $\{f < \bar{\gamma}\}$, G is homologous in $\{f < \bar{\gamma}\}$ to a combination of the chains of (11) and (12). Therefore,

$$\partial F + T + \sum_{i=1}^{\sigma} \zeta_i^q + \sum_{i=1}^{\epsilon} \Gamma_i^q + \sum_{i=1}^{\tau} \partial \eta_i^{q+1} + \partial H = \emptyset,$$

where H is a q-chain in $\{f < \bar{\gamma}\}$. This may be written

$$\partial\left[F + \sum_{i=1}^{\tau} \eta_i^{q+1} + H\right] = \sum_{i=1}^{\sigma} \zeta_i^q + \sum_{i=1}^{\epsilon} \Gamma_i^q + T,$$

and so

$$\sum_{i=1}^{\sigma} \zeta_i^q + \sum_{i=1}^{\epsilon} \Gamma_i^q + T \sim \emptyset \quad (\text{in } \{f < \bar{\gamma}\} \cup g).$$

This last expression implies that, if we add the cycle T to the set (14), the resulting set is not homologically independent in $\{f < \bar{\gamma}\} \cup g$.

Thus, (14) is a maximal, homological independent set and so the saltus of $R^q(\gamma)$ at $\bar{\gamma}$ is

$$r^q - s^{q+1}.$$

If follows that

$$R^q = \sum_{\gamma \in Q} \left[r^q(\gamma) - s^{q+1}(\gamma) \right]$$

$$= M_+^q - M_-^{q+1}$$

$$= M^q - M_-^q - M_-^{q+1}.$$

Hence,

$$\sum_{i=0}^{q} (M^i - R^i)(-1)^{q-i} = \sum_{i=0}^{q} (-1)^{q-i}(M_-^i + M_-^{i+1})$$

$$= (-1)^q M_-^0 + M_-^{q+1}.$$

But $M_-^0 = 0$, since every 0-cycle is an absolute cycle.

Q.E.D.

<u>Corollary</u>

$$M^q \geq R^q, \quad q = 0,1,\ldots,n.$$

<u>Proof</u>:

Since M_-^{q+1} is a non-negative integer,

$$M^q - R^q = \sum_{i=0}^{q} (M^i - R^i)(-1)^{q-i} + \sum_{i=0}^{q-1} (M^i - R^i)(-1)^{q-i-1} \geq 0.$$

Q.E.D.

CHAPTER 3

CRITICAL CONFIGURATIONS OF CHARGES ON AN M-TORUS

3.1 STATEMENT OF THE PROBLEM

The problem to be considered in this chapter has its origins in the following physical problem.

Let S be a conducting surface (in E^3) and let

$$p^i = (x^i, y^i, z^i) \quad ; \quad i = 1, 2, \ldots, n,$$

be the locations of n electrons constrained to lie on S. In a dynamic situation, the charges will remain at these locations if the net force acting on each charge (due to the presence of the other charges) is normal to the surface at the location of that charge. If this condition is satisfied, the charges are said to be in equilibrium on S, and the set of points, $p^1, p^2, \ldots, p^n$, forms a critical configuration of charges on S. The problem is to determine all of the critical configurations of charges on S.

Under certain conditions this problem may be formulated in terms of finding the stationary points of a real valued function. Suppose S may be represented by the equation $f(x,y,z) = 0$, where the first partial derivatives of f are continuous in some open subset of E^3 which contains S, and, at each point of S, at least one of the partial derivatives, $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, does not vanish. This condition will enable us to solve $f(x,y,z) = 0$ (locally) for one of the variables in terms of the other two, and to define a tangent plane at each point of S.

The net force acting on the charge located at p^i due to the presence of the other charges is given by

$$(1) \qquad F^i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{C\,\alpha_i\,\alpha_j}{\left|p^i - p^j\right|^3}\,(p^i - p^j)$$

where $\left|p^i - p^j\right|$ denotes the Euclidean distance between p^i and p^j, α_j is the magnitude of the charge located at p^j, $j = 1, 2, \ldots, n$, and C is a constant. (Peck, 1952, pg. 3). Let

$$V(p^1, p^2, \ldots, p^n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \frac{C\,\alpha_k\,\alpha_j}{\left|p^k - p^j\right|}$$

(V represents the total potential of the configuration.)

Equation (1) may now be wirtten:

$$F^i = \left(\partial V/\partial x^i,\ \partial V/\partial y^i,\ \partial V/\partial z^i \right)$$

Suppose that $\partial f/\partial z$ does not vanish at p^i. Then there exists a function, u, and an open neighborhood, N, of (x^i, y^i) such that the first partial derivatives of u are continuous on N and $F(x,y,u(x,y))=0$ for all $(x,y) \in N$. (Buck, 1956, pg. 222.) Hence, the tangent plane to S at p^i is given by

$$z - z^i = \left|\partial u/\partial x\right|(x - x^i) + \left|\partial u/\partial y\right|(y - y^i)$$

where $\partial u/\partial x$ and $\partial u/\partial y$ are evaluated at (x^i,y^i).

For equilibrium, F^i must be normal to this plane. Hence, if p is any point in the tangent plane, the vector, $p - p^i$, must be orthogonal to F^i at p^i. This condition may be written:

$$0 = F^i \cdot (p - p^i)$$

$$= \frac{\partial V}{\partial x^i}(x - x^i) + \frac{\partial V}{\partial y^i}(y - y^i) + \frac{\partial V}{\partial z^i}(z - z^i)$$

$$= \left(\frac{\partial V}{\partial x^i} + \frac{\partial V}{\partial z^i}\frac{\partial u}{\partial x}\right)(x - x^i) + \left(\frac{\partial V}{\partial y^i} + \frac{\partial V}{\partial z^i}\frac{\partial u}{\partial y}\right)(y - y^i)$$

where the dot, "$\cdot$", denotes inner product. Since this equation must hold for any p in the tangent plane, the equations

(2a) $$\frac{\partial V}{\partial x^i} + \frac{\partial V}{\partial z^i}\frac{\partial u}{\partial x} = 0$$

and

(2b) $$\frac{\partial V}{\partial y^i} + \frac{\partial V}{\partial z^i}\frac{\partial u}{\partial y} = 0$$

must be satisfied if the points p^1, p^2, ..., p^n are to form a critical configuration.

By repeating this procedure, two equilibrium conditions resembling (2a) and (2b) may be derived at each charge location. The equations for a different location will be identical in form with (2a) and (2b) with the exception that it may be necessary to replace x, y and z by some permutation of these symbols. For example, if $\partial f/\partial z = 0$ and $\partial f/\partial y \neq 0$ at the location p^k, the conditions at p^k are given by

$$\frac{\partial V}{\partial x^k} + \frac{\partial V}{\partial y^k}\frac{\partial v}{\partial x} = 0$$

and

$$\frac{\partial V}{\partial z^k} + \frac{\partial V}{\partial y^k}\frac{\partial v}{\partial z} = 0$$

where v(x,z) is a local solution of f(x,y,z) = 0 for y in terms of x and z.

Hence, if the points p^1, p^2, ..., p^n form a critical configuration, then these points satisfy 2n equations of the form

(3a) $$\frac{\partial V}{\partial \xi^i} + \frac{\partial V}{\partial \zeta^i}\frac{\partial w^i}{\partial \xi} = 0$$

and

(3b) $$\frac{\partial V}{\partial \eta^i} + \frac{\partial V}{\partial \zeta^i}\frac{\partial w^i}{\partial \eta} = 0,$$

$i = 1, 2, \ldots, n$, where ξ^i, η^i, ζ^i is some permutation of x^i, y^i, z^i and $w^i(\xi,\eta)$ is a local solution of $f(x,y,z) = 0$ for one of the variables in terms of the other two (in some neighborhood of (ξ^i, η^i)).

But, if $V(p^1, p^2, \ldots, p^n)$ is interpreted as a real valued function defined on a subset of S^n, the n-fold cartesian product of S with itself, then (3a) and (3b) are the conditions for a stationary point of V. In other words, if the set of points, $\{p^1, p^2, \ldots, p^n\}$, forms a critical configuration of charges on S, then the ordered n-tuple $(p^1, p^2, \ldots, p^n)$ is a stationary point of V on S^n, and so the problem of determining all critical configurations of charges on S reduces to that of finding all stationary points of V.

The problem just described may be generalized by replacing S by an m-dimensional surface in $(m + n)$-dimensional space and by replacing the Newtonian law of mutual repulsion, $1/|p^i - p^j|$, by a more general function. A variation of this problem, the determination of total number of critical configurations of charges on an m-torus, will be considered in this chapter.

Let T^m be the m-dimensional torus:

$$T^m = \left\{(x_1,x_2,\ldots,x_{2m}) \;\middle|\; x_{2i-1}^2 + x_{2i}^2 = 1, \quad i = 1,2,\ldots,m\right\} \cap E^{2m}$$

Let g be a real valued function which is strictly decreasing on $[0, \infty)$ and which has continuous third order derivatives at every point of $[0,\infty)$.

Let

$$V(p^1,p^2,\ldots,p^n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_i \alpha_j \; g(|p^i - p^j|^2),$$

where $p^i = (x_1^i, x_2^i, \ldots, x_{2m}^i) \in T^m$, α_i is a positive real number, $(i = 1,2,\ldots,n)$ and

$$|p^i - p^j|^2 = \sum_{k=1}^{2m} (x_k^i - x_k^j)^2 .$$

We define the critical configurations of n charges on T^m to be the stationary points of V on $(T^m)^n$.

Since T^1 is the m-fold cartesian product of T^1 with itself, $(T^m)^n = T^{mn}$. Therefore, the domain of V is an mn-dimensional manifold of class C^{∞}; i.e., the mn-dimensional torus. Moreover, the conditions imposed on g ensure that the third order partial derivatives of V are continuous on T^{mn}. However, before the theory of Chapter 2 can be applied, the potential function, V, must be modified.

To simplify the notation, the points of E^{mn} will be denoted by $\{y_{i,j}\}$ instead of

$$\left\{ y_1^1,\ y_2^1,\ \ldots,\ y_m^1;\ \ y_1^2,\ y_2^2,\ \ldots,\ y_m^2;\ \ldots\ ;\ \ y_1^n,\ y_2^n,\ \ldots,\ y_m^n \right\}.$$

Let $\left\{ y_{i,j} \right\} \in E^{mn}$ such that

(4a) $$\cos y_{i,j} = x_{2i-1}^{j}$$

and

(4b) $$\sin y_{i,j} = x_{2i}^{j}.$$

Then,

$$\begin{aligned} \left| p^i - p^j \right|^2 &= \sum_{k=1}^{m} \left[(x_{2k-1}^i - x_{2k-1}^j)^2 + (x_{2k}^i - x_{2k}^j)^2 \right] \\ &= 2 \sum_{k=1}^{m} \left[1 - \cos (y_{k,i} - y_{k,j}) \right] \\ &= 4 \sum_{k=1}^{m} \sin^2 \tfrac{1}{2}(y_{k,i} - y_{k,j}). \end{aligned}$$

Therefore,

$$V(p^1, p^2, \ldots, p^n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_i \alpha_j g(4 \sum_{k=1}^{m} \sin^2 (y_{k,i} - y_{k,j})).$$

Clearly, if each $y_{k,i}$ is increased by an amount $\varepsilon \neq 0$, then V is unchanged. Hence, since V has at least one stationary point (T^{mn} is compact), V has an uncountably infinite number of stationary points. Therefore, the theory of Chapter 2 does not apply.

This situation may be avoided by placing a fixed charge on T^m. Let p^o be the location of a fixed charge on T^m. By symmetry, p^o may be located anywhere on T^m. Therefore, let

$$p^o = (1, 0, 1, 0, \ldots, 1, 0).$$

Then the potential function of the configuration now takes the form:

$$(5) \quad V(p^1, p^2, \ldots, p^n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_i \alpha_j \, g\left(4 \sum_{k=1}^{m} \sin^2 \tfrac{1}{2}(y_{k,i} - y_{k,j})\right) + \sum_{j=1}^{n} \alpha_0 \alpha_j \, g\left(4 \sum_{k=1}^{m} \sin^2 \tfrac{1}{2} y_{k,j}\right)$$

Now if each $y_{i,j}$ in (4a) and (4b) is restricted to a sufficiently small interval, (4a) and (4b) define a homeomorphism of T^{mn} into E^{mn}. Hence, the conditions for a stationary point of V are:

$$(6)\quad \frac{\partial V}{\partial y_{r,s}} = -\sum_{i=1}^{s-1} 2\alpha_i\alpha_s g'\left(4\sum_{k=1}^{m}\sin^2\tfrac{1}{2}(y_{k,i}-y_{k,s})\right)\sin(y_{r,i} - y_{r,s})$$
$$+\sum_{i=s+1}^{n} 2\alpha_s\alpha_i g'\left(4\sum_{k=1}^{m}\sin^2\tfrac{1}{2}(y_{k,s}-y_{k,i})\right)\sin(y_{r,s} - y_{r,i})$$
$$+\ 2\alpha_0\alpha_s g'\left(4\sum_{k=1}^{m}\sin^2\tfrac{1}{2}y_{k,s}\right)\sin y_{r,s}$$

$$= 0,$$

$r = 1,2,\ldots,m;\quad s = 1,2,\ldots,n.$

For each $y \in E$, let

$$\bar{y} = \left\{z \,\middle|\, z-y \text{ is a multiple of } 2\pi\right\}.$$

Let

$$\bar{E} = \left\{\bar{y} \,\middle|\, y \in E\right\}.$$

As in the case of E^{mn}, the elements of $\bar{E}^{mn}$ will be denoted by $\left\{\bar{y}_{i,j}\right\}$.

An element $\left\{\bar{y}_{i,j}\right\}$ of $\bar{E}^{mn}$ will be called a <u>solution</u> of (6) if for any $\left\{z_{i,j}\right\} \in E^{mn}$ such that

$$\bar{z}_{i,j} = \bar{y}_{i,j}, \ (i = 1,2,\ldots,m, \ j = 1,2,\ldots,n),$$

$$0 = \sum_{\substack{i=0 \\ i \neq s}}^{n} 2\alpha_i\alpha_s \ g'\left(4\sum_{k=1}^{m} \sin^2\tfrac{1}{2}(z_{k,s} - z_{k,i})\right) \ \sin(z_{r,s} - z_{r,i}),$$

$r = 1,2,\ldots,m; \quad s = 1,2,\ldots,n.$

It follows from (4a) and (4b) and (6), that there exists a 1-1 correspondence between the stationary points of V on T^{mn} and the solutions of (6) in $\bar{E}^{mn}$. Therefore, in the sequel the solutions of (6) in $\bar{E}^{mn}$ will sometimes be called stationary points of V in T^{mn}.

The remainder of this chapter will be devoted to the enumeration of the stationary points of the potential given by (5) or, alternatively, to the enumeration of the solutions of (6) in $\bar{E}^{mn}$. The problem of finding the total number of stationary points with a given index will also be considered.

3.2 - CRITICAL CONFIGURATIONS OF TWO CHARGES ON AN m-TORUS

When there are but two charges on T^m, one of which is fixed, equation (6) becomes

$$(7) \qquad 2\,\alpha_0\,\alpha_1\,g'\left(4\sum_{k=1}^{m}\sin^2\tfrac{1}{2}y_{k,1}\right)\sin y_{r,1} = 0;$$

$r = 1,2,\ldots,m.$

In this case, the critical configurations may be enumerated without the aid of the Morse Theory.

Now,

$$g'\left(4\sum_{k=1}^{m}\sin^2\tfrac{1}{2}y_{k,1}\right) \neq 0,$$

since g is strictly increasing. Therefore, the only solutions of (7) are those for which

$$\sin y_{i,1} = 0, \quad i = 1,2,\ldots,m.$$

Hence, (7) is satisfied if and only if

$$y_{i,1} \in \overline{\pi} \text{ or } y_{i,1} \in \overline{0}, \quad i = 1,2,\ldots,m.$$

Theorem 3.1

Let K^m be the number of critical configurations of two charges on T^m. Then,

$$K^m = 2^m.$$

Proof

It suffices to show that the set of all solutions of (7) in $\overline{E}^{ml}$ may be put into one to one correspondence with the set of all subsets of the first m integers. But, if $\{\overline{y}_{i,1}\} \in \overline{E}^{ml}$ is a solution of (7), then the set of all i such that $\overline{y}_{i,1} = \overline{0}$ forms a (unique) subset of the first m integers. On the other hand, let A be a subset of the first m integers, and let $\{\overline{y}_{i,1}\} \in \overline{E}^{ml}$ such that

$$\overline{y}_{i,1} = \overline{0} \text{ if } i \in A$$

and

$$\overline{y}_{i,1} = \overline{\pi} \text{ if } i \notin A.$$

Then $\{\overline{y}_{i,1}\}$ is a solution of (7).

Q.E.D.

Suppose $\{\overline{y}_{i,1}\}$ is a solution of (7). Then

$$\frac{\partial^2 V}{\partial y_{i,1}^2} = 2\alpha_0\alpha_1 \cos y_{i,1}\, g'\left(4\sum_{k=1}^{m}\sin^2\frac{1}{2}y_{k,1}\right)$$

$$+ \; 4\alpha_0\alpha_1 (\sin y_{i,1})^2 \; g''\left(4\sum_{k=1}^{m}\sin^2\frac{1}{2}y_{k,1}\right)$$

$$= 2\alpha_0\alpha_1 \cos y_{i,1} \; g'\left(4\sum_{k=1}^{m}\sin^2\frac{1}{2}y_{k,1}\right), \quad i = 1,2,\ldots,m,$$

and

$$\frac{\partial^2 V}{\partial y_{i,1}\partial y_{j,1}} = 4\alpha_0\alpha_1 \sin y_{i,1} \sin y_{j,1} \; g''\left(4\sum_{k=1}^{m}\sin^2\frac{1}{2}y_{k,1}\right)$$

$$= 0, \qquad i,j = 1,2,\ldots,m; \; i \neq j.$$

Hence, the Hessian of V at $\{\bar{y}_{i,1}\}$ is of the form

$$\prod_{i=1}^{m} 2\alpha_0\alpha_1 \; g'\left(4\sum_{k=1}^{m}\sin^2\frac{1}{2}y_{k,1}\right) \cos y_{i,1} = (-1)^r\left[2\alpha_0\alpha_1 g'\left(4\sum_{i=1}^{m}\sin^2\frac{1}{2}y_{i,1}\right)\right]^m$$

where r is some integer. Thus, the stationary points of V on T^m are non-degenerate.

The next theorem may be proved by combinatorial methods similar to those used to prove Theorem 3.1. However, the proof given here is based on some results of the preceeding chapter.

Theorem 3.2

$$M^k = \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

where M^k is the number of stationary points of index k.

Proof

Let $R^k_{(m)}$ denote the k-th connectivity number of T^m. Then

$$R^k_{(m)} = R^k_{(m-1)} R^0_{(1)} + R^{k-1}_{(m-1)} R^1_{(1)}$$

(Alexandroff and Hopf, 1935, pg. 309), since $T^m = T^{m-1} \times T^1$. But $R^0_{(1)} = R^1_{(1)} = 1$. (Pontryagin, 1952, pg. 47, Theorem 1 , T^1 is homeomorphic to the polyhedron denoted by $|S^1|$ in this reference). Therefore, by a simple inductive argument,

$$R^k_{(m)} = \binom{m}{k}$$

By corollary 1 of Theorem 2.7,

$$\binom{m}{k} = R^k_{(m)} \geq M^k .$$

Now, if $\binom{m}{k} \neq M^k$, for some k, then

$$2^m = \sum_{k=0}^{m} \binom{m}{k} < \sum_{k=0}^{m} M^k$$

which contradicts Theorem 3.1.

Q.E.D.

3.3 THREE CHARGES ON AN m-TORUS

For three charges on T^m, equation (6) becomes

$$(8)\qquad 2\alpha_0\alpha_1\, g'(4\rho_{01}) \sin y_{i,1} = -2\alpha_1\alpha_2\, g'(4\rho_{12}) \sin(y_{i,1} - y_{i,2})$$
$$= -2\alpha_0\alpha_2\, g'(4\rho_{02}) \sin y_{i,2}$$

where

$$(9)\qquad \rho_{i,j} = \sum_{k=1}^{m} \sin^2\tfrac{1}{2}(y_{k,i} - y_{k,j}),\quad i,j = 1,2,3.$$

Clearly, those values of $\bar{y}_{i,1}$ and $\bar{y}_{i,2}$ for which $\sin y_{i,1} = 0$ and $\sin y_{i,1} = 0$ satisfy (8). However, there may be other solutions since (8) does not imply $\sin y_{i,j} = 0$. Hence, the straight-forward approach used in the two charge problem no longer applies.

The three charge problem, i.e., the enumeration of the stationary points of V on T^{2m}, will now be reduced to the enumeration of the stationary points of real valued functions defined on T^2.

Let $\hat{m}$ be the set of all positive integers which do not exceed m:

$$\hat{m} = \left\{ i \,\middle|\, i \text{ is an integer and } 0 < i \leq m \right\}.$$

Lemma 3.1

Let $A \subset \hat{m}$. Let $B_1 \subset \hat{m} - A$ and $B_2 \subset \hat{m} - A$. Let k, b_{01}, b_{02}, and b_{12} be, respectively, the number of elements in A, B_1, B_2 and $(B_1 \cup B_2) - (B_1 \cap B_2)$. Let $(u,v) \in E^2$ such that $\sin u \neq 0$ and

$$(10) \quad \alpha_0\alpha_1 g'\left(4b_{01}+4k\sin^2\tfrac{1}{2}u\right)\sin u = -\alpha_1\alpha_2 g'\left(4b_{12}+4k\sin^2\tfrac{1}{2}(u-v)\right)\sin(u-v)$$
$$= -\alpha_0\alpha_2 g'\left(4b_{02}+4k\sin^2\tfrac{1}{2}v\right)\sin v$$

Let Γ be a function on A such that $\Gamma_i^2 = 1$ for each $i \in A$.

Let $\{y_{i,j}\} \in E^{m2}$ such that

(i) $y_{i,1} \in \Gamma_i\bar{u}$ and $y_{i,2} \in \Gamma_i\bar{v}$ if $i \in A$

(ii) $y_{i,1} \in \bar{\pi}$, if $i \in B_1$

(iii) $y_{i,1} \in \bar{0}$, if $i \in (\hat{m} - A) - B_1$

(iv) $y_{i,2} \in \bar{\pi}$, if $i \in B_2$

and (v) $y_{i,2} \in \bar{0}$, if $i \in (\hat{m} - A) - B_2$

Then $\{\bar{y}_{i,j}\}$ is a solution of (8).

Proof

If $i \in \hat{m}-A$, then $y_{i,1}$ and $y_{i,2}$ satisfy (8) since

$$\sin y_{i,1} = \sin y_{i,2} = \sin (y_{i,1} - y_{i,2}) = 0.$$

If $i \in A$ then, since $y_{i,1} \in \Gamma_i \bar{u}$ and $y_{i,2} \in \Gamma_i \bar{v}$,

$$\alpha_0\alpha_1\, g'\left(4b_{01} + 4k \sin^2\tfrac{1}{2}u\right) \sin y_{i,1}$$

$$= -\alpha_1\alpha_2 g'\left(4b_{12} + 4k \sin^2\tfrac{1}{2}(u-v)\right) \sin(y_{i,1}-y_{i,2})$$

$$= -\alpha_0\alpha_2 g'\left(4b_{02} + 4k \sin^2\tfrac{1}{2}v\right) \sin y_{i,2}$$

Hence, to complete the proof, we need only show

$$\rho_{01} = \sum_{i=1}^{m} \sin^2\tfrac{1}{2}y_{i,1} = b_{01} + k \sin^2\tfrac{1}{2}u$$

$$\rho_{02} = \sum_{i=1}^{m} \sin^2\tfrac{1}{2}y_{i,2} = b_{02} + k \sin^2\tfrac{1}{2}v$$

and

$$\rho_{12} = \sum_{i=1}^{m} \sin^2\tfrac{1}{2}(y_{i,1} - y_{i,2}) = b_{12} + k \sin^2\tfrac{1}{2}(u - v).$$

Since $A \cap B = \emptyset$,

$$\rho_{01} = \sum_{i\in(\hat{m}-A-B_1)} \sin^2\tfrac{1}{2}y_{i,1} + \sum_{i\in B_1} \sin^2\tfrac{1}{2}y_{i,1} + \sum_{i\in A} \sin^2\tfrac{1}{2}y_{i,1}$$

$$= (m - k - b_{01})\cdot 0 + b_{01}\cdot 1 + \sum_{i\in A} \Gamma_i^2 \sin^2\tfrac{1}{2}u$$

$$= b_{01} + k \sin^2\tfrac{1}{2}u.$$

Similarly,

$$\rho_{02} = \sum_{i\in(\hat{m}-A-B_2)} \sin^2\tfrac{1}{2}y_{i,2} + \sum_{i\in B_2} \sin^2\tfrac{1}{2}y_{i,2} + \sum_{i\in A} \sin^2\tfrac{1}{2}y_{i,2}$$

$$= b_{02} + k \sin^2\tfrac{1}{2}v \quad ,$$

and since $(B_1-B_2)\cup(B_2-B_1) = (B_1\cup B_2) - (B_1\cap B_2)$ and $(B_1-B_2)\cap(B_2-B_1) = \emptyset$,

$$\rho_{12} = \sum_{i\in K} \sin^2\tfrac{1}{2}(y_{i,1}-y_{i,2}) + \sum_{i\in(B_2-B_1)} \sin^2\tfrac{1}{2}(y_{i,1}-y_{i,2})$$

$$+ \sum_{i\in(B_1-B_2)} \sin^2\tfrac{1}{2}(y_{i,1}-y_{i,2}) + \sum_{i\in A} \sin^2\tfrac{1}{2}(y_{i,1}-y_{i,2})$$

$$= b_{12} + k \sin^2\tfrac{1}{2}(u - v) \ ,$$

where $K = \hat{m} - (B_1-B_2) - (B_2-B_1)$.

Q.E.D.

The preceeding lemma suugests a procedure for constructing solutions of (8). We first select the sets A, B_1, and B_2 described in the lemma and then seek solutions of (10). The method of constructing solutions of (10) from solutions of (8) is clear from the lemma.

We now show that <u>all</u> solutions of (8) may be constructed in this manner.

<u>Lemma 3.2</u>

Let $\left\{\bar{y}_{i,j}\right\} \in \bar{E}^{m2}$ be a solution of (9). Suppose $\sin y_{r,1} \neq 0$ and $\sin y_{s,1} \neq 0$ for $y_{r,1} \in \bar{y}_{r,1}$ and $y_{s,1} \in \bar{y}_{s,1}$. Then there exists $\mu \in E$ such that $\mu^2 = 1$,

$$\bar{y}_{r,1} = \mu \bar{y}_{s,1} \quad \text{and} \quad \bar{y}_{r,2} = \mu \bar{y}_{s,2}$$

<u>Proof</u>

By hypothesis,

$$\begin{aligned}\alpha_0\alpha_1 g'(4\rho_{01}) \sin y_{r,1} &= -\alpha_1\alpha_2 g'(4\rho_{12}) \sin(y_{r,1} - y_{r,2}) \\ &= -\alpha_0\alpha_2 g'(4\rho_{02}) \sin y_{r,2}\end{aligned}$$

and

$$\begin{aligned}\alpha_0\alpha_1 g'(4\rho_{01}) \sin y_{s,1} &= -\alpha_1\alpha_2 g'(4\rho_{12}) \sin(y_{s,1} - y_{s,2}) \\ &= -\alpha_0\alpha_2 g'(4\rho_{02}) \sin y_{s,2},\end{aligned}$$

where

$$\rho_{ij} = \sum_{k=1}^{m} \sin^2\frac{1}{2}(y_{k,i} - y_{k,j}), \quad i,j = 0,1,2.$$

Then, since $g'(4\rho_{01}) \neq 0$ and $\sin y_{s,1} \neq 0$,

$$\text{(11)} \qquad \frac{\sin y_{r,1}}{\sin y_{s,1}} = \frac{\sin y_{r,2}}{\sin y_{s,2}} = \frac{\sin(y_{r,1} - y_{r,2})}{\sin(y_{s,1} - y_{s,2})}$$

Let

$$\text{(12)} \qquad \lambda = \frac{\sin(y_{r,1} - y_{r,2})}{\sin(y_{s,1} - y_{s,2})}$$

Then,

$$\sin y_{r,1}\cos y_{r,2} - \sin y_{r,2}\cos y_{r,1} = \lambda\sin y_{s,1}\cos y_{s,2} - \lambda\sin y_{s,2}\cos y_{s,1}$$

also, from (11) and (12),

$$\sin y_{r,1} = \lambda\sin y_{s,1} \quad \text{and} \quad \sin y_{r,2} = \lambda\sin y_{s,2}.$$

Therefore, since $\lambda = 0$,

(13) $(\cos y_{r,2} - \cos y_{s,2}) \sin y_{s,1} = (\cos y_{r,1} - \cos y_{s,1}) \sin y_{s,2}$.

Let $a = \sin y_{s,1}$ and let $b = \sin y_{s,2}$. Then equation (13) may be written:

(14) $\varepsilon_1 (\sqrt{1 - \lambda^2 b^2} - \varepsilon_2 \sqrt{1 - b^2}) a = \varepsilon_3 (\sqrt{1 - \lambda^2 a^2} - \varepsilon_4 \sqrt{1 - a^2}) b$

where $\varepsilon_i^2 = 1$, $i = 1,2,3,4$. Squaring (14) and simplifying, we obtain

$$a^2 - b^2 = \varepsilon_2 a^2 \sqrt{1 - \lambda^2 b^2} \sqrt{1 - b^2} - \varepsilon_4 b^2 \sqrt{1 - \lambda^2 a^2} \sqrt{1 - a^2}$$

Again squaring and simplifying, we obtain

$$-1 + \frac{1}{2}(a^2 + b^2)(1 - \lambda^2) - a^2 b^2 \lambda^2$$

$$= -\varepsilon_2 \varepsilon_4 \sqrt{1 - \lambda^2 a^2} \sqrt{1 - a^2} \sqrt{1 - \lambda^2 b^2} \sqrt{1 - b^2}$$

Squaring once more and simplifying, we finally arrive at

$$(\lambda^2 - 1)^2 (a^2 - b^2)^2 = 0$$

Thus,

$$\sin^2 y_{s,1} = \sin^2 y_{s,2}$$

(from (11) and (12)), or

$$\sin^2 y_{s,1} = \sin^2 y_{r,1} .$$

Suppose $\sin^2 y_{s,2} = \sin^2 y_{s,1}$. From equation (10), $\sin y_{s,1}$ and $\sin y_{s,2}$ are opposite in sign. Therefore,

$$\sin y_{s,1} = -\sin y_{s,2}.$$

Hence, $\overline{y}_{s,1} = -\overline{y}_{s,2}$ or $\overline{y}_{s,1} = \overline{y}_{s,2} - \overline{\pi}$. But, if $\overline{y}_{s,1} = \overline{y}_{s,2} - \overline{\pi}$, then,

$$\sin(y_{s,2} - y_{s,1}) = 0,$$

which is impossible since $\sin y_{s,2} \neq 0$. Therefore,

$$\overline{y}_{s,1} = -\overline{y}_{s,2}. \tag{15}$$

From equation (11), $\sin y_{r,1} = -\sin y_{r,2}$, and so, by a similar argument,

(16) $$\bar{y}_{r,1} = -\bar{y}_{r,2}.$$

Therefore, from (11),

$$\frac{\sin y_{r,1}}{\sin y_{s,1}} = \frac{\sin 2y_{r,1}}{\sin 2y_{s,1}} = \frac{\sin y_{r,1} \cos y_{r,1}}{\sin y_{s,1} \cos y_{s,1}}$$

Hence,

$$\cos y_{r,1} = \cos y_{s,1}.$$

This implies,

$$\bar{y}_{r,1} = \bar{y}_{s,1} \quad \text{or} \quad \bar{y}_{r,1} = -\bar{y}_{s,1}.$$

If $\bar{y}_{r,1} = \bar{y}_{s,1}$, then, by (15) and (16), the lemma is true for $\mu = 1$. Similarly, if $\bar{y}_{r,1} = -\bar{y}_{s,1}$, the lemma is true for $\mu = -1$.

Suppose $\sin^2 y_{s,1} = \sin^2 y_{r,1}$. Then,

$$\sin y_{s,1} = \sin y_{r,1} \quad \text{or} \quad \sin y_{s,1} = -\sin y_{r,1}.$$

Hence, there are two cases to consider.

Case 1

Suppose $\sin y_{s,1} = \sin y_{r,1}$. Then, from (11), $\sin y_{s,2} = \sin y_{r,2}$. Hence,

$$\bar{y}_{r,1} = \bar{y}_{s,1} \text{ or } \bar{y}_{r,1} = \bar{\pi} - \bar{y}_{s,1},$$

and

$$\bar{y}_{r,2} = \bar{y}_{s,2} \text{ or } \bar{y}_{r,2} = \bar{\pi} - \bar{y}_{s,2}.$$

Now, from (11),

$$\sin(y_{r,1} - y_{r,2}) = \frac{\sin y_{r,1}}{\sin y_{s,1}} \sin(y_{s,1} - y_{s,2}) = \sin(y_{s,1} - y_{s,2})$$

Hence, if $\bar{y}_{r,1} = \bar{\pi} - \bar{y}_{s,1}$ and $\bar{y}_{r,2} = \bar{\pi} - \bar{y}_{s,2}$, then

$$\sin(y_{r,1} - y_{r,2}) = -\sin(y_{s,1} - y_{s,2}),$$

which is impossible.

If $\bar{y}_{r,1} = \bar{\pi} - \bar{y}_{s,1}$ and $\bar{y}_{r,2} = \bar{y}_{s,2}$, then

$$\sin(y_{r,1} - y_{r,2}) = \sin(y_{s,1} - y_{s,2})$$

$$= \sin(y_{s,1} + y_{s,2}).$$

Hence,

$$\bar{y}_{s,1} - \bar{y}_{s,2} = \bar{y}_{s,1} + \bar{y}_{s,2}$$

or

$$\bar{y}_{s,1} - \bar{y}_{s,2} = \bar{\pi} - \bar{y}_{s,1} - \bar{y}_{s,2}.$$

But $2y_{s,2} \in \tilde{0}$, since $\sin y_{s,2} \neq 0$. Therefore, $2\bar{y}_{s,1} = \bar{\pi}$ and so,

$$y_{s,1} \in \overline{\pi/2} \quad \text{or} \quad y_{s,1} \in -\overline{\pi/2}.$$

If $y_{s,1} \in \overline{\pi/2}$, then

$$\bar{y}_{r,1} = \bar{\pi} - \overline{\pi/2} = \overline{\pi/2} = \bar{y}_{s,1}.$$

Similarly, if $y_{s,1} \in -\overline{\pi/2}$, then

$$\bar{y}_{r,1} = \bar{y}_{s,1}.$$

By a similar argument, if $\bar{y}_{r,1} = \bar{y}_{s,2}$, and $\bar{y}_{r,2} = \bar{\pi} - \bar{y}_{s,2}$, then

$$\bar{y}_{r,2} = \bar{y}_{s,2}.$$

Thus, for case 1, the lemma holds for, $\mu = 1$.

Case 2

Suppose $\sin y_{r,1} = -\sin y_{s,1}$. Then, as in Case 1, there are four possibilities to consider: (i), $\bar{y}_{r,1} = -\bar{y}_{s,1}$ and $\bar{y}_{r,2} = -\bar{y}_{s,2}$; (ii), $\bar{y}_{r,1} = -\bar{y}_{s,1}$ and $\bar{y}_{r,2} = \bar{y}_{s,2} - \bar{\pi}$; (iii), $\bar{y}_{r,1} = \bar{y}_{s,1} - \bar{\pi}$ and $\bar{y}_{r,2} = -\bar{y}_{s,2}$; (iv), $\bar{y}_{r,1} = \bar{y}_{s,1} - \bar{\pi}$ and $\bar{y}_{r,2} = \bar{y}_{s,2} - \bar{\pi}$.

As in case 1, $\bar{y}_{r,1} = \bar{y}_{s,1} - \bar{\pi}$ and $\bar{y}_{r,2} = \bar{y}_{s,2} - \bar{\pi}$ is impossible since

$$\begin{aligned}\sin(y_{r,1} - y_{r,2}) &= \frac{\sin y_{r,1}}{\sin y_{s,1}} \sin(y_{s,1} - y_{s,2}) \\ &= -\sin(y_{s,1} - y_{s,2}) \\ &\neq \sin(y_{s,1} - y_{s,2})\end{aligned}$$

If $\bar{y}_{r,1} = -\bar{y}_{s,1}$ and $\bar{y}_{r,2} = \bar{y}_{s,2} - \bar{\pi}$, then

$$\begin{aligned}-\sin(y_{r,1} - y_{r,2}) &= -\sin(y_{s,1} + y_{s,2}) \\ &= \sin(y_{s,1} - y_{s,2}).\end{aligned}$$

Hence $\bar{y}_{s,2} - \bar{y}_{s,1} = \bar{y}_{s,1} + \bar{y}_{s,2}$, or $\bar{y}_{s,1} - \bar{y}_{s,2} = \bar{y}_{s,1} + \bar{y}_{s,2} - \bar{\pi}$.

and, as in case 1, $2y_{s,1} \notin \bar{0}$, and so

$$\bar{y}_{s,2} = \overline{\pi/2} \quad \text{or} \quad \bar{y}_{s,2} = -\overline{\pi/2}$$

Therefore,

$$\bar{y}_{r,2} = \overline{\pi/2} - \bar{\pi} = -\overline{\pi/2} = -\bar{y}_{s,2}$$

or $$\bar{y}_{r,2} = -\overline{\pi/2} - \bar{\pi} = \overline{\pi/2} = -\bar{y}_{s,2}.$$

Similarly, if $\bar{y}_{r,1} = \bar{y}_{s,1} - \bar{\pi}$ and $\bar{y}_{r,2} = -\bar{y}_{s,2}$, then, $\bar{y}_{r,1} = -\bar{y}_{s,1}$.

Thus, for case 2, the lemma is true for $\mu = -1$.

Q.E.D.

Definition 3.1

Let $A \subset \hat{m}$, $B_1 \subset \hat{m} - A$, and $B_2 \subset \hat{m} - A$. Then, for each $(p^1, p^2) \in T^2$, let

$$\begin{aligned} W(p^1,p^2;A,B_1,B_2) &= \alpha_0\alpha_1 g\left(4b_{01} + k\left|p^1-p^0\right|^2\right) \\ &+ \alpha_0\alpha_2 g\left(4b_{02} + k\left|p^2-p^0\right|^2\right) \\ &+ \alpha_1\alpha_2 g\left(4b_{12} + k\left|p^1-p^2\right|^2\right), \end{aligned}$$

where k, b_{01}, b_{02}, and b_{12} are, respectively, the number of elements in A, B_1, B_2, and $(B_1 \cup B_2) - (B_1 \cap B_2)$, and $p^0 = (1,0,1,0) \in T^2$.

Clearly, the third order partial derivatives of $W(p^1,p^2;A,B_1,B_2)$ are continuous on T^2. In terms of local coordinates,

$$\begin{aligned} W(p^1,p^2;A,B_1,B_2) &= \alpha_0\alpha_1\ g\left(4b_{01} + 4k\ \sin^2\tfrac{1}{2}u\right) \\ &+ \alpha_0\alpha_2\ g\left(4b_{02} + 4k\ \sin^2\tfrac{1}{2}v\right) \\ &+ \alpha_1\alpha_2\ g\left(4b_{12} + 4k\ \sin^2\tfrac{1}{2}(u-v)\right), \end{aligned}$$

where $(u,v) \in E^2$ such that

$$p^1 = (\cos u,\ \sin u)$$

and

$$p^2 = (\cos v,\ \sin v).$$

Following the convention introduced in section 3.1, a point, $(\bar{u},\bar{v}) \in \bar{E}^2$, will be called a stationary point of $W(p^1,p^2;A,B_1,B_2)$ if, for each $u \in \bar{u}$ and $v \in \bar{v}$,

$$\begin{aligned} (17)\quad 2\alpha_0\alpha_1 g'\left(4b_{01}+4k\ \sin^2\tfrac{1}{2}u\right)\sin u &= -2\alpha_0\alpha_2 g'\left(4b_{02}+4k\sin^2\tfrac{1}{2}v\right)\sin v \\ &= -2\alpha_1\alpha_2 g'\left(4b_{12}+4k\ \sin^2\tfrac{1}{2}(u-v)\right)\sin(u-v). \end{aligned}$$

Theorem 3.3

Let $\{\bar{y}_{i,j}\} \in \tilde{E}^{m2}$. Let

$$A = \{ i \mid \bar{y}_{i,1} \neq \bar{0} \text{ and } \bar{y}_{i,1} \neq \bar{\pi} \}$$

$$B_1 = \{ i \mid \bar{y}_{i,1} = \bar{\pi} \}$$

$$B_2 = \{ i \mid \bar{y}_{i,2} = \bar{\pi} \}.$$

Then $\{\bar{y}_{i,j}\}$ is a stationary point of V if and only if

(i) $(\bar{y}_{i,1}, \bar{y}_{i,2})$ is a stationary point of $W(p^1, p^2; A, B_1, B_2)$, $i = 1, 2, \ldots, m$,

and (ii) there exists a function Ω on $A \times A$ to $\{-1, 1\}$ such that, if $i \in A$ and $j \in A$, then

$$\bar{y}_{i,1} = \Omega_{i,j} \bar{y}_{j,1} \text{ and } \bar{y}_{i,2} = \Omega_{i,j} \bar{y}_{j,2}.$$

Proof:

Suppose $\{\bar{y}_{i,j}\}$ is a stationary point of V. Then, by lemma 3.2, there exists a function Ω on A to $\{-1, 1\}$ such that $\bar{y}_{i,1} = \Omega_{i,j} \bar{y}_{j,1}$ and $\bar{y}_{i,2} = \Omega_{i,j} \bar{y}_{j,2}$; $i,j \in A$.

Clearly, if $i \in \hat{m} - A$, then $(\bar{y}_{i,1}, \bar{y}_{i,2})$ is a stationary point of $W(p^1, p^2; A, B_1, B_2)$, since

$$\sin y_{i,1} = \sin y_{i,2} = \sin(y_{i,1} - y_{i,2}) = 0.$$

Let $i \in A$. Then, by an argument similar to the one used in the proof of lemma 3.1,

(18a)
$$\sum_{j=1}^{m} \sin^2\frac{1}{2}y_{j,1} = b_{01} + \sum_{i\in A} \sin^2\frac{1}{2}\Omega_{i,j}y_{i,1}$$
$$= b_{01} + k \sin^2\frac{1}{2}y_{i,1}.$$

Similarly,

(18b)
$$\sum_{j=1}^{m} \sin^2\frac{1}{2}y_{j,2} = b_{02} + k \sin^2\frac{1}{2}y_{i,2},$$

and

(18c)
$$\sum_{j=1}^{m} \sin^2\frac{1}{2}(y_{j,1} - y_{j,2}) = b_{12} + k \sin^2\frac{1}{2}(y_{i,1} - y_{i,2}).$$

But, since $\{\bar{y}_{i,j}\}$ is a stationary point of V,

(19)
$$2\alpha_0\alpha_1 \, g'\left(4\sum_{j=1}^{m}\sin^2\frac{1}{2}y_{j,1}\right)\sin y_{i,1}$$
$$= -2\alpha_0\alpha_2 \, g'\left(4\sum_{j=1}^{m}\sin^2\frac{1}{2}y_{j,2}\right)\sin y_{i,2}$$
$$= -2\alpha_1\alpha_2 \, g'\left(4\sum_{j=1}^{m}\sin^2\frac{1}{2}(y_{j,1}-y_{j,2})\right)\sin(y_{i,1}-y_{i,2}).$$

Thus, by (18a), (18b), (18c) and (19), $(\bar{y}_{i,1}, \bar{y}_{i,2})$ is a stationary point of $W(p^1, p^2; A, B_1, B_2)$.

Suppose conditions (i) and (ii) of the theorem are satisfied.

By definition, $\bar{y}_{i,1} = \bar{\pi}$, if $i \in B_1$ and $\bar{y}_{i,2} = \bar{\pi}$, if $i \in B_2$. Also, if $i \in \hat{m} - A - B_1$, then $\sin y_{i,1} = 0$ and $\bar{y}_{i,1} \neq \bar{\pi}$. Therefore, $\bar{y}_{i,1} = \bar{0}$ if $i \in \hat{m} - A - B_1$. Furthermore, if $i \in \hat{m} - A - B_2$, then $\sin y_{i,2} = 0$ and $\bar{y}_{i,2} \neq \bar{\pi}$, since $\sin y_{i,1} \doteq 0$ and $(\bar{y}_{i,1}, \bar{y}_{i,2})$ is a stationary point of $W(p^1, p^2; A, B_1, B_2)$. Therefore, $\bar{y}_{i,2} = \bar{0}$ if $i \in \hat{m} - A - B_2$.

Thus, $\{\bar{y}_{i,j}\}$ satisfies conditions (ii), (iii), (iv) and (v) of lemma 3.1.

Let $i \in A$ and let $(u,v) = (y_{i,1}, y_{i,2})$. Let Γ be on A to $\{-1, 1\}$ such that if $j \in A$, then $\Gamma_j = \Omega_{ij}$. Then (u,v) and Γ satisfy the conditions of lemma 3.1. Also, if $j \in A$ then $\bar{y}_{j,1} = \Omega_{i,j}\bar{y}_{i,1} = \Gamma_j\bar{u}$ and $\bar{y}_{j,2} = \Omega_{i,j}\bar{y}_{i,2} = \Gamma_j\bar{v}$.

Thus condition (i) of lemma 3.1 is also satisfied, and therefore, $\{\bar{y}_{i,j}\}$ is a stationary point of V.

Q.E.D.

Let A, B_1, and B_2 be fixed. The stationary points of $W(p^1, p^2; A, B_1, B_2)$ will be investigated in the next three lemmas. To simplify the notation, we shall write $W(u,v)$ instead of $W(p^1, p^2; A, B_1, B_2)$.

A stationary point, $(\bar{u},\bar{v}) \in \bar{E}^2$, will be called an <u>interior</u> stationary point if $\sin \bar{u} \neq 0$.

<u>Lemma 3.3</u>

Let $(\bar{u},\bar{v})$ be an interior stationary point of W. Then, $(\bar{u},\bar{v})$ is non-degenerate, and the index of $(\bar{u},\bar{v})$ is even.

<u>Proof</u>:

Let

$$C_{01} = 2k\alpha_0\alpha_1 \; g'\left(4b_{01} + 4k \sin^2\tfrac{1}{2}u\right)$$
$$C_{02} = 2k\alpha_0\alpha_2 \; g'\left(4b_{02} + 4k \sin^2\tfrac{1}{2}v\right)$$
$$C_{12} = 2k\alpha_1\alpha_2 \; g'\left(4b_{12} + 4k \sin^2\tfrac{1}{2}(u-v)\right)$$
$$D_{01} = k^2\alpha_0\alpha_1 \; g''\left(4b_{01} + 4k \sin^2\tfrac{1}{2}u\right)$$
$$D_{02} = k^2\alpha_0\alpha_2 \; g''\left(4b_{02} + 4k \sin^2\tfrac{1}{2}v\right)$$
$$D_{12} = k^2\alpha_1\alpha_2 \; g''\left(4b_{12} + 4k \sin^2\tfrac{1}{2}(u-v)\right)$$

Then,

$$\text{(20a)} \quad \frac{\partial^2 W}{\partial u^2} = D_{01}\sin^2 u + C_{01}\cos u + D_{12}\sin^2(u-v) + C_{12}\cos(u-v)$$

$$\text{(20b)} \quad \frac{\partial^2 W}{\partial v^2} = D_{02}\sin^2 v + C_{02}\cos v + D_{12}\sin^2(u-v) + C_{12}\cos(u-v)$$

$$\text{(20c)} \quad \frac{\partial^2 W}{\partial u \partial v} = - D_{12}\sin^2(u-v) - C_{12}\cos(u-v).$$

Hence, the Hessian of W at $(\bar{u},\bar{v})$ is given by:

$$\det \begin{bmatrix} \frac{\partial^2 W}{\partial u^2} & \frac{\partial^2 W}{\partial u \partial v} \\ \frac{\partial^2 W}{\partial u \partial v} & \frac{\partial^2 W}{\partial v^2} \end{bmatrix}$$

$$\begin{aligned} &= D_{01}D_{02}\sin^2 u \sin^2 v + D_{01}D_{12}\sin^2 u \sin^2(u-v) \\ &+ D_{02}D_{12}\sin^2 v \sin^2(u-v) + D_{01}C_{02}\sin^2 u \cos v \\ &+ D_{01}C_{12}\sin^2 u \cos(u-v) + D_{02}C_{12}\sin^2 v \cos(u-v) \\ &+ D_{02}C_{01}\sin^2 u \cos u + D_{12}C_{01}\sin^2(u-v) \cos u \\ &+ D_{12}C_{02}\sin^2(u-v) \cos v + C_{01}C_{02}\cos u \cos v \\ &+ C_{01}C_{12}\cos u \cos(u-v) + C_{02}C_{12}\cos v \cos(u-v). \end{aligned}$$

Let Δ denote this determinant.

Since $(\bar{u},\bar{v})$ is a stationary point of W,

(21) $$C_{01}\sin u = -C_{12}\sin(u-v) = -C_{02}\sin v.$$

Hence,

$$C_{12} = \frac{C_{01}\sin u}{\sin(u-v)} = -\frac{C_{02}\sin v}{\sin(u-v)}.$$

Therefore,

$$C_{01}C_{02}\cos u \cos v + C_{02}C_{12}\cos u \cos(u-v)$$
$$+ C_{02}C_{12}\cos v \cos(u-v)$$

$$= C_{01}C_{02}\Big[\cos u \cos v + \sin v \cos u \cot(u-v) - \sin u \cos v \cot(u-v)\Big]$$

$$= C_{01}C_{02}\Big[\cos u \cos v - \cos(u-v)\Big]$$

$$= -C_{01}C_{02}\sin u \sin v.$$

But, $(-C_{01}C_{02}\sin u \sin v)$ is positive since $C_{01} < 0$, $C_{02} < 0$, and $\sin u$ and $\sin v$ are opposite in sign (from (21)). Also, $D_{01} \geqq 0$, $D_{02} \geqq 0$, $D_{12} \geqq 0$, $\sin^2 u > 0$, $\sin^2 v > 0$, and $\sin^2(u-v) > 0$. Therefore,

$$\Delta > D_{01}\sin^2 u\Big[C_{02}\cos v + C_{12}\cos(u-v)\Big] + D_{02}\sin^2 v\Big[C_{01}\cos u + C_{12}\cos(u-v)\Big] + D_{12}\sin^2(u-v)\Big[C_{01}\cos u + C_{02}\cos v\Big]. \tag{22}$$

To complete the proof, we need only show that the bracketed terms in (22) are non-negative.

For each $x, y \in E$, let

$$\langle \bar{x}, \bar{y} \rangle = \left\{ z \mid \text{for some integer } k,\ x < z + 2k\pi < y \right\}.$$

Since $\sin u$ and $\sin v$ are opposite in sign, we may suppose $u \in \langle \bar{0}, \bar{\pi} \rangle$ and $v \in \langle -\bar{\pi}, \bar{0} \rangle$. From (21), $\sin u$ and $\sin(u-v)$ are also opposite in sign. Therefore, $(u-v) \in \langle -\bar{\pi}, \bar{0} \rangle$.

Now, if $u \in \langle \bar{0}, \overline{\pi/2} \rangle \cup \{\overline{\pi/2}\}$ and $-v \in \langle \bar{0}, \overline{\pi/2} \rangle \cup \{\overline{\pi/2}\}$, then $u-v \notin \langle -\bar{\pi}, \bar{0} \rangle$. Therefore, there are three cases to consider:

<u>Case 1</u>

Suppose $u \in \langle \overline{\pi/2}, \bar{\pi} \rangle$ and $-v \in \langle \overline{\pi/2}, \bar{\pi} \rangle$. Then,

$$\sin u > 0,\quad -\cos u > 0,\quad -\cos v > 0,\quad \text{and} \quad -\sin v > 0. \tag{23}$$

Now, if $u-v \in \langle -\bar{\pi}, -\overline{\pi/2} \rangle \cup \{\overline{\pi/2}\}$, then $\cos(u-v) \leq 0$, and since C_{01}, C_{02}, C_{12}, $\cos u$, and $\cos v$ are negative, the bracketed terms in (22) are non-negative.

Suppose $u-v \in \langle -\overline{\pi/2}, \bar{0} \rangle$. Then, from (23),

$$- \sin(u-v) = - \sin u \cos v + \sin v \cos u$$

$$> - \sin u \cos v$$

$$> \sin u.$$

Similarly,

$$- \sin(u-v) > - \sin v.$$

Therefore, since $C_{12} < 0$, we have by (21),

$$C_{01} \sin u = - C_{12} \sin(u-v) < C_{12} \sin u$$

and

$$- C_{02} \sin v = - C_{12} \sin(u-v) < - C_{12} \sin v.$$

Hence, $- C_{01} > - C_{12}$ and $- C_{02} > - C_{12}$.

Also,

$$\cos(u-v) = \cos u \cos v - \sin u \sin v$$

$$< \cos u \cos v.$$

Therefore,

$$\cos(u-v) < -\cos u \quad \text{and} \quad \cos(u-v) < -\cos v$$

Hence,

$$C_{01}\cos u > -C_{12}\cos(u-v)$$

and

$$C_{02}\cos v > -C_{12}\cos(u-v).$$

Thus, the first two bracketed terms in (22) are non-negative, and since C_{01}, C_{02}, $\cos u$ and $\cos v$ are negative, the right hand side of (22) is non-negative.

<u>Case 2</u>

Suppose $u \in \langle \overline{\pi/2}, \overline{\pi} \rangle$ and $-v \in \langle \overline{0}, \overline{\pi/2} \rangle \cup \{\overline{\pi/2}\}$. Then, $u-v \in \langle -\overline{\pi}, -\overline{\pi/2} \rangle$.

Let $u' = u$, $v' = u-v$, $C'_{01} = C_{01}$, $C'_{02} = C_{12}$, and $C'_{12} = C_{02}$. Then, $u' \in \langle \overline{\pi/2}, \overline{\pi} \rangle$ and $-v' \in \langle \overline{\pi/2}, \overline{\pi} \rangle$ and

$$C'_{01}\sin u' = -C'_{12}\sin(u-v) = -C'_{02}\sin v .$$

Hence, by the argument used in case 1, with u, v, C_{01}, C_{02}, and C_{12} replaced by u', v', C'_{01}, C'_{02} and C'_{12} ,

$$C_{01}\cos u + C_{02}\cos v = C'_{01}\cos u' + C'_{12}\cos(u'-v') > 0,$$

$$C_{02}\cos v + C_{12}\cos(u-v) = C'_{02}\cos v' + C'_{12}\cos(u'-v') > 0,$$

and

$$C_{01}\cos v + C_{12}\cos(u-v) = C'_{01}\cos u' + C'_{02}\cos v' > 0.$$

Therefore, the bracketed terms in (22) are non-negative for Case 2.

<u>Case 3</u>

Suppose $u \in \langle \bar{0}, \overline{\pi/2} \rangle \cup \{\overline{\pi/2}\}$ and $-v \in \langle \overline{\pi/2}, \bar{\pi} \rangle$. Then, $-(u-v) \in \langle -\bar{\pi}, -\overline{\pi/2} \rangle$. Let $u'' = -v$, $v'' = u-v$, $C''_{01} = C_{02}$, $C''_{02} = C_{12}$, and $C''_{12} = C_{01}$. Then, by the argument used in case 2, the bracketed terms in (22) are again non-negative.

Q.E.D.

<u>Lemma 3.4</u>

(0,0) is a maximum of W.

<u>Proof:</u>

Let

$$H = \begin{bmatrix} \dfrac{\partial^2 W}{\partial v^2}(0,0) & \dfrac{\partial^2 W}{\partial u \partial v}(0,0) \\ \\ \dfrac{\partial^2 W}{\partial v \partial u}(0,0) & \dfrac{\partial^2 W}{\partial v^2}(0,0) \end{bmatrix}$$

Then, from (20a), (20b), and (20c),

$$H = 2k \begin{bmatrix} \alpha_0\alpha_1 g'(4b_{01}) + \alpha_1\alpha_2 g'(4b_{12}) & -\alpha_1\alpha_2 g'(4b_{12}) \\ \\ -\alpha_1\alpha_2 g'(4b_{12}) & \alpha_0\alpha_2 g'(4b_{01}) + \alpha_1\alpha_2 g'(4b_{12}) \end{bmatrix}$$

Therefore,

$$\det H = (2k)^2 \Big\{ \alpha_1\alpha_2 \left[\alpha_0 g'(4b_{01}) + \alpha_2 g'(4b_{12})\right]\left[\alpha_0 g'(4b_{02}) + \alpha_1 g'(4b_{12})\right] - \left[\alpha_1\alpha_2 g'(4b_{12})\right]^2 \Big\}$$

$$> 0 .$$

Hence, either the eigenvalues of H are both negative or both positive. But the sum of the eigenvalues of H is equal to the trace of H, and the trace of H is nagative since $g' < 0$.

Therefore, H is negative definite.

Q.E.D.

Lemma 3.5

If the stationary points of W are non-degenerate, and if $(\bar{u},\bar{v})$ is an interior stationary point of W, then $(\bar{u},\bar{v})$ is a minimum of W. Moreover, if $(\bar{u}',\bar{v}')$ is an interior stationary point of W, and if $(\bar{u}',\bar{v}') = (\bar{u},\bar{v})$, then

$$(\bar{u}',\bar{v}') = (-\bar{u},-\bar{v}).$$

Proof:

Let F be the set of all stationary points of W in $\bar{E}^2$. Let

$$F^+ = \left\{ (\bar{u},\bar{v}) \;\middle|\; \sin u \neq 0 \right\} \cap F ,$$

and let

$$F^- = F - F^+.$$

(F^+ is the set of interior stationary points of W.)

We note first that F^+ contains an even number of elements. For, if $(\bar{u},\bar{v}) \in F^+$, then so is $(-\bar{u},-\bar{v})$, from (17), and $(\bar{u},\bar{v}) \neq (-\bar{u},-\bar{v})$.

Now, the only elements of F^- are $(\bar{0},\bar{0})$, $(\bar{0},\bar{\pi})$, $(\bar{\pi},\bar{0})$, and $(\bar{\pi},\bar{\pi})$.

Let M^i denote the number of stationary points of W of index i. Let $2t$ be the number of elements in F^+, where t is an integer. Then, from lemma 3.3 and lemma 3.4,

$$M^0 + M^2 \geq 2t + 1.$$

Also,

$$M^1 \leq 3,$$

since the only stationary points of W which may have an odd index are the points $(\bar{0},\bar{\pi})$, $(\bar{\pi},\bar{0})$ and $(\bar{\pi},\bar{\pi})$. Hence, if the stationary points of W are non-degenerate, then

$$\begin{aligned} 0 &= R^0_{(2)} - R^1_{(2)} + R^2_{(2)} \\ &= M^0 - M^1 + M^2 \\ &\geq 2t + 1 - 3 = 2t - 2, \end{aligned}$$

where $R^{i}_{(2)}$ is the i-th connectivity number of T^2. Thus, $t \leq 1$, and so, either F^+ contains two elements or F^+ is empty.

Now, it is clear from (20a), (20b), and (20c) that $\partial^2 W/\partial u^2$, $\partial^2 W/\partial v^2$, and $\partial^2 W/\partial u \partial v$ are unchanged if we substitute $(-u,-v)$ for (u,v). Therefore, if $F^+ \neq \emptyset$, then F^+ contains either two maxima or two minima. But, F^- contains no minima, and F must contain at least one minima since T^2 is compact. Thus, all interior stationary points of W are minima.

Q.E.D.

We may now compute an upper bound on the number of stationary points of V on T^{2m} provided we assume that all stationary points of V on T^{2m} are non-degenerate. The conditions under which this assumption is valid will be considered later.

Theorem 3.4

If the stationary points of V on T^{2m} are non-degenerate, then there are at most 6^m stationary points of V on T^{2m}.

Proof:

Let $K \subset \bar{E}^{m2}$ denote the set of all stationary points of V on T^{2m}, and let G be the set of ordered 4-tuples of the form (A, B_1, B_2, λ), where $A \subset \hat{m}$, $B_1 \subset \hat{m} - A$, $B_2 \subset \hat{m} - A$, and λ is a function on A such that $\lambda_i^2 = 1$ for all $i \in A$.

Assume that all stationary points of V on T^{2m} are non-degenerate.

We now construct a univalent function, F, on K into G. An element $p = \{\bar{y}_{i,j}\} \in K$ uniquely determines the three sets:

$$A(p) = \{i \mid \sin y_{i,1} \neq 0\} \subset \hat{m}$$

$$B_1(p) = \{i \mid \bar{y}_{i,1} = \bar{\pi}\} \subset \hat{m} - A$$

$$B_2(p) = \{i \mid \bar{y}_{i,2} = \bar{\pi}\} \subset \hat{m} - A.$$

If $A(p) = \emptyset$, let

$$F(p) = (A(p), B_1(p), B_2(p), \emptyset) \in G.$$

Suppose $A(p) \neq \emptyset$. Let

$$W_p(u,v) = W(p^1, p^2; A(p), B_1(p), B_2(p)).$$

where $(u,v) \in E^2$ and $p^i \in T^1$, $i = 1,2$. By Theorem 3.3, $(\bar{y}_{i,1}, \bar{y}_{i,2})$ is an interior stationary point of W_p for each $i \in A(p)$.

Let $j \in A(p)$. Let $u(p) = y_{j,1}$ and $v(p) = y_{j,2}$ if $y_{j,1} \in \langle \bar{0}, \bar{\pi} \rangle$. Otherwise, let $u(p) = -y_{j,1}$ and $v(p) = -y_{j,2}$. By lemma 3.2, there exists a function $\lambda(p)$ on $A(p)$ such that $\lambda_i^2(p) = 1$, and, for each $i \in A(p)$,

$$\bar{y}_{i,1} = (\lambda(p))_i \bar{u}(p)$$

and

$$\bar{y}_{i,2} = (\lambda(p))_i \bar{v}(p).$$

Let

$$F(p) = (A(p), B_1(p), B_2(p), \lambda(p)) \in G.$$

Then, F is on K into G. To show that F is univalent, suppose $F(p) = F(p')$, $p, p' \in K$. Then $A(p) = A(p')$, $B_1(p) = B_1(p')$, $B_2(p) = B_2(p')$, and hence, $W_p = W_{p'}$. Also, $\lambda(p) = \lambda(p')$. Therefore, if $p \neq p'$, then, $(\bar{u}(p), \bar{v}(p)) \neq (\bar{u}(p'), \bar{v}(p'))$. In other words, if $p \neq p'$ and $F(p) = F(p')$, then W_p has at least four interior stationary points. Thus, by lemma 3.5, to show F is univalent, it suffices to show that the stationary points of W_p are non-degenerate.

Suppose $(\bar{u}'', \bar{v}'')$ is a degenerate stationary point of W_p on T^2. Let $p'' = \left\{ \bar{y}''_{i,j} \right\} \in \bar{E}^{m2}$ such that

$$\bar{y}''_{i,1} = \bar{\bar{\pi}} \quad \text{for } i \in B_1(p)$$

$$\bar{y}''_{i,2} = \bar{\bar{\pi}} \quad \text{for } i \in B_2(p)$$

$$\bar{y}''_{i,1} = \bar{y}_{i,2} = \bar{0} \quad \text{for } i \in (\hat{m} - A(p) \cup B_1(p) \cup B_2(p))$$

and $\bar{y}''_{i,1} = \bar{u}''$, $\bar{y}''_{i,2} = \bar{v}''$ for $i \in A(p)$.

Then, by theorem 3.3, p'' is a stationary point of V on T^{2m}, and, by hypothesis, is non-degenerate.

Since,

$$\frac{\partial^2 V}{\partial y_{i,1} \partial y_{j,1}} = \frac{\partial^2 V}{\partial y_{i,1} \partial y_{j,2}} = 0,$$

for $i \neq j$, the Hessian of V at p'' is the determinant of the matrix:

$$H = \begin{bmatrix} M_1 & O_2 & O_2 & \cdots & O_2 \\ O_2 & M_2 & O_2 & \cdots & O_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & O_2 & \cdots & M_m \end{bmatrix}$$

where O_2 is the two by two matrix:

$$O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$M_i = \begin{bmatrix} \partial^2 V/\partial y_{i,1}^2 & \partial^2 V/\partial y_{i,1}\partial y_{i,2} \\ \partial^2 V/\partial y_{i,1}\partial y_{i,2} & \partial^2 V/\partial y_{i,2}^2 \end{bmatrix}$$

$$i = 1,2,\ldots,m.$$

The determinant of H is given by:

$$\det H = \prod_{i=1}^{m} \det M_i .$$

Since p'' is non-degenerate, $\det H \neq 0$. Hence, $\det M_i \neq 0$, $i = 1,2,\ldots,m$. In particular, for each $i \in A(p)$,

$$0 \neq M_i = \det \begin{bmatrix} \frac{\partial^2 W_p}{\partial u^2}(u'',v'') & \frac{\partial^2 W_p}{\partial u \partial v}(u'',v'') \\ \\ \frac{\partial^2 W_p}{\partial u \partial v}(u'',v'') & \frac{\partial^2 W_p}{\partial v^2}(u'',v'') \end{bmatrix}$$

which implies $(\bar{u}'',\bar{v}'')$ is not a degenerate stationary point of W_p.

Thus, F is univalent, and so, the number of elements of K is at most equal to the number of elements of G.

To complete the proof, we now show that the number of elements in G is 6^m.

For a fixed $A \subset \hat{m}$, there are exactly 2^k functions on A to $\{-1, 1\}$, where k is the number of elements in A. For a given $b_{01} \leq m - k$ and $b_{02} \leq m - k$, there are $\binom{m-k}{b_{01}}$ subsets of $\hat{m} - A$ which contain b_{01} elements and $\binom{m-k}{b_{02}}$

subsets of $m - A$ which contain b_{02} elements. Thus, there are

$$\binom{m-k}{b_{01}} \binom{m-k}{b_{02}} 2^k$$

elements in G of the form: (A, B_1, B_2, λ), where B_1 contains b_{01} elements, B_2 contains b_{02} elements and λ is a function on A to $\{-1, 1\}$. It follows that the total number of elements in G of the form: (A, B_1, B_2, λ), where the number of elements in B_1 and B_2 is unspecified, is given by

$$\sum_{b_{01}=0}^{m-k} \sum_{b_{02}=0}^{m-k} \binom{m-k}{b_{01}} \binom{m-k}{b_{02}} 2^k = 2^{2m-k},$$

Since there are $\binom{m}{k}$ subsets of $\hat{m}$ which contain k elements, the total number of elements in G is

$$\sum_{k=0}^{m} \binom{m}{k} 2^{2m-k} = 2^{2m}(1 + \tfrac{1}{2})^m = 6^m .$$

Q.E.D.

To answer the question as to whether or not the upper bound given by theorem 2.4 is ever attained, we return our attention to the stationary points of W.

Lemma 3.6

Suppose $(\bar{\pi},\bar{0})$, $(\bar{0},\bar{\pi})$, and $(\bar{\pi},\bar{\pi})$ are non-degenerate stationary points of W and suppose that the index of each is 1. Then there exists an interior stationary point of W.

Proof:

Let M^i $(i = 0,1,2)$ denote the number of stationary points of W of index i. Then, $M^1 \geq 3$, by hypothesis, and, since T^2 is compact, $M^0 \geq 1$ and $M^2 \geq 1$. Hence, there are at least five stationary points of W. But there are only four stationary points of W which are not interior stationary points.

Q.E.D.

Lemma 3.7

Suppose

(24a) $$\alpha_2/g'(4b_{01}) - \alpha_1/g'(4b_{02}+4k) - \alpha_0/g'(4b_{12}+4k) > 0,$$

(24b) $$\alpha_1/g'(4b_{02}) - \alpha_2/g'(4b_{01}+4k) - \alpha_0/g'(4b_{12}+4k) > 0,$$

(25c) $$\alpha_0/g'(4b_{12}) - \alpha_2/g'(4b_{01}+4k) - \alpha_1/g'(4b_{02}+4k) > 0.$$

Then there exists an interior stationary point of W.

Proof:

By lemma 3.6, it suffices to show that (24a), (24b), and (24c) imply that $(\bar{\pi},\bar{0})$, $(\bar{0},\bar{\pi})$, and $(\bar{\pi},\bar{\pi})$ are non-degenerate stationary points of index 1.

For (φ,ψ) equal to (1,0), (0,1) or (1,1), let

$$Q(\varphi,\psi) = \begin{bmatrix} C_{01}(-1)^{\varphi} + C_{12}(-1)^{(\varphi-\psi)} & -C_{12}(-1)^{(\varphi-\psi)} \\ \\ -C_{12}(-1)^{(\varphi-\psi)} & C_{02}(-1)^{\psi} + C_{12}(-1)^{(\varphi-\psi)} \end{bmatrix}$$

where

$$C_{01} = 2k\alpha_0\alpha_1 \; g'\left(4b_{01} + 4k \sin^2\tfrac{1}{2}\pi\varphi\right)$$

$$C_{02} = 2k\alpha_0\alpha_2 \; g'\left(4b_{02} + 4k \sin^2\tfrac{1}{2}\pi\varphi\right)$$

and

$$C_{12} = 2k\alpha_1\alpha_2 \; g'\left(4b_{12} + 4k \sin^2\tfrac{1}{2}\pi(\varphi-\psi)\right) .$$

The determinant of $Q(\varphi,\psi)$ is given by:

$$(25) \qquad \det Q(\varphi,\psi) = C_{01}C_{02}(-1)^{\varphi+\psi} + C_{01}C_{12}(-1)^{2\varphi-\psi} + C_{02}C_{12}(-1)^{\varphi-2\psi}$$

$$= C_{01}C_{02}(-1)^{\varphi+\psi} + C_{01}C_{12}(-1)^{\psi} + C_{02}C_{12}(-1)^{\varphi}$$

$$= K\left\{ \frac{\alpha_0(-1)^{\varphi+\psi}}{g'\left|4b_{12} + 4k\sin^2\frac{\pi}{2}(\varphi-\psi)\right|} + \frac{\alpha_1(-1)^{\psi}}{g'\left|4b_{01} + 4k\sin^2\frac{\pi}{2}\varphi\right|} + \frac{\alpha_2(-1)^{\varphi}}{g'\left|4b_{02} + 4k\sin^2\frac{\pi}{2}\psi\right|}\right\}$$

where

$$K = \frac{2k\alpha_0\alpha_1\alpha_2}{g'\left|4b_{12}+ 4k\sin^2\frac{\pi}{2}(\varphi-\psi)\right| g'\left|4b_{01}+ 4k\sin^2\frac{\pi}{2}\varphi\right| g'\left|4b_{02}+ 4k\sin^2\frac{\pi}{2}\psi\right|}$$

Since g' is negative, we have from (24a),

$$\det Q(0,1) < 0.$$

But the Hessian of W at $(\bar{0},\bar{\pi})$ is $\det Q(0,1)$, and since $\det Q(0,1)$ is negative, $Q(0,1)$ must have one positive and one negative eigenvalue. Therefore, $(\bar{0},\bar{\pi})$ is non-degenerate stationary point of W of index 1.

By (24b) and (24c), det $Q(1,0) < 0$, and det $Q(1,1) < 0$, and so, by the same argument, $(\bar{0},\bar{\pi})$ and $(\bar{\pi},\bar{\pi})$ are non-degenerate stationary points of W and each had index 1.

Q.E.D.

Theorem 3.5

Suppose that, for each k, b_{01}, b_{02}, and b_{12}, such that $k \leq m$, $b_{01} \leq m - k$, $b_{02} \leq m - k$, $b_{12} \leq m - k$, and $b_{12} \leq b_{02} + b_{01}$, equations (24a), (24b), and (24c) are satisfied. Then there are exactly 6^m stationary points of V on T^{2m}.

Proof:

Let K, G, and F be the same as in the proof of Theorem 3.4. Then, it is sufficient to show that F is onto G.

Let $(A,B_1,B_2,\lambda) \in G$. Then, by hypothesis and lemma 3.7, there exists $(\bar{u},\bar{v}) \in \bar{E}^2$ which is an interior stationary point of $W(p^1,p^2;A,B_1,B_2)$.

Let $\{\bar{y}_{i,1}\} \in \bar{E}^{m2}$ such that

$$\bar{y}_{i,1} = \lambda_i\bar{u} \text{ and } \bar{y}_{i,2} = \lambda_i\bar{v}, \quad i \in A$$

$$\bar{y}_{i,1} = \bar{\pi} \text{ for } i \in B_1$$

$$\bar{y}_{i,1} = \bar{0} \text{ for } i \in (\hat{m} - A - B_1)$$

$$\bar{y}_{i,2} = \bar{\pi} \text{ for } i \in B_2$$

and

$$\bar{y}_{i,2} = \bar{0} \text{ for } i \in (\hat{m} - A - B_2).$$

Then $\{\bar{y}_{i,j}\}$ is a stationary point of V by Theorem 3.3.

Let $p = \{\bar{y}_{i,j}\}$. Then, $A(p) = A$, $B_1(p) = B_1$, $B_2(p) = B_2$ and $\lambda(p) = \lambda$. Therefore, $F(p) = (A, B_1, B_2, \lambda) \in G$.

Q.E.D.

The next theorem settles the question as to whether or not the conditions of Theorem 3.5 are ever satisfied

Theorem 3.6

There exists g, α_0, α_1, and α_2 such that the number of stationary points of V on T^{2m} is 6^m.

Proof:

Let $\alpha_0 = \alpha_1 = \alpha_2 = 1$, and let $g' = -1$. Then, (24a), (24b), and (24c) are satisfied for all $k \leq m$, $b_{01} \leq m - k$, $b_{02} \leq m - k$, $b_{12} \leq m - k$, and $b_{12} \leq b_{01} + b_{02}$.

Q.E.D.

We now develope analogs to theorems 3.4, 3.5, and 3.6 for a lower bound on the number of stationary points of V on T^{2m}.

As noted in the beginning of this section, $\{\bar{y}_{i,j}\} \in \bar{E}^{m2}$ is a stationary point of V if $\sin y_{i,1} = \sin y_{i,2} = 0$. By the same argument used in the proof of Theorem 3.1, there are 4^m elements in the set,

$$\left\{\{\bar{y}_{i,j}\} \;\middle|\; \sin y_{i,1} = \sin y_{i,2} = 0\right\} \cap \bar{E}^{m2}.$$

Thus, the following theorem is established.

Theorem 3.7

There are at least 4^m stationary points of V on T^{2m}.

Before stating an analog to Theorem 3.5, we need an analog to lemma 3.7.

Lemma 3.8

Suppose that strict inequality holds between the right and left hand sides of (24a), (24b), and (24c), but that one of the inequality signs is reversed. Then there are no interior stationary points of W, and, moreover, one of the points $(\bar{0},\bar{\pi})$, $(\bar{\pi},\bar{0})$, or $(\bar{\pi},\bar{\pi})$ is the unique minimum of W on T^2.

Proof:

Let $Q(\varphi,\psi)$ be the same as in the proof of lemma 3.7. Then, by (25), $(\bar{\pi},\bar{0})$, $(\bar{0},\bar{\pi})$, and $(\bar{\pi},\bar{\pi})$ are non-degenerate stationary points of W, since strict inequalities hold between the left and right sides of (24a), (24b), and (24c). Hence, by lemma 3.3 and lemma 3.4, all stationary points of W are non-degenerate. Therefore,

$$M^1 = M^2 + M^0,$$

where M^i is the number of stationary points of index i ($i = 0,1,2$).

Since one of the inequalities in (24a), (24b), or (24c) is reversed, one of the stationary points, $(\bar{\pi},\bar{0})$, $(\bar{0},\bar{\pi})$ or $(\bar{\pi},\bar{\pi})$ is either a maximum or a minimum, and, since these are the only stationary points which may have an odd index,

$$M^1 \leq 2.$$

Since T^2 is compact, $M^0 + M^2 \geq 2$. Therefore,

$$M^1 = M^0 + M^2 = 2,$$

and so, there are but fcur stationary points of W, none of which is an interior stationary point.

Since T^2 is compact and $(\bar{0},\bar{0})$ is not a minimum of W, one of the three remaining stationary points must be a minimum, and moreover must be unique since

$$M^2 = 4 - M^1 - M^0 \leq 4 - R^1_{(2)} - R^0_{(2)} = 1.$$

Q.E.D.

Theorem 3.8

Suppose, for all integers k, b_{01}, b_{02} and b_{12} such that $0 \leq k \leq m$, $0 \leq b_{01} \leq m-k$, $0 \leq b_{02} \leq m-k$, $0 \leq b_{12} \leq m-k$ and $0 \leq b_{12} \leq b_{01} + b_{02}$, strict inequalities hold between the left and right hand sides of (24a), (24b) and(24c) and one of the inequalities is reversed. Then there are exactly 4^m stationary points of V on T^{2m}.

Proof

Let K, G, and F be as in the proof of Theorem 3.4. Let G^- denote the set of all elements of G of the form $(\emptyset, B_1, B_2, \emptyset)$. Recalling the proof of Theorem 2.4, the number of elements in G^- is

$$\sum_{b_{01}=0}^{m} \sum_{b_{02}=0}^{m} \binom{m}{b_{01}} \binom{m}{b_{02}} = 4^m$$

Therefore, to complete the proof, we need only show that F is univalent on K into G^-.

Let $p = \{\bar{y}_{i,j}\} \in K$. Let $k(p)$, $b_{01}(p)$ $b_{02}(p)$ and $b_{12}(p)$ be the number of elements in $A(p)$ $B_1(p)$, $B_2(p)$ and $(B_1(p) \cap B_2(p)) - (B_1(p) \cup B_2(p))$. Then, since strict inequalities hold between the right and left hand sides of (24a), (24b) and(24c) and one of the inequalities is reversed (for $b_{01} = b_{01}(p)$, $b_{02} = b_{02}(p)$, and $b_{12} = b_{12}(p)$), there are no interior stationary points of W_p.

Therefore, by Theorem 3.3, $A(p) = \emptyset$, and so $F(p) \in G^-$.

Suppose $p' = \{\bar{y}_{i,j}\} \in K$ and $F(p) = F(p')$. Then $A(p) = A(p') = \emptyset$, $B_1(p) = B_1(p')$ and $B_2(p) = B_2(p')$. Hence,

$$\bar{y}_{i,1} = \bar{\pi} = \bar{y}'_{i,1} \text{ for } i \in B_1(p)$$

$$\bar{y}_{i,1} = \bar{0} = \bar{y}'_{i,1} \text{ for } i \in \hat{m} - B_1(p)$$

$$\bar{y}_{i,2} = \bar{\pi} = \bar{y}_{i,2} \text{ for } i \in B_2(p)$$

and

$$\bar{y}_{i,2} = \bar{0} = \bar{y}_{i,2} \text{ for } i \in \hat{m} - B_2(p)$$

Thus, $p = p'$.

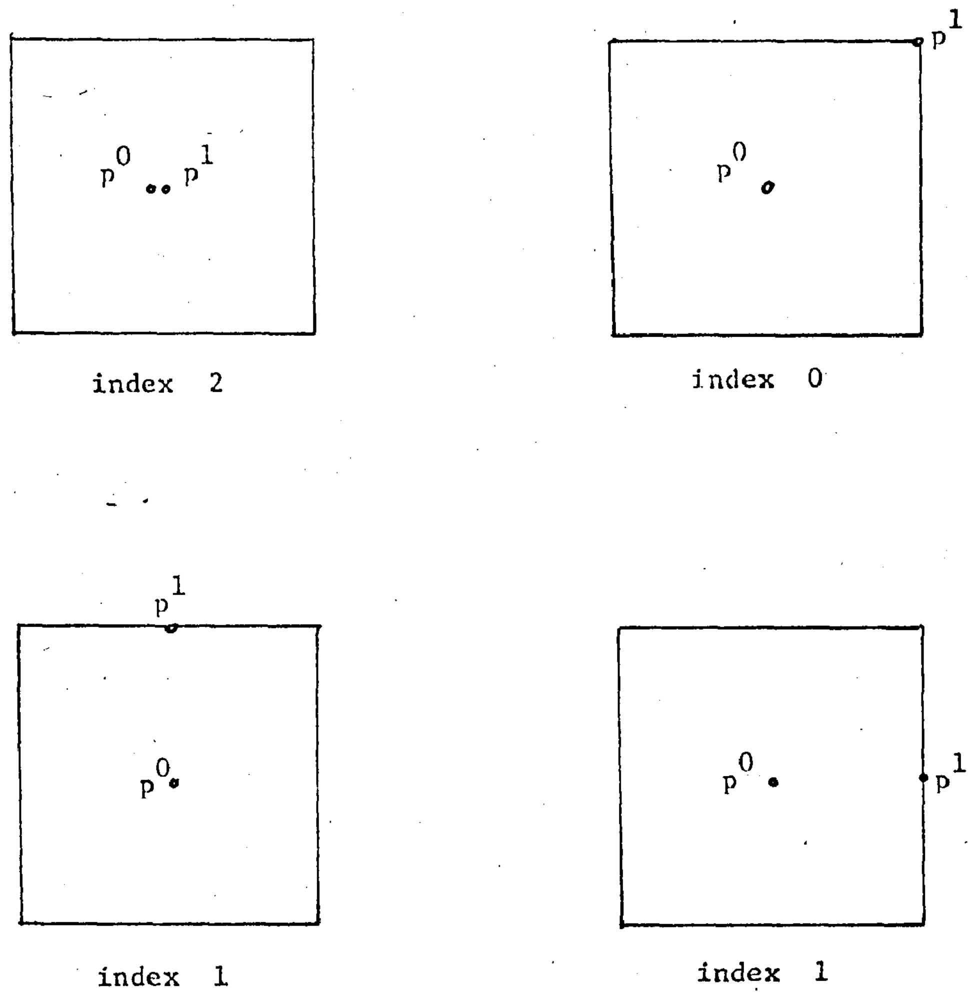
Q.E.D.

Clearly, if we let $g' = -1$, and $\alpha_0 = \alpha_1 = 1$, then α_2 may be chosen so large that the inequality sign in (24c) is reversed for all integers k, b_{01}, b_{02} and b_{12} such that $0 \leq k \leq m$, $0 \leq b_{01} \leq m$, $0 \leq b_{02} \leq m$ and $0 \leq b_{12} \leq m$. Moreover, increasing α_2 increases the left hand sides of (24a) and (24b). This proves the following theorem.

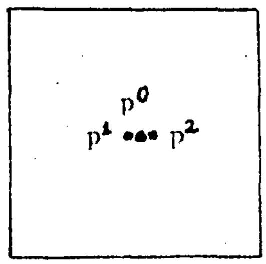
Theorem 3.9

There exists α_0, α_1, α_2 and g such that the number of stationary points of V on T^{2m} is 4^m.

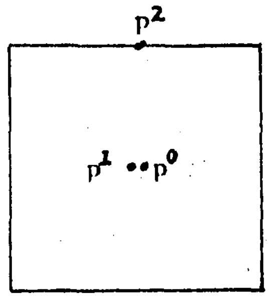
Critical configurations for two and three charges on a 2-torus are shown in figures 4, 5, and 6.



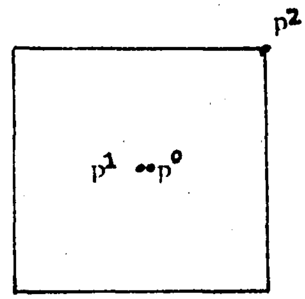
<u>Figure 4</u> Critical Configurations of Two Charges on T^2.



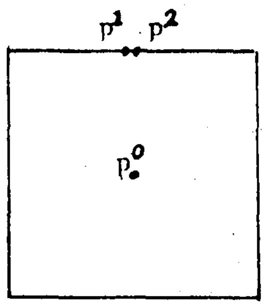
(1 configuration)



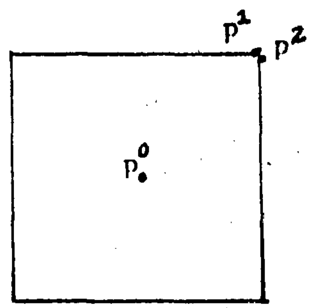
(4 configurations)



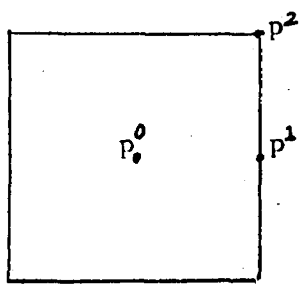
(2 configurations)



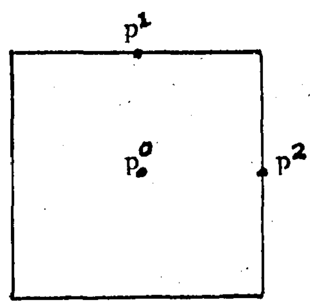
(2 configurations)



(1 configuration)



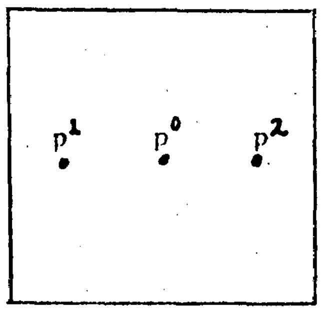
(4 configurations)



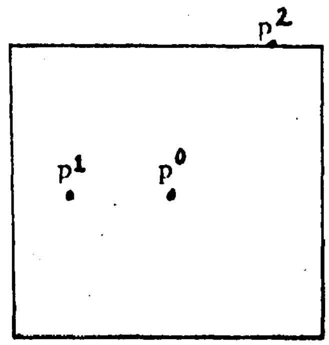
(2 configurations)

<u>Figure 5</u> Critical Configurations of Three Charges on T^2 as Predicted by Theorem 3.9.

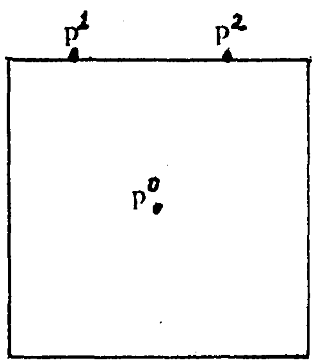
Where more than one critical configuration is indicated, the others may be obtained from the one given by rotating the axes 90 degrees or by interchanging p^1 and p^2.



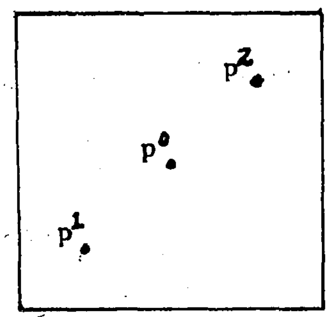
(4 configurations)



(8 configurations)



(4 configurations)



(4 configurations)

Figure 6 Additional Critical Configurations of Three Charges on T^2 as Predicted by Theorem 3.6.

Where more than one critical configuration is indicated, the others may be obtained from the one given by rotating the axes or by interchanging p^1 and p^2.

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