Summary. The detection of two dimensional optical signals which have been corrupted by noise is considered. Discussion is limited to the detection of a known object in a known location.

The problem is approached from the classical statistical technique of hypothesis testing. Initially the solution is formulated in very general terms. The decision rule is derived for a signal distorted by noise of an unspecified type which may include signal dependent noise. Once the decision rule is obtained, the probabilities of false alarm and detection are evaluated from a priori knowledge of the noise and imaging system.

The general results are applied to Poisson noise and signal dependent Gaussian noise.

Introduction. The application of the principles of communication science to optics is not a recent development. Optical imaging has been examined in the light of information theory [1,2], and the modern theory of image formation is clearly expressed in terms of Fourier analysis and linear systems theory [3,4,5].

Several authors have evaluated photographic and electronic imaging on an absolute scale [6,7,8,9,10,11,12]. Signal dependent models for the grain noise in photographic film have been suggested by Walkup and Choens [13] and others. The material available on restoration of degraded optical signals is vast and is not reviewed here. There has been some work on object detection in signal dependent noise. The problem has been solved for the case of signals in photon noise detected by an ideal receptor [14].

The work was supported by ESL, Inc., Sunnyvale, California
A set of operating characteristics applicable to detection in a broad class of noise is developed in this paper.

**The Decision Rule.** The problem of detecting a known object in an image degraded by arbitrary noise and sampling is considered first. The object location, size, and intensity in the image plane, as well as the background, are assumed known a priori. A likelihood ratio test is used because for a specified maximum probability of false alarm, the probability of detection is maximized [15].

Let \( R_k (x,y) \) be the available signal at the point \((x,y)\) in the image plane. The signal \( R(x,y) \) sampled obviously depends upon whether or not the object in question is present, and is given by

\[
H_0: \quad R(x,y) = R_0(x,y) \tag{1}
\]

or

\[
H_1: \quad R(x,y) = R_1(x,y) \tag{2}
\]

Assume that the image is sampled to form an \( M \times N \) array of data samples using a sampling function \( g_{ij}(x,y) \) to form the sample \( R_{ij} \). Then

\[
R_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(x,y) g_{ij}(x,y) \, dx \, dy \tag{3}
\]

\( R_{ij} \) will be a random variable with probability density function

\[
P_{R_{ij}|H_k}(R_{ij}|H_k) \tag{4}
\]

The likelihood ratio test is

\[
\Lambda(R) = \frac{P_{R|H_1}(R|H_1)}{P_{R|H_0}(R|H_0)} \begin{cases} 1 & \text{if } H_1 \\ \gg & \text{if } H_0 \end{cases} \eta. \tag{5}
\]

where \( \eta \) is a threshold value determined by the maximum acceptable false alarm probability.

If the sampling functions \( g_{ij}(x,y) \) are non-zero, or of significant value, only over disjoint areas in the \( x,y \) plane, the individual density functions can reasonably be assumed conditionally independent, and the decision rule is
Taking the natural log of the likelihood ratio defines the random variable

$$ L(R) = \sum_{j=1}^{N} \sum_{i=1}^{M} \ln[p_{r_{ij}|H_1}(R_{ij}|H_1)] - \sum_{j=1}^{N} \sum_{i=1}^{M} \ln[p_{r_{ij}|H_0}(R_{ij}|H_0)]. $$

(7)

Now the decision rule may be stated as

$$ L(R) \gtrsim \gamma = \ln \eta. $$

(8)

Notice that L(R) is formed by the addition and subtraction of conditionally independent random variables. If the object is large enough to require a reasonably large number of samples, say 50, in the image plane, the central limit theorem implies that L(R) will be approximately normally distributed.

The probabilities of false alarm and detection are given by

$$ P_F = \int_{\gamma}^{\infty} p_{L|H_0}(L|H_0) dL, $$

(9)

and

$$ P_D = \int_{\gamma}^{\infty} p_{L|H_1}(L|H_1) dL. $$

(10)

If the conditional probability density functions in Equations 9 and 10 are assumed to be normal, it is necessary only to determine the conditional means and variances of L(R). These are designated by \( \mu^2 \) and \( \sigma_0^2 \), respectively. The relation of \( P_F \) and \( P_D \) to the conditional density functions of L can be clearly illustrated by sketching \( p_{L|H_0}(L|H_0) \) and \( p_{L|H_1}(L|H_1) \).

However, it is more useful to standardize one of the curves by making the following change of variables:

$$ u = \frac{L - \mu_0}{\sigma_0}. $$

(11)
Also let

\[ \mu = \frac{\mu_1 - \mu_0}{\sigma_0}, \quad (12) \]

\[ \sigma^2 = \frac{\sigma_1^2}{\sigma_0^2}, \quad (13) \]

and

\[ \gamma' = \frac{\gamma - \mu_0}{\sigma_0}. \quad (14) \]

thus

\[ P_F = \text{erfc}(\gamma'), \quad (15) \]

or

\[ \gamma' = \text{erfc}^{-1}(P_F), \quad (16) \]

where

\[ \text{erfc}(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} \, dx. \quad (17) \]

Therefore, if a maximum allowable \( P_F \) is specified, \( \gamma' \) is fixed, and

\[ P_D = \text{erfc}\left(\frac{\gamma' - \mu}{\sigma}\right) \]

\[ = \text{erfc}\left(\frac{\sigma_0}{\sigma_1} \gamma' - \frac{\mu_1 - \mu_0}{\sigma_1}\right). \quad (18) \]

A decision rule based on \( U \) which is equivalent to that based on \( L \) (Equation 8) is

\[ H_1 \]

\[ U \gtrsim \gamma'. \quad (19) \]

\[ H_0 \]
Thus $P_D$ has been expressed as a function of $\gamma'$ (or $P_F$) and the two parameters $\sigma_1/\sigma_0$ and $(\mu_1 - \mu_0)/\sigma_0$. These equations are presented in graphical form in Figure 1. It should be stressed that these results are quite general. Other than the requirement that the samples $R_{ij}$ be conditionally independent, no restrictions have been placed on the noise model used or the sampling technique. If the values $\mu_k$ and $\sigma_k^2$, can be found, the curves of Figure 1 and the equations on which they are based may be used.

**Example**

Assume that $\sigma_1/\sigma_0 = 1.2$ and $(\mu_1 - \mu_0)/\sigma_1 = 3.0$. If the maximum allowable $P_F = .1$ (on 10% of tests made with the object absent the detector will indicate it, is present), the problem proceeds as follows.

From the intersection of the $P_F = .1$ line and curve corresponding to $\sigma_1/\sigma_0 = 1.2$ in Figure 1, draw a vertical line intersecting the $\gamma'\sigma_0/\sigma_1$ axis and the curve for $(\mu_1 - \mu_0)/\sigma_1 = 3.0$ on the $P_D$ graph. The values $\gamma'\sigma_0/\sigma_1 = 1.06$ and $P_D = .973$ are found. Solving for $\gamma'$, the decision rule is

$$H_1 \quad U \geq 1.27 \quad H_0$$

(20)

If it is required that the maximum $P_F = .01$, $\gamma'\sigma_0/\sigma_1$ is found to be 1.93 and $P_D = .86$. Thus the more stringent requirement on $P_F$ reduces significantly the probability of detecting the object.

**Noise Models.** The general results of the preceding section will now be applied to detection problems using different models for the noise content of the received signal $R(x,y)$. In each case the conditional means $\mu_k$ and variances $\sigma_k^2$, necessary to use the curves are calculated.

**A. Poisson Noise**

An ideal diode array is basically a photon counter. Photon emission is by nature a discrete random process in which the noise level depends on the level of intensity [8] and is often modeled by a Poisson process.

The expected value $S_k(x,y)$ of the received signal is now defined as the photon flux per unit area per second. The output $R_{ij}$ of the diode which senses area $A_{ij}$ in the image plane will be a Poisson random variable with mean
Figure 1. Detector operating characteristics.
\[ H_k : m_{ijk} = \beta_{ij} T_0 \int \int S_k(x,y) dx dy, \]  

(21)

where \( \beta_{ij} \) is the quantum efficiency (the probability with which an incident photon is counted) and \( T_0 \) is the exposure time. Because \( R_{ij} \) is Poisson, its conditional probability density function is

\[ p_{R_{ij} \mid H_k}(R_{ij} \mid H_k) = \frac{m_{ijk}^{R_{ij}}}{R_{ij}!} e^{-m_{ijk}} \quad R_{ij} = 0, 1, 2, \ldots \]  

(22)

From Equation 9 the decision rule becomes

\[ N \sum_{j=1}^N \sum_{i=1}^M R_{ij} \ln \left( \frac{m_{ij1}}{m_{ij0}} \right) - N \sum_{j=1}^N \sum_{i=1}^M (m_{ij1} - m_{ij0}) \gtrless \gamma. \]  

(23)

If \( L(R) \) is assumed to be normally distributed, its conditional mean and variance are

\[ \mu_k = \sum_{j=1}^N \sum_{i=1}^M m_{ijk} \ln \left( \frac{m_{ij1}}{m_{ij0}} \right) - \sum_{j=1}^N \sum_{i=1}^M (m_{ij1} - m_{ij0}) \]  

(24)

and

\[ \sigma_k^2 = \sum_{j=1}^N \sum_{i=1}^M m_{ijk} \left[ \ln \left( \frac{m_{ij1}}{m_{ij0}} \right) \right]^2. \]  

(25)

These values of \( \mu_k \) and \( \sigma_k^2 \), may be used with the curves of Figure 1.

The upper bound on diode array performance for the case of Poisson noise can be found by examining \( \mu_k \) and \( \sigma_k^2 \), when the diodes have unity quantum efficiency, are contiguous and the diode area shrinks to zero as the number of diodes grows without bound to cover the image area \( I \). For these conditions,

\[ \mu_k = T_0 \int \int I S_k(x,y) \ln \left[ \frac{S_k(x,y)}{S_0(x,y)} \right] dx dy \]

\[ - T_0 \int \int I \left[ S_k(x,y) - S_0(x,y) \right] dx dy \]  

(26)
This is the same result obtained by Helstrom [14]. These values $\mu_k$ and $\sigma^2_k$ can be used with the graphs in Figure 1, and the results compared with those obtained for a discrete array. The degradation in delectability of a specified object caused by sampling and diode quantum efficiency can thus be determined.

**B. Signal Dependent Gaussian Noise**

One possible noise model is that the signal $R(x,y)$ will be normally distributed about the expected value $S_k(x,y)$ with the variance deterministically related to the signal [13].

The above model may be formulated as follows

$$H_k: \quad R(x,y) = S_k(x,y) + Q_k(x,y)N(x,y)$$  \hspace{1cm} (28)

where $N(x,y)$ is a zero-mean white Gaussian noise with autocorrelation $\sigma(x,y)$, which is independent of $S_k(x,y)$; and $Q_k(x,y)$ is a deterministic function of $S_k(x,y)$ and modulates the noise. An individual sample $R_{ij}$ is assumed to be the average intensity $R(x,y)$, weighted by the sampling function $g_{ij}(x,y)$ and normalized by the volume under the sampling function curve

$$R_{ij} = \frac{\int \int R(x,y)g_{ij}(x,y)dx dy}{\int \int g_{ij}(x,y)dx dy} \quad \hspace{1cm} (29)$$

For convenience, define $A = \int \int g_{ij}(x,y)dx dy$. The random variables $R_{ij}$ will be normal with means

$$m_{ijk} = \frac{1}{A} \int \int S_k(x,y)g_{ij}(x,y)dx dy$$

and variances

$$\sigma_{ijk}^2 = \frac{1}{A^2} \int \int Q_k^2(x,y)g_{ij}^2(x,y)dx dy \quad \hspace{1cm} (30)$$

After simplification the decision rule reduces to

$$L(R) = \sum_{j=1}^{N} \sum_{i=1}^{M} \left[ a_{ij} R_{ij}^2 + b_{ij} R_{ij} + c_{ij} \right] \overset{H_1}{\gtrsim} \gamma, \quad \overset{H_0}{\leq}$$  \hspace{1cm} (31)
where

\[
a_{ij} = \frac{1}{2\sigma_{ij0}^2} - \frac{1}{2\sigma_{ij1}^2},
\]

\[
b_{ij} = \frac{m_{ij1}^2}{\sigma_{ij1}^2} - \frac{m_{ij0}^2}{\sigma_{ij0}^2},
\]

and

\[
c_{ij} = \frac{m_{ij0}^2}{2\sigma_{ij0}^2} - \frac{m_{ij1}^2}{2\sigma_{ij1}^2} + \ln\left(\frac{\sigma_{ij0}}{\sigma_{ij1}}\right).
\]

As before, \( L \) will be assumed normal by the central limit theorem, and only the conditional means and variances of \( L \) are needed to use the curves. Its conditional mean and variance are found to be

\[
\mu_k = \sum_{j=1}^{N} \sum_{i=1}^{M} [a_{ij}(\sigma_{ijk}^2 + m_{ijk}^2) + b_{ij}m_{ijk} + c_{ij}],
\]

and

\[
\sigma_k^2 = \sum_{j=1}^{N} \sum_{i=1}^{M} [a_{ij}^2 (4m_{ijk}^2 \sigma_{ijk}^2 + 2\sigma_{ijk}^4) + (4a_{ij}b_{ij}m_{ijk} + b_{ij}^2)\sigma_{ijk}^2].
\]

A particular type of signal dependent Gaussian noise has been used to model the grain noise of photographic film. For this case let \( Q_k(x,y) = \sqrt{\alpha' S_k(x,y)} \), and assume either \( S_k(x,y) \) is essentially constant over a sample region, or \( g_{iy}(x,y) \) takes on only two values, zero and a positive constant. From this definition the following results are obtained:

\[
\sigma_{ijk}^2 = \frac{\alpha}{A} m_{ijk}.
\]

with \( \alpha \) determined from \( \alpha' \) and \( g_{iy}(x,y) \). Also,

\[
a_{ij} = \frac{A}{2\alpha m_{ij0}} - \frac{A}{2\alpha m_{ij1}},
\]

\[
b_{ij} = 0,
\]

and
Therefore, for this case

\[ L(R) = \sum_{j=1}^{N} \sum_{i=1}^{M} \left[ \frac{A}{2\alpha} \left( \frac{1}{m_{ij0}} - \frac{1}{m_{ij1}} \right) \right] R_{ij}^2 + \frac{A}{2\alpha} (m_{ij0} - m_{ij1}) + \frac{1}{2} \ln \left( \frac{m_{ij0}}{m_{ij1}} \right), \]  

(41)

with mean

\[ \mu_k = \sum_{j=1}^{N} \sum_{i=1}^{M} \left[ \frac{A}{2\alpha} \left( \frac{1}{m_{ij0}} - \frac{1}{m_{ij1}} \right) \right] (A m_{ijk} + m_{ijk}^2) + \frac{A}{2\alpha} (m_{ij0} - m_{ij1}) + \frac{1}{2} \ln \left( \frac{m_{ij0}}{m_{ij1}} \right), \]  

(42)

and variance

\[ \sigma_k^2 = \sum_{j=1}^{N} \sum_{i=1}^{M} \left[ \frac{1}{m_{ij0}} - \frac{1}{m_{ij1}} \right]^2 \left( \frac{A}{\alpha} m_{ijk}^3 + \frac{1}{2} m_{ijk}^2 \right). \]  

(43)

Thus the parameters needed to use Figure 1 can be calculated.

**Conclusions.** It is clear that statistical techniques can be applied to the detection of two dimensional signals in signal dependent noise. The method appears to be useful in many applications. The added a priori knowledge necessary for detection improves performance beyond visual detection of an unrestored image, and the performance of no restoration scheme can be expected to equal that of the likelihood ratio test. Extensive use as a detection technique may be limited by the need of accurate knowledge required of the signal and image sensing system. The method, however, may be used quite effectively to evaluate the relative performance of various systems.

Throughout this paper it has been assumed that sufficient samples of the image have been taken for L to be considered normal. Therefore caution should be observed in applying these results when the available samples are few and when dealing with extremely small probabilities of false alarm.
BIBLIOGRAPHY


