PERFORMANCE BOUNDS ON SPREAD SPECTRUM MULTIPLE ACCESS COMMUNICATION SYSTEMS

KUNG YAO
University of California
Los Angeles, California

Summary. Several approaches for the evaluation of upper and lower bounds on error probability of spread spectrum multiple access communication systems are presented. These bounds are obtained by utilizing an isomorphism theorem in the theory of moment spaces. From this theorem, we generate closed, compact, and convex bodies, where one of the coordinates represents error probability, while the other coordinate represents a generalized moment of the multiple access interference random variable. Derivations for the second moment, fourth moment, single exponential moment, and multiple exponential moment are given in terms of the partial cross correlations of the codes used in the system.

Introduction. Spread spectrum multiple access technique is of use in a multi-user computer communication network system [1] as well as in a more conventional satellite communication system with a single wide-band repeater [2]. In such situations, a code modulation spread spectrum multiple access (SSMA) system is considered suitable for a network of low-cost mobile ground based and airbourned users requiring no network control. In any case, for this and related SSMA systems, the exact evaluation of error probability has been considered a formidable task. Error probability obtained by complete simulation of such systems may involve considerable computational cost.

In this paper, we present several approaches based on the theory of moment spaces to obtain upper and lower bounds on the error probability of a SSMA system. As expected, bounds that use moments that require more computational effort are generally tighter than those that require less. As to be seen, the second moment, fourth moment, single exponential moment, and multiple exponential moment require increasing computational effort. Indeed, by taking a sufficiently large number of terms in the multiple exponential moment case, we can make the upper and lower bounds arbitrarily tight.

This work was supported in part by ONR Contract N0014-75-C-0528 under Task NR-042-285 and by the Electronics Program of ONR.
SSMA Model. There are various forms of SSMA communications systems. In a direct code modulation SSMA system, the data stream of each user modulates a shift-register (SR) generated sequence code to obtain the spread spectrum effect. The multiple access capability is achieved by requiring each user’s code word to be near orthogonal. In this paper, we shall only consider the model of an asynchronous SSMA system as discussed in [3].

Thus, we allow the time delay’s \( \tau_i \) and phase angles \( \theta_i \) of different users to be r.v.’s. The input to each receiver consists of the sum of all \( K \) users’ signals and additive white Gaussian noise. Each receiver consists of a matched filter matched to its corresponding code word. Without loss of generality, we consider the first receiver. Then we assume it is completely synchronized to its own code word. Thus, \( \theta_i = \tau_i = 0 \). But \( \theta_2, \ldots, \theta_K \) and \( \tau_2, \ldots, \tau_K \) are independent and uniformly distributed r.v.’s. Thus, the output of the matched filter of the 1st receiver is given by

\[
y = \sqrt{\frac{P}{2}} \int_0^T a_1^2(t) b_1(t) dt + \sum_{i=2}^{K} \sqrt{\frac{P}{2}} \int_0^T a_1(t-\tau_i) b_i(t-\tau_i) a_1(t) dt \cos \theta_i \\
+ \int_0^T n(t) a_1(t) \cos \omega_c t dt = \text{hb}_1,0 + Z+n . \tag{1}
\]

The first term of \( y \) represents the desired signal, the second term, \( Z \), represents the interference from the other \( (K-1) \) users, and the last term, \( n \), is a Gaussian r.v. of zero-mean and variance \( \sigma^2 \). All users are assumed to have equal power \( P \). Here the information data of the \( i \)-th user is defined by

\[
b_i(t) = \sum_{n=-\infty}^{\infty} b_{i,n} P_T(t-nT) ,
\]

where the \( b_{i,n} \)’s are i.i.d. r.v.’s taking values +1 and -1 with equal probability, and \( P_T(\cdot) \) is a unit height rectangular window from 0 to \( T \). The code waveform is defined by

\[
a_i(t) = \sum_{j=-\infty}^{\infty} a_j^{(i)} P_T(t-jT_c) ,
\]

where \( a_j^{(i)} \) is the SR code sequence of the \( i \)-th user and consists of +1 and -1 which is periodic with period \( p \) and of chip length \( T_c \).
Now, consider the error probability of the first user, assuming all K user’s code words have been specified. Denote this error probability by \( P_e \). Then

\[
P_e = \frac{1}{2} \Pr\{h+Z+n<0\} + \frac{1}{2} \Pr\{-h+Z+n>0\}
\]

\[
= \mathbb{E}\{Q\left(\frac{h+Z}{\sigma}\right) + Q\left(\frac{-h+Z}{\sigma}\right)\}/2
\]

\[
= \mathbb{E}\{Q\left(\frac{h+Z}{\sigma}\right)\}
\]

(2a) (2b)

where

\[
Q(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(-t^2/2) \, dt
\]

\( P_e \) in (2a) or (2b) is expressed as a generalized moment where the expectation operation, \( \mathbb{E} \), is taken over all the r.v.’s \( \tau_i, b_{i-1}, b_{i0}, \) and \( \theta_i \) in \( Z \). As to be seen, depending on specific cases, sometimes we prefer to use \( P_e \) given by (2a), while other times we prefer to use \( P_e \) given by (2b). In general, since \( Z \) is extremely complicated, it is not possible to evaluate this moment analytically. However, suppose it is possible to evaluate some other moments of \( Z \). Then if we can find relationships among the moments defined by \( P_e \) and other moments that we can evaluate readily, then we can obtain information about the error probability. In the next section, we shall state an isomorphism theorem from the theory of moment spaces that shall yield a precise geometric interpretation of this concept.

**Moment Space Error Bounds.** The following theorem which was originally developed in the theory of games, shall provide relationships among arbitrary moments of a random variable.

Let \( Z \) be a random variable with a probability distribution function \( G_Z(z) \) defined over a finite closed interval \( I = [a,b] \). Let \( k_1(z), k_2(z), \ldots, k_N(z) \) be a set of \( N \) continuous functions defined on \( I \) The generalized moment of the random variable \( Z \) induced by the function

\[
m_i = \int_I k_i(z) dG_Z(z) = \mathbb{E}_Z\{k_i(z)\}, \quad i = 1, \ldots, N
\]

(3)

We denote the \( N \)-th moment space \( \mathcal{M} \) as

\[
\mathcal{M} = \{ m = (m_1, \ldots, m_N) \in \mathbb{R}^N | m_i = \int_a^b k_{i}(z) dG_{z}(z), \quad i = 1, \ldots, N \}
\]
where $G_z$ ranges over the set of probability distributions defined on $I = [a, b]$ and $\mathbb{R}^N$ denotes N-dimensional Euclidean space. Then $M$ is a closed, bounded, and convex set. Now let $C$ be the curve $r = (r_1, \ldots, r_N)$ traced out in $\mathbb{R}^N$ by $r_1 = k_1(z)$ for $z$ in $I$. Let $\mathcal{H}$ be the convex hull of $C$. Then

$$\mathcal{H} = M.$$ (4)

This theorem has been used in [4] to bound $P_e$ encountered in intersymbol interference problems with $N = 2$. In order to explain the application of this theorem in obtaining $P_e$ bounds, consider a plot of $k_2(z)$ versus $k_1(z)$. Now, we take $k_2(z)$ to be the expression inside the curly bracket given either by (2a) or (2b). Thus, $m_2 = E\{k_2(z)\} = P_e$. We shall consider several different $k_1(z)$. In any case, $k_1(z)$ will be chosen such that, $m_1 = E\{k_1(z)\}$ is evaluable. When we plot specific $k_2(z)$ versus $k_1(z)$, the curve $C$ typically turns out to be given by parts of the curve shown in Figure 1. The curve in Figure 1 consists of several sections that are convex $\cup$ or $\cap$ functions. Let the points $C$, $E$, $G$, $I$, etc, be points of infection of the curve. Then curve $ABC$ is convex $\cup$, $CDE$ is convex $\cap$, $EFG$ is convex $\cup$, $GHI$ is convex $\cap$, etc. Suppose a plot of $k_2(z)$ versus $k_1(z)$ yields the curve $C$ given by $ABB'$. Then the upper envelope of this body $\mathcal{H}$, is given by the straight line $AB'$, while the lower body is given just by $ABB'$. Thus, from the above theorem and (4), we can obtain upper and lower bounds on $P_e$. Suppose $E\{k_1(z)\}$ yields a number $m_1$ (which has to be less or equal to $k_1(z_B)$). Then the lower bound is given by $P_e(L) = k_2(k_1^{-1}(m_1))$ while the upper bound is given by $P_e(U) = k_2(z_A) + (m_1 - k_1(z_A)) \times \{k_2(z_B) - k_2(z_A)/k_1(z_B) - k_1(z_A)\}$. Here, we used the notation of $A = (k_1(z_A), k_2(z_A))$, $B = (k_1(z_B), k_2(z_B))$, etc.

Now, suppose a different $k_2(z)$ versus $k_1(z)$ yields a curve $C'$ given by $ABCD$. Thus, parts of $C'$ is convex $\cup$ while parts of $C'$ is convex $\cap$. Then the first part of the upper envelope of $\mathcal{H}$, is given by the straight line $AC'$, while the second part is given by the curve $C'D$. The point $C'$ is defined by equating the slope of the chord $AC'$ to the derivation of the curve $C$ at $c'$. That is,

$$\frac{k_2(z_c') - k_2(z_A)}{k_1(z_c') - k_1(z_A)} = \frac{k_2(z_c')}{k_1(z_c')}.$$ (5)

We note, since $k_2(z)$, $k_1(z)$, $k_1(z)$, and $z_A$ are known, we can solve for $z_c'$ in (5) easily by using any single root solution technique (i.e. Newton method, Regula Falsi Method, etc.). Thus, if $k_1(z_A) \leq m_1 \leq k_1(z_C')$, we have $P_e(U) = k_2(z_A) + (m_1 - k_1(z_A)) \times \{(k_2(z_c') - k_2(z_A))/k_1(z_c') - k_1(z_A)\}$. If $k_1(z_c') \leq m_1 \leq k_1(z_D)$, we have $P_e(U) = k_2(k_1^{-1}(m_1))$. By similar arguments, the lower envelope of $\mathcal{H}$, is given by the curve $ABB''$ and the straight line $B''D$. The point $B''$ or $Z_B''$ can be obtained from

$$\frac{k_2(z_D) - k_2(z_B'')}{k_1(z_D) - k_1(z_B'')} = \frac{k_2(z_B'')}{k_1(z_B'')}.$$ (6)
Thus, if \( k_1(z_A) \leq m_1 \leq k_1(z_B) \), we have \( P_e^{(L)} = k_2(k^{-1}_1(m_1)) \). If \( k_2(z_B) \leq m_1 \leq k_1(z_B) \), then we have \( P_e^{(L)} = k_2(z_B) + (m_1 - k_1(z_B)) \times \left\{ \frac{k_2(z_D) - k_2(z_B)}{(k_1(z_D) - k_1(z_B))} \right\} \).

Another interesting situation is when \( \mathcal{C} \) is given by or contains the curve \( CDEFGHI \). Then the upper envelope is given by the curve \( CD' \), the straight line \( D'H \) and the curve \( HI \). In particular, the points \( D' \) (or \( z_D' \)) and \( H \) (or \( z_H \)) are obtained from

\[
\frac{k_2(z_H) - k_2(z_D)}{k_1(z_H) - k_1(z_D)} = \frac{k_2'(z_D)}{k_1'(z_D)} = \frac{k_2'(z_H)}{k_1'(z_H)}.
\] (7)

From (7), we can try to obtain the two unknowns \( z_D \) and \( z_H \) from two non-linear equations. In practice, because of the local convex \( U \) properties of the curve near \( D' \) and \( H \), we can use iterative solution approach to find \( z_D \) and \( z_H \) quite readily. Thus, for \( k_1(z) \leq m_1 \leq k_1(z_D) \), \( P_e(U) = k_2(k^{-1}_1(m_1)) \), for \( k_1(z_D) \leq m_1 \leq k_1(z_H) \), \( P_e(U) = k_2(z_D) + (m_1 - k_1(z_D)) \times \left\{ \frac{k_2(z_H) - k_2(z_D)}{(k_1(z_H) - k_1(z_D))} \right\} \); for \( k_1(z_H) \leq m_1 \leq k_1(z) \), \( P_e(U) = k_2(k^{-1}_1(m_1)) \). The evaluation of lower envelopes and lower error bounds are similar as that for upper envelopes and upper error bounds.

From all the above discussions, it is clear that given any \( k_2(z) \) and \( k_1(z) \), explicit evaluation of upper and lower error bounds are possible as soon as the moment \( m_1 = E[k_1(Z)] \) and the domain of \( z \) are available.

Evaluation of Moments and Maximum Distortion. -Now, we consider the explicit evaluations of several different moments \( m_1 = E[h_1(Z)] \) as well as the maximum distortion \( D \).

In order to use \( m_1 = E[Z^2] \), let \( k_1(z) = z^2 \) and \( k_2(z) \) be given by expression (2a). Then the domain of \( k_1(z) \) is \([0,D]\), when \( D = \text{Max } Z \). From the definition of the r.v. \( Z \) in (1), it is clear that \( E[Z] = 0 \). From (19) of [3], it is seen that for fixed \( d_1,0 \),

\[
m_1 = E[Z^2] = \frac{PT^3}{12T} \sum_{i=2}^{K} r_{i1},
\] (8)

where

\[
r_{ij} = \sum_{n=0}^{N-1} \{ \rho_{i,j}(n) + \rho_{i,j}(n) + \rho_{i,j}(n+1) + \rho_{i,j}(n+1) + \delta_{i,j}(n) + \delta_{i,j}(n) \}
\]

\[
+ \delta_{i,j}(n+1),
\] (9)

\[
\rho_{i,j}(n) = \sum_{m=0}^{n-1} a_m a_{m-n},
\] (10)

\[
\delta_{i,j}(n) = \sum_{m=n}^{N-1} a_m a_{m-n}.
\] (11)
The period of the codes is denoted by \( p \) and \( N = T/T_c \) is assumed to be an integral multiple of \( p \).

Now, consider the evaluation of \( D \) defined as the maximum value of the r.v. \( Z \), where the maximization is considered over \( \theta_i \), \( \tau_i \), \( b_i \), and \( b_{i,0} \). After some algebra, it can be seen that

\[
D = \left( \frac{p}{2} \right)^2 T_c \sum_{i=2}^{K} \max \left\{ |\rho_{i1}(n_i)| + |\hat{\rho}_{i1}(n_i)| \right\} .
\]  

(12)

Thus, \( m_i \) and \( D \) can be readily evaluated and depend on the partial cross correlations (given by (10) and (11)) of the code words chosen by the \( K \) users.

In order to use \( m_i = \mathbb{E}[Z^4] \), let \( k_1(z) = z^4 \) and \( k_2(z) \) be given by expression (2a). Then the domain of \( k_i(z) \) is also \([0,D]\). In order to evaluate \( m_i = \mathbb{E}[Z^4] \), let

\[
z_i = \int_0^T a_i(t-\tau_i)b_i(t-\tau_i)a_1(t)dt\cos \theta_i = c_i \cos \theta_i ,
\]  

(13)

Then

\[
\mathbb{E}[Z^4] = \left( \frac{p}{2} \right)^2 \mathbb{E}\left\{ \sum_{i=2}^{K} z_i \right\}^4 = \left( \frac{p}{2} \right)^2 \mathbb{E}[z_i^4] + 6 \sum_{i=2}^{K-1} \mathbb{E}[z_j^2] \sum_{j=i+1}^{K} \mathbb{E}[z_j^2] .
\]  

(14)

Here, each \( \mathbb{E}[z_j^2] \) is in the form of (8) and can be evaluated by using (8) - (11). Now,

\[
\mathbb{E}[z_i^4] = \mathbb{E}[\cos^4 \theta_i] \mathbb{E}[c_i^4] = \left( \frac{3}{8} \right) \left( \frac{1}{T} \right) \sum_{n=0}^{N-1} \int_0^{T_c} \{ R_{i1}^4 (\lambda+nT_c) + \hat{R}_{i1}^4 (\lambda+nT_c) \}
\]

\[
+ 6R_{i1}^2 (\lambda+nT_c) \hat{R}_{i1}^2 (\lambda+nT_c) \} d\lambda .
\]  

(15)

Furthermore,

\[
R_{i1} (\lambda+nT_c) = A_{i1} (\lambda+nT_c)
\]

\[
\hat{R}_{i1} (\lambda+nT_c) = \hat{A}_{i1} (\lambda+nT_c)
\]  

(16)
where

\[
A_n = \rho_{i1}(n), \quad B_n = \rho_{i1}(n+1) - \rho_{i1}(n)
\]

\[
A_n = \beta_{i1}(n), \quad B_n = \beta_{i1}(n+1) - \beta_{i1}(n)
\]

(17)

By using (16) and (17), each \( E[z_i^4] \) is given by

\[
E[z_i^4] = \left( \frac{3T^5}{8T} \right) \sum_{n=0}^{N-1} \left( \begin{align*}
(A_n^4 + \hat{A}_n^4) + & 2(A_n^3 \hat{A}_n + \hat{A}_n^3) + 2(A_n^2 \hat{A}_n^2 + \hat{A}_n^2) + 2(A_n \hat{A}_n^2 + \hat{A}_n) + \\
& 2(A_n^2 + \hat{A}_n^2) + 4A_n \hat{A}_n + \hat{A}_n^2
\end{align*} \right).
\]

(18)

Thus, each term in (14) for \( m_i \) can be evaluated.

In order to use \( m_i = E\{\exp[c(h+Z)]\} \), let \( k_1(z) = \exp[c(h+Z)] \) and \( k_2(z) \) be given by expression (2b). Then the domain of \( k_i(z) \) is \([h-D, h+D]\). From the definition of \( Z \) in (1), we have

\[
m_1 = \exp(ch) \sum_{i=1}^{K} \sum_{b_i = -1}^{1} a_{i}(t-\tau_i) \exp(c \cos\theta_i) \left( \int_0^T a_{i}(t) \, dt \right)
\]

(19)

Thus, we need to consider the evaluation of the expression

\[
E_{\theta_i, \tau_i, b_i = -1} \{ \cdot \}
\]

for each \( 1 \leq i \leq K \).

\[
E_{\theta_1, \tau_1, b_1, -1} \{ \cdot \} = E_{\theta_1, \tau_1} \{ \exp[c \cos\theta_1] (R_{i1}(\tau_1) + \hat{R}_{i1}(\tau_1)) \}
\]

\[
+ \exp[c \cos\theta_1] (R_{i1}(\tau_1) - \hat{R}_{i1}(\tau_1)) + \exp[c \cos\theta_1] (-R_{i1}(\tau_1) + \hat{R}_{i1}(\tau_1))
\]

\[
+ \exp[c \cos\theta_1] (-R_{i1}(\tau_1) - \hat{R}_{i1}(\tau_1)) / 4
\]

\[
= \frac{1}{4} \sum_{n=0}^{N-1} a_{n, i} \left( \exp[c \cos\theta_1] (A_n + \hat{A}_n) + (B_n + \hat{B}_n) \lambda \right)
\]

(19)
where \( R_{11}(\cdot) \) and \( \hat{R}_{11}(\cdot) \) are given by (16) and \( A_n, B_n, \hat{A}_n, \) and \( \hat{B}_n \) are given by (17). After performing the integration in (20), we have

\[
E_{\theta_i, t_i, b_i, -1, b_i, 0} \{ \cdot \} = \frac{1}{4N} E_{\theta_i} \sum_{n=0}^{N-1} \sum_{m=1}^{4} \left\{ \frac{\exp(\alpha_i^{(m)}(n)\cos\theta_i) - \exp(\beta_i^{(m)}(n)\cos\theta_i)}{\alpha_i^{(m)}(n) - \beta_i^{(m)}(n)\cos\theta_i} \right\}
\]

where

\[
\begin{align*}
\alpha_i^{(1)}(n) &= c_T c_i (\rho_{i1}(n+1) + \beta_{i1}(n+1)) = -\alpha_i^{(4)}(n) \\
\alpha_i^{(2)}(n) &= c_T c_i (\rho_{i1}(n+1) - \beta_{i1}(n+1)) = -\alpha_i^{(3)}(n) \\
\beta_i^{(1)}(n) &= c_T c_i (\rho_{i1}(n) + \beta_{i1}(n)) = -\beta_i^{(4)}(n) \\
\beta_i^{(2)}(n) &= c_T c_i (\rho_{i1}(n) - \beta_{i1}(n)) = -\beta_i^{(3)}(n) .
\end{align*}
\]

From an integral representation of \( I_0(z) \) given by (8.431.1) in [5], we obtain

\[
2\pi \int_{-1}^{1} I_0(z) dz = 2 \int_{-1}^{1} \left\{ \frac{\exp(\alpha t) - \exp(\beta t)}{t(1-t^2)^{1/2}} \right\} dt \]

\[
= \int_{0}^{2\pi} \left\{ \frac{\exp(\alpha \cos\theta) - \exp(\beta \cos\theta)}{\cos\theta} \right\} dt
\]

Thus, substituting (23) in (21) and noting \( I_0(z) \) is an even function,

\[
E_{\theta_i, t_i, b_i, -1, b_i, 0} \{ \cdot \} = \frac{1}{2N} \sum_{n=0}^{N-1} \sum_{m=1}^{2} \left[ \frac{1}{(\alpha_i^{(m)}(n) - \beta_i^{(m)}(n))} \right] \times \int_{0}^{2\pi} I_0(z) dz
\]

Thus, substituting (24) in (19), we obtain the single exponential moment,
Finally, let

$$k_1(z) = \sum_{j=1}^{J} d_j \exp[c_j(h+z)],$$

where $d_j$ and $c_j$ are real-value numbers, and $k_2(z)$ be given by expression (2b), then the multiple exponential moment is given by

$$m_1 = \sum_{j=1}^{J} d_j m_1(c_j),$$

where $m_1(c_j)$ is given by (25) with $c = c_j$.

**Conclusions.** In the previous section, we derived the second moment, fourth moment, single exponential moment, and multiple exponential moment for the spread spectrum multiple access system. These moments are given in (8), (14), (25), and (27), respectively. Clearly, the computational efforts involved in evaluating these moments are of increasing order. However, in general, we can show the resulting upper and lower error bounds become tighter. In the last two cases, we assume the parameters $c$, $c_j$, and $d_j$ in (25) and (27) are selected appropriately.

There are several approaches in obtaining explicit moment space error bounds. From our earlier discussions, it is clear that by plotting $k_2(z)$ versus $k_1(z)$ graphically and then finding its convex hull, we can obtain the upper and lower error bound immediately. Indeed, we have used computer-controlled plotter and obtained satisfactory bounds. Furthermore, this approach is applicable to any $k_2(z)$ and $k_1(z)$ functions. An alternative approach is clearly the use of analytical solution to the upper and lower envelopes of the convex body. Bounds based on the second moment are discussed and given in [4], [7], and [8]. Similar arguments in [4] and [6] lead to bounds based on the fourth moment. Various possible error bounds and associated regions of $c$ for the single exponential moment are given in [4]. In particular, the selection of the optimum values of $c$ for the upper and the lower
bounds are treated. Finally, for systems with large maximum distortion $D$, the range of possible values of $k_2(z)$ is large. In order to obtain tight error bounds, $k_1(z)$ must approximate $k_2(z)$ closely. In (26), by using large numbers of terms $J$ with proper $d_j$ and $c_j$'s, we can make $k_1(z)$ close to $k_2(z)$. Unfortunately, the explicit solution of this approximation problem for $d_j$ and $c_j$ with finite $J$ is unknown for Chebychev norm [8]. From an efficiency point of view for a fixed $J$, this norm is the most appropriate one to use for minimizing the distance between the upper and lower envelopes. However, in $L_2$ norm, it is easy to select a set of $c_j$'s such that $k_1(z)$ converges to $k_2(z)$ as $J$ becomes unbounded. Then $k_1(z)$ can approximate $k_2(z)$ arbitrarily well for some finite $J$. Thus, the associated upper and lower bounds using $m_1$ given by (27) can be made arbitrarily tight for some finite $J$.

References


Figure 1. $k_2(z)$ versus $k_1(z)$