Summary.—We present simple upper and lower bounds to the distribution function of the sum of two random variables in terms of the marginal distribution functions of the variables. These bounds are then used to obtain upper and lower bounds to the error probability of a coherent digital system in the presence of intersymbol interference and additive gaussian noise. The bounds are expressed in terms of the error probability obtained with a finite pulse train, and the bounds to the marginal distribution function of the residual pulse train. Since the difference between the upper and lower bounds can be shown to be monotonically decreasing function of the number of pulses in the finite pulse train, the bounds can be used to compute the error probability of the system with arbitrarily small error.

Introduction.—In digital transmission systems, the transfer characteristics of the transmitting and receiving filters are far from ideal, and the real transmission channel usually exhibits some form of time dispersion [1,2]. When an ideal digital signal is passed through such filters or is transmitted through such a channel, the signal suffers a distortion which takes the form of an overlap in time between pulses; this form of distortion is usually known as intersymbol interference. Intersymbol interference may also result from the choice of nonoptimum sampling instants, imperfect demodulating carrier phase, improper pulse design, etc. In addition, the signal may be corrupted by thermal noise, co-channel and adjacent channel interference and other forms of noise that may be present in the channel or in the system used to transmit the information.

In digital transmission systems, one of the main performance characteristics is the probability of error; this probability of error can often be expressed as a finite weighted sum of one or more distribution functions. Various authors have tried to evaluate this probability of error by a variety of methods, but the exact computation of this highly complex probability distribution can seldom be carried out.

We first present simple upper and lower bounds to the distribution function of the sum of two random variables $z_N$ and $z_R$ in terms of the marginal distribution functions of the variables. If the spread or dispersion of the random variable $z_R$ is small compared to the spread of the random variable $z_N$, one can show that these two bounds are fairly close to each other and that one can evaluate the distribution function of the sum of the variables in terms of the distribution function of $z_N$ and the bounds to the distribution function of $z_R$.

We then use these bounds to obtain upper and lower bounds to the error probability of a binary coherent digital system in the presence of intersymbol interference and additive gaussian noise. Since the difference between the upper and lower bounds can be shown to be a monotone decreasing function
of the number $N$ of pulses in the finite pulse train, the bounds can be used to compute the error probability of the system with arbitrarily small error.

Distribution Function and its Evaluation.-Let us assume that a random variable $z$ is the sum of two random variables $z_N$ and $z_R$, ($z_N$ and $z_R$ are assumed to be independent)

$$z = z_N + z_R,$$  \hspace{1cm} (1)

and that we are interested in the distribution function of $z$

$$F_z(a) = \Pr[z \leq a] = \Pr[z_N + z_R \leq a].$$ \hspace{1cm} (2)

From (2) we have

$$F_z(a) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} dF_N(z) dF_R(x) = \int_{-\infty}^{\infty} F_N(a-y) dF_R(y),$$ \hspace{1cm} (3)

or

$$F_z(a) = \langle F_N(a-y) \rangle_R.$$ \hspace{1cm} (4)

Let us now select an interval $(-\Delta \epsilon, \Delta u)$ from the range of the random variable $z_R$. From (4) we can show [3] that

$$F_{z_N}(a-\Delta u)[F_{z_R}(\Delta u)-F_{z_R}(-\Delta \epsilon)]$$

$$\leq F_z(a) \leq F_{z_R}(-\Delta \epsilon)+F_{z_N}(a+\Delta \epsilon)[1-F_{z_R}(-\Delta \epsilon)]$$

$$\leq F_{z_R}(-\Delta \epsilon)+F_{z_N}(a+\Delta \epsilon).$$ \hspace{1cm} (5)

In general it is not easy to compute $F_z(y)$. However we may be able to bound $F_{z_R}(y)$ so that

$$0 \leq F_{z_R}(-\Delta \epsilon) = \Pr[z_R \leq -\Delta \epsilon] \leq F_{z_R}(-\Delta \epsilon) \leq 1,$$ \hspace{1cm} (6)

$$0 \leq 1-F_{z_R}(\Delta u) = \Pr[z_R > \Delta u] \leq F_{z_R}(\Delta u) \leq 1,$$ \hspace{1cm} (7)

and

$$1 - F_{z_R}(\Delta u) - F_{z_R}(-\Delta \epsilon) = \Pr[-\Delta \epsilon < z_R \leq \Delta u]$$

$$\geq 1-L_{z_R}(-\Delta \epsilon)-U_{z_R}(\Delta u) \geq 0.$$ \hspace{1cm} (8)

If these bounds can be found, (5)-(8) can be made to yield

$$F_{z_N}(a-\Delta u)[1-L_{z_R}(-\Delta \epsilon)-U_{z_R}(\Delta u)]$$

$$\leq F_z(a)$$

$$\leq F_{z_N}(a+\Delta \epsilon)+L_{z_R}(-\Delta \epsilon).$$ \hspace{1cm} (9)
Evaluation of Another Upper Bound to $F_z(a)$.—Often we find that $z$ contains a gaussian random variable $n$ and can be written as

$$z = n + w_N + z_R = z_N + z_R, \quad z_N = n + w_N,$$  \hspace{1cm} (10)

where $n$, $w_N$, and $z_R$ are statistically independent random variables. We have already assumed that the mean of $z_R$ is zero. Without loss of generality we shall now assume that the mean of $n$ is zero, and its variance is $\sigma^2$.

From (10) one can show [3] that

$$F_{z_N}(a) = \frac{1}{2} \text{erfc} \left( \frac{-a + x_{w_N}}{\sigma \sqrt{2}} \right),$$  \hspace{1cm} (11)

$$F_{z}(a) \leq B_{z_N}(a, m_R, \sigma^2_R) = (1 - \sigma^2_R/\sigma^2)^{-1/2} \exp \left[ \frac{m_R^2}{2(1 - \sigma^2_R/\sigma^2)} \right]$$

$$\frac{1}{2} \text{erfc} \left[ \frac{-a + m_R/\sigma^2_R + x_{w_N}}{\sigma^2_R/\sigma^2 - 1/2} \right],$$  \hspace{1cm} (12)

$$\sigma^2_R/\sigma^2 < 1,$$

$$\Phi_{z_R}(t) = \int_{-\infty}^{\infty} \exp (ty) dF_{z_R}(y) \leq \exp (tm_R + \sigma^2_R t^2/2).$$  \hspace{1cm} (13)

If $z_R$ is an even random variable we have

$$F_{z}(a) = \frac{1}{4} \text{erfc} \left[ \frac{-a + x_{w_N} + y_{z_R}}{\sigma \sqrt{2}} \right]_{w_N, z_R}$$

$$+ \frac{1}{4} \text{erfc} \left[ \frac{-a + x_{w_N} - y_{z_R}}{\sigma \sqrt{2}} \right]_{w_N, z_R}.$$  \hspace{1cm} (14)

In this case one can also show [3] that

$$F_{z}(a) \geq \frac{1}{2} \text{erfc} \left[ \frac{-a + x_{w_N}}{\sigma \sqrt{2}} \right]_{w_N}, -a + x_{w_N} \geq 0,$$

$$= F_{z_N}(a), -a + x_{w_N} \geq 0, \forall x_{w_N}.$$  \hspace{1cm} (15)

In this case we then have

$$F_{z_N}(a) \leq F_{z}(a) \leq B_{z_N}(a, m_R, \sigma^2_R),$$

$$-a + x_{w_N} \geq 0, \sigma^2_R/\sigma^2 < 1, \forall x_{w_N}.$$  \hspace{1cm} (16)

Error Bounds with Intersymbol Interference.—The methods presented in Secs. 2 and 3 are now applied to the analysis of a binary coherent digital system subject to intersymbol interference and additive gaussian noise.

Let us now assume that the signal at the input to the receiver detector can be represented as
\[ y(t) = \sum_{\ell = -\infty}^{\infty} a_\ell p(t - \ell T) + n(t), \]  
(17)

where \( n(t) \) is a gaussian random variable with mean zero and variance \( \sigma^2 \). We shall also assume that \( \{a_\ell\} \) is a sequence of independent random variables, and \( a_\ell = \pm 1 \) with equal probability.

If the zeroth transmitted symbol \( a_0 = 1 \) and if it is detected by sampling \( y(t) \) at \( t = t_0 \), we can show that the probability of error \( P_2 \) can be written as

\[ P_2 = \text{Pr}[n + \sum_{\ell} a_\ell p_\ell < -p_0], \]  
(18)

where

\[ p_\ell = |p(t - \ell T)|, \]  
(19)

and

\[ n = n(t_0). \]  
(20)

Without loss of generality we shall now reorder sequence \( \{p_\ell\} \) in such a way that the terms of the sequence are non-increasing with increasing \( \ell \) and let us denote this new sequence by \( \{r_k\} \). Hence we can write

\[ P_2 = \text{Pr}[n + \sum_{\ell=1}^{\infty} a_k r_k < -p_0]. \]  
(21)

Equation (9) can be shown [3] to yield

\[ F_Z^{-\Delta u}[1 - 2 \exp \{-(\Delta u)^2/2\beta^2_R\}] \leq F_Z(-p_o) \leq F_Z(-p_o + \Delta u) + \exp \{-(\Delta u)^2/2\beta^2_R\}, \]  
(22)

\[ \beta^2_R = \sum_{A} r^2_{\ell}, \quad \ell \in \Lambda \text{ if } \ell > N. \]  
(23)

Note that any number \( \beta^2_R > \sum_{A} r^2_{\ell} \) may be used in computing the bounds in (22). This may be done in order to simplify computing \( \sum_{A} r^2_{\ell} \).

Since \( F_Z(-p_o) \leq 1 \), we assume that there exists an \( N \) such that

\[ |F_Z^{-\Delta u}[1 - 2 \exp \{-(\Delta u)^2/2\beta^2_R\}] - F_Z(-p_o) - F_Z(-p_o)| < \epsilon_2, \]  
(24)

where

\[ \epsilon_2 \leq \frac{1}{2} \epsilon_1 \min. \{F_Z^{-\Delta u}[1 - 2 \exp \{-(\Delta u)^2/2\beta^2_R\}] - F_Z(-p_o)\}, \]  
(25)

For this \( N \) we calculate \( \beta^2_R \) and choose \( \Delta u \) so that

\[ \exp \{-(\Delta u)^2/2\beta^2_R\} = \epsilon_2/3. \]  
(26)

We then calculate \( D_N(\Delta u, \Delta u) \) and compare it with

\[ X_N = \epsilon_1 F_Z^{-\Delta u}[1 - 2 \exp \{-(\Delta u)^2/2\beta^2_R\}]. \]  
(27)
We increase $N$ so that
\[ D_N, (\Delta u, \Delta u) \leq X_N', \quad N' \geq N. \] (28)

It is not necessary to increase $N$ in steps of one. The step size can be chosen to suit particular examples.

From (9) and (28) we can write
\[ A_N', (-p_0) \leq F_z(-p_0) \leq B_N', (-p_0), \] (29)
\[ A_N', (-p_0) \leq F_z_{N'}(-p_0 - \Delta u)[1 - 2 \exp \{-(\Delta u)^2/2\beta^2_R\}], \] (30)
\[ B_N', (-p_0) \geq F_z_{N'}(-p_0 + \Delta u) + \exp \{-(\Delta u)^2/2\beta^2_R\}, \] (31)
\[ B_N', (-p_0) - A_N', (-p_0) \leq \varepsilon_1 A_N', (-p_0). \] (32)

It is evident from (29) and (32) that $F_z(-p_0)$ is equal to $A_N', (-p_0)$ or $B_N', (-p_0)$ with an error less \( \varepsilon_1 \).

We have programmed this algorithm on a digital computer and we have been very successful in evaluating $F_z(-p_0)$ from this algorithm.

Applications.-Let us now assume that $p(t)$ is obtained by passing a square pulse through a single-pole RC-filter or that
\[ p(t) = \begin{cases} 0, & t < 0 \\ 1 - \exp (-2\pi Wt), & 0 \leq t \leq T, \\ \exp [-2\pi W(t-T)], & t > T. \end{cases} \] (33)
(34)
(35)

For this pulse we can write
\[ p_0 = 1 - \exp (-2\pi W t_0), \quad 0 \leq t_0 \leq T, \] (36)

and
\[ r_k = [1 - \exp (-2\pi W T)] \exp [-2\pi W(t_0 + (k-1)T)], \quad k \geq 1. \] (37)

For $2W T = 0.5$, and $t_0 = T$, we plot in Fig. 1 $F_z(-p_0)$ with an error less than 0.2 percent. In this figure we also plot $N'$, the number of terms required in estimating $F_z(-p_0)$.

Let us now consider the ideal bandlimited pulse $p(t)$ where
\[ p(t) = \frac{\sin \pi t/T}{\pi t/T}, \] (38)
\[ p_0 = \frac{\sin \pi \delta}{\pi \delta}, \quad \delta = t_0 T < 1, \quad t_0 \geq 0 \] (39)
\[ r_{2k-1} = \frac{\sin \pi \delta}{\pi[k-\delta]}, \quad k \geq 1, \] (40)
\[ r_{2k} = \frac{\sin \pi \delta}{\pi[k+\delta]}, \quad k \geq 1. \] (41)
This system does not have an "open eye pattern". Also we shall assume that we take an even number of terms in \( w_N \) in estimating \( F_Z(-p_0) \).

We have

\[
\beta_R^2 = \sum_{k=N+1}^{\infty} \frac{r_k^2}{\pi^2} \sin^2 \pi \delta \left[ \frac{1}{(k-\delta)^2} + \frac{1}{(k+\delta)^2} \right]
\]

\[
\leq 2 \frac{(1+\delta^2)}{(1-\delta^2)^2} \frac{\sin^2 \pi \delta}{\pi^2} \left[ \frac{\pi^2}{\delta^2} - \sum_{k=1}^{\infty} \frac{1}{k^2} \right] = \alpha_R^2.
\]

Since \( \alpha_R^2 \) is more easily computed than \( \beta_R^2 \), we shall use \( \alpha_R^2 \) in (22).

For \( \delta = 0.05 \), we plot in Fig. 2, \( F_Z(-p_0) \) with an error less than 50 percent when \( F_Z(-p_0) \geq 2 \times 10^{-6} \) and less than 100 percent when \( F_Z(-p_0) < 2 \times 10^{-6} \). In this figure, we also plot \( N \), the number of terms required in estimating \( F_Z(-p_0) \). Since \( \alpha_R^2 \) is a slowly decreasing function of \( N \), the number of terms required for estimating \( F_Z(-p_0) \) is much larger than that in the earlier example.

Since \( z_N \) contains a gaussian random variable and since \( z_R \) is an even random variable, (12) can also be used to obtain upper and lower bounds to \( F_Z(-p_0) \).

Conclusions. - We have presented simple upper and lower bounds to the distribution function of the sum of two random variables in terms of the marginal distribution functions of the variables.

These bounds are then applied to the error rate analysis of a binary coherent digital system subject to intersymbol interference and additive gaussian noise. Since the difference between the upper and lower bounds is a monotone decreasing function of the number of pulses in the finite pulse train, the bounds can be used to compute the error probability with arbitrarily small error. Application of these bounds is illustrated by two examples.

Many other applications including the analysis of co-channel and adjacent channel interference in communication systems will be evident to the reader. Some such novel applications will be given in a future publication.

REFERENCES


Fig. 1 - Probability of error of binary coherent digital system with intersymbol interference and additive gaussian noise. The received pulse is an exponential pulse, and $2WT = 0.5$. The upper bound $B_N(-p_0)$ is plotted in this figure and $N$ was increased in steps of one. $[B_N(-p_0) - P_Z(-p_0)] / P_Z(-p_0) < 0.002$.

Fig. 2 - Probability of error of binary coherent digital system with intersymbol interference and additive gaussian noise. The received pulse is an ideal bandlimited pulse, and it is sampled at $t_0$, $t_0T = 0.05$. The upper bound $B_N(-p_0)$ is plotted in this figure and $N$ was increased in steps of 100. $[B_N(-p_0) - P_Z(-p_0)] / P_Z(-p_0) < 0.5$, $P_Z(-p_0) > 2\times10^{-6}$, $[B_N(-p_0) - P_Z(-p_0)] / P_Z(-p_0) < 1$, $P_Z(-p_0) < 2\times10^{-6}$. 