PERMUTATION AND CIRCULANT MATRICES AND THE FAST
FOURIER TRANSFORM

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Summary  This paper provides a description of the Fast Fourier Transform and its
connection with the circulant and permutation matrices. It is written for the case where
the number of discrete time samples is equal to the number of discrete frequency samples
but is otherwise not restricted. The paper demonstrates that since the modal matrix of a
permutation matrix contains only one bit of information, the evaluation of the discrete
Fourier Transform involves considerably fewer than $N^2$ multiplications where $N$ is the
number of samples involved and is also the order of the matrices involved.

Circulant and Permutation Matrices  An $n \times n$ matrix of the following form is known
as a circulant matrix.

$$C = \begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & c_3 & \cdots & c_0
\end{bmatrix}$$

It is noted that there are at most $n$ distinct elements, all elements on the main diagonal
are equal, all elements on the off-diagonals are equal. Each row is a cyclic permutation
of the row immediately preceding it.

It is well known that the eigenvalues of $C$ are determined readily through the use of the
permutational matrix $P$ defined as:
Restricting $C$ to a $3 \times 3$ matrix, we have, without loss of generality:

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

where $I$ is the identity matrix. The eigenvalues of $P$ can be obtained by solving the eigenvalue problem:

\[
PY = \lambda Y
\]

where $Y$ is a $3 \times 1$ column vector and $\lambda$ is a scalar. The eigenvalues are the cube-roots of unity given by $\lambda^3 = 1$ and are denoted by $1, a, a^2$ where $a = e^{i2\pi/3}$ and $a^2 = e^{i4\pi/3}$. For each of these eigenvalues the three components of a direction defining the eigenvector of $P$ are given by $1, a, a^2$. Hence, the modal matrix \(^1\) of $P$ is given by:

\[
M = \begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a
\end{bmatrix}
\]

As in the case of $C$ both $P$ and $M$ have at most $n$ distinct elements and not $n^2$ elements as might be found in a matrix of order $n$. Furthermore, there are $\frac{n-1}{2}$, when $n$ is odd, and $\frac{n-2}{2}$, when $n$ is even, complex conjugate pairs of elements in $M$. The $n$-roots of unity reduce the plane to $n$ pie-shaped sectors each subtending the angle $\frac{2\pi}{n}$. Since the first root has the value 1, it is only necessary to know the angle $\frac{2\pi}{n}$ to determine all
other roots. Thus only one distinct element besides 1 is needed to completely determine N.

The inverse of the third order modal matrix is $M^{-1} = 1/3 M^*$, where $M^*$ is the complex conjugate of $M$.

Circulant Matrix in Diagonal Form. - The spectral matrix of $P$, $S$, is related to $P$ as follows:

$$P = \begin{pmatrix} 1 & a & \lambda_1 \\ a & a^2 & \lambda_2 \\ \lambda_3 & \end{pmatrix}$$

Similarly the spectral matrix $[2]$ of a polynomial in $P$, $aI + bP + cP^2$ is given by $aI + bS + cS^2$.

We note that since $P^n = I$ polynomials of degree $n - 1$ span the entire space represented by the modal columns of $M$. In the above the spectral matrix of $P^2$ is $S^2$.

The spectral matrix $S_0$ of the circulant matrix $C = c_0I + c_1P + c_2p^2$ can be written immediately as:

$$S_0 = c_0I + c_1S + c_2S^2 = \begin{pmatrix} c_0 + c_1 + c_2 \\ c_0 + c_1a + c_2a^2 \\ c_0 + c_1a^2 + c_2a \end{pmatrix}$$

Note that when these eigenvalues are complex they occur in complex conjugate pairs. The three eigenvalues of $S_0$ can also be thought of as the elements of the following row and column matrices:
The General Circulant Matrix of order $n$. The following extends the $3 \times 3$ results above to the cases of $n \times n$ matrices:

$$\begin{pmatrix} \lambda c_0, \lambda c_1, \lambda c_2 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} M \quad \text{and} \quad \begin{pmatrix} \lambda c_0 \\ \lambda c_1 \\ \lambda c_2 \end{pmatrix} = M \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

The General Circulant Matrix of order $n$ is described by:

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & & & & \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

is the $n^{\text{th}}$ order circulant.

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the $n^{\text{th}}$ order permutation matrix.

The eigenvalues of $P$ are the $n^{\text{th}}$ roots of unity, i.e., $\lambda^n = 1$ and $\lambda_k = a_k = \exp\left(\frac{2\pi ik}{n}\right)$, $k = 0, 1, 2, \ldots n - 1$. The modal matrix of $P$ is given by:
where $M^*$ is the complex conjugate of $M$.

The spectral matrices of $P$, $P^m$ and $C$ are:

\[
M = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & a_1 & a_2 & & a_{n-1} \\
1 & a_1^2 & a_2^2 & & a_{n-1}^2 \\
& \cdots & \cdots & \cdots & \cdots \\
1 & a_1^{n-1} & a_2^{n-1} & & a_{n-1}^{n-1}
\end{bmatrix}
\quad \text{and} \quad M^{-1} = \frac{1}{n} M^*
\]

and

\[
S = \begin{bmatrix} a_0 & & & \\ a_1 & a_2 & & \\ & \cdots & \cdots & \cdots \\ & & a_{n-1} & \end{bmatrix} \quad ; \quad S^m = \begin{bmatrix} a_0^m & & & \\ a_1^m & a_2^m & & \\ & \cdots & \cdots & \cdots \\ & & a_{n-1}^m & \end{bmatrix}
\]

and

\[
S_0 = \text{diag} \left[ \lambda_{c_k} \right]
\]
where

$$\lambda_{c_k} = \sum_{j=0}^{n-1} \begin{bmatrix} M_{k_j} \end{bmatrix} c_j$$

$$= c_0 + c_1 a_1^{k-1} + c_2 a_2^{k-1} + c_3 a_3^{k-1} + \ldots + c_{n-1} a_n^{k-1}$$

$$= \left| \lambda_{c_k} \right| \exp(i\psi_{c_k})$$

**The Discrete Fourier Transform** The above treatment of circulant matrices indicated that an arbitrary vector C could be represented in terms of basis vectors consisting of the n-roots of unity. This fact is of use in the evaluation of discrete fourier transforms of arbitrary waveforms.

The fourier pair

$$F(t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} \, d\omega$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} \, dt$$

can be represented as one-sided sums of weighted discrete samples as follows:

$$F(t_k) = \Delta\omega \sum_{n=0}^{N-1} G(\omega_n) e^{i\omega_n t_k}$$

$$G(\omega_n) = \frac{\Delta t}{2\pi} \sum_{k=0}^{K-1} F(t_k) e^{-i\omega_n t_k}$$
where the total frequency interval \( \omega_N \) is divided into \( N \) equal increments of width \( \Delta \omega \) and the time interval \( t_k = K \Delta t \). The following relations are used or implied in the above equations:

\[
\Delta \omega = \frac{2\pi K}{NT}
\]

\[
\omega_n = n \Delta \omega = \frac{2\pi n K}{NT}
\]

\[
t_k = k \Delta t = k \frac{T}{K}
\]

If the following change in notation is made

\[
F_k = F(t_k), \quad G_n = \frac{2\pi G(\omega_n)}{\Delta t} \quad \text{and} \quad a_{nk} = \exp\left(-\frac{ink 2\pi}{N}\right) \quad (nk = 0, 1, 2, \ldots N-1)
\]

the discrete Fourier transform pair can be written:

\[
F_k = \frac{1}{N} \sum_{n=0}^{N-1} G_n a^*_{nk}
\]

\[
G_n = \sum_{k=0}^{K-1} F_k a_{nk}
\]

where \( a^*_{nk} \) is the complex conjugate of \( a_{nk} \). In matrix notation these equations are given by:

\[
G = MF
\]

\[
F = M^{-1}G
\]

where \( F \) is a \( K \times 1 \) column matrix, \( G \) is an \( N \times 1 \) column matrix and \( M \) is an \( N \times K \) matrix and \( M^{-1} = \frac{1}{N} M^* \). There are three cases of interest for \( M \). When \( N = K \), the usual case,
This is the Modal Matrix for the $n \times n$ permutation matrix discussed in Section 1 of the paper. When $K > N$ the modal columns repeat after every $N$ columns. When $K < N$ some of the modal columns are missing. For the rest of the discussion we restrict the discussion to $N = K$ since very little is lost in doing so\[^3\].

The elements of the $G$ vector ($G_0, G_1, G_2, \ldots G_{N-1}$) are given by the eigenvalues of the Circulant matrix whose first row is $F_0, F_1, \ldots F_{N-1}$. We noted in section one that these eigenvalues are obtained as the inner product of $M$ and the first row of the circulant. Hence, we have directly

$$
\begin{pmatrix}
G_0 \\
G_1 \\
G_2 \\
\vdots \\
G_{N-1}
\end{pmatrix} = \left(\begin{pmatrix}
\lambda F_0 \\
\lambda F_1 \\
\lambda F_2 \\
\vdots \\
\lambda F_{N-1}
\end{pmatrix}
\right) = \begin{pmatrix}
F_0 + F_1 + \ldots + F_{N-1} \\
F_0 + a_1 F_1 + \ldots + a_{N-1} F_{N-1} \\
F_0 + a_2 F_1 + \ldots + a_{2(N-1)} F_{N-1} \\
\vdots \\
F_0 + a_{N-1} F_1 + \ldots + (N-1)^2 F_{N-1}
\end{pmatrix}
$$

where the coefficients are the $n$-roots of unity and complex eigenvalues occur in conjugate pairs.

The calculate $G$, therefore, from $N$ given equispaced samples of $F(t)$ extending over the time $T$, one forms the modal matrix $M$ and calculates the first $N/2$, for $N$ even, and the first $N-1/2$ for $N$ odd values of $G$. The remaining values are $-G_0$ and $G_{N/2}^*$ to $G_1^*$ for $N$ even and $G_{N-1/2}^*$ to $G_1^*$.
for N odd. There are $N - 1$ products required to find $G_1$ and any of the other N/2, G components required. Hence, there are $\frac{N}{2} (N - 1) \approx \frac{N^2}{2}$ product required. Since approximately N of these products involve real numbers and the imaginary numbers involved require only 2 multiplications one can say that an upper bound to the number of multiplying operations is $N^2$. For N not too large the number of multiplications can be considerably reduced from $N^2$.

A similar approach can be used when discrete samples of $G(t)$ are available and $F$ is required. In this case the equation to be used is $F = M^{-1}G = \frac{1}{N} M^*G$.

The Fast Fourier Transform. -If we restrict $K = N = 2^\gamma$, $\gamma$ an integer, the modal matrix of $P$, $M$, can be written as a product of $\gamma$ square $(N \times N)$ matrices. Consider the $4 \times 4$ modal matrix involving the four fourth roots of unity $1, a, a^2, a^3$. In this case,

$$M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & a & a^2 & a^3 \\
1 & a^2 & 1 & a \\
1 & a^3 & a^2 & a
\end{bmatrix}$$

This can be written $M = [a_{nk}]$ where $a_{nk} = \exp\left(-\frac{2\pi ink}{N}\right) = \exp\left(-\frac{2\pi ink}{4}\right)$ when $N = 4$. The subscript $nk$ denotes the row and column as well as the power to which $a_1 = a = \exp\left(-\frac{2\pi i}{4}\right)$ is raised. We note in passing that $a_0 = 1$.

If the second and third rows of $M$ are interchanged, the resulting matrix can be expressed as a product of two matrices as follows:

$$M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & a^2 & 1 & a^2 \\
1 & a & a^2 & a^3 \\
1 & a^3 & a^2 & a
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & a^2 & 0 & 0 \\
0 & 0 & 1 & a \\
0 & 0 & 1 & a^3
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & a^2 & 0 \\
0 & 1 & 0 & a^2
\end{bmatrix}$$

This is still written as $M$ since the modal columns are unaffected by an interchange of rows. This will result only in the eigenvalues coming out in a different order in the spectral matrix.
In the case of an 8 x 8 matrix M can be written as:

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & a^4 & 1 & a^4 & 1 & a^4 & 1 & a^4 \\
1 & a^2 & a & 1 & a^6 & 1 & a^6 & a \\
1 & a^6 & a^2 & 1 & a^2 & 1 & a^4 & a^2 \\
1 & a & a^2 & 3 & a & 5 & 6 & 7 \\
1 & a^5 & a^2 & 7 & a & 1 & 6 & a^3 \\
1 & a^3 & 6 & 1 & a & 4 & 7 & 2 \\
1 & a^7 & a & 6 & 5 & a & 4 & 3 \\
1 & a^2 & a & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

where the rows have been placed in the order 0, 4, 2, 6, 1, 5, 3, 7 as is obvious from the second modal column. Since \( N = 8 = 2^3 \), this matrix can be written as the product of three matrices. Although we shall later state general rules from which these three matrices can be found we shall indicate here the way in which the 3 matrices entering the product are formulated. The Modal matrix M can be written as a partitioned matrix as follows:

For a 16 x 16 matrix the partitioned matrices are as follows:
\[
\begin{array}{cccc}
\hline
B & a^4B & a^2B & a^2B \\
\hline
C & a^2C & a^4C & a^6C \\
\hline
C & a^6D & a^{12}D & a^{18}D \\
\end{array}
\]

\[
\begin{array}{cccc}
\hline
B & a^4B & a^2B & a^2B \\
\hline
C & a^2C & a^4C & a^6C \\
\hline
C & a^6D & a^{12}D & a^{18}D \\
\end{array}
\]

\[
\begin{array}{cccc}
I & I & I & I \\
\hline
I & a^4I & I & a^4I \\
\hline
I & a^2I & a^4I & a^6I \\
\hline
I & a^6I & a^{12}I & a^{18}I \\
\end{array}
\]

\[
\begin{array}{cccc}
\hline
B & a^4B & a^2B & a^2B \\
\hline
C & a^2C & a^4C & a^6C \\
\hline
C & a^6D & a^{12}D & a^{18}D \\
\end{array}
\]

\[
\begin{array}{cccc}
I & a^1I & I & a^1I \\
\hline
I & a^4I & I & a^4I \\
\hline
I & a^2I & a^4I & a^6I \\
\hline
I & a^6I & a^{12}I & a^{18}I \\
\end{array}
\]

\[
\begin{array}{cccc}
I & I & I & I \\
\hline
I & a^4I & I & a^4I \\
\hline
I & a^2I & a^4I & a^6I \\
\hline
I & a^6I & a^{12}I & a^{18}I \\
\end{array}
\]

\[
\begin{array}{cccc}
I & I & I & I \\
\hline
I & a^4I & I & a^4I \\
\hline
I & a^2I & a^4I & a^6I \\
\hline
I & a^6I & a^{12}I & a^{18}I \\
\end{array}
\]
For a 16 x 16 matrix the partitioned matrices are as follows:
where

\[
A = \begin{pmatrix}
1 & a^8 \\
1 & a^8 \\
\end{pmatrix} \quad B = \begin{pmatrix}
1 & a^4 \\
1 & a^{12} \\
\end{pmatrix} \quad C = \begin{pmatrix}
1 & a^2 \\
1 & a^{10} \\
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
1 & a^6 \\
1 & a^{14} \\
\end{pmatrix} \quad E = \begin{pmatrix}
1 & a^9 \\
1 & a \\
\end{pmatrix} \quad H = \begin{pmatrix}
1 & a^5 \\
1 & a^{13} \\
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
1 & a^3 \\
1 & a^{11} \\
\end{pmatrix} \quad L = \begin{pmatrix}
1 & a^7 \\
1 & a^{15} \\
\end{pmatrix}
\]

M for the 16 x 16 case is shown as a single partitioned matrix on the following page. The following relationships should be noted:

a) \(-a^8 = a^0 = 1\)

b) \(-a^{4} = a^{12} = i\)

c) \(a^2 = -a^{10} = a^{14*} = -a^{6*}\)

d) \(a = -a^9 = a^{15*} = -a^{7*}\)

e) \(a^5 = -a^{13} = a^{11*} = -a^{3*}\)
A more accurate count of the different multiplying operations involved in finding \( G = MF \) can be made by noting the above enumerated relationships and the following facts:

\[
G_0 = M_{ij} F_j \text{ is real}
\]
\[
G_1 = M_{2j} F_j \text{ is real}
\]
\( G_2 \) to \( G_7 \) are comprised of 3 complex conjugate pairs
\( G_8 \) to \( G_{15} \) are comprised of 4 complex conjugate pairs

With these facts in mind there are 52 multiplying operations involved in calculating the components \( G_0 \) to \( G_{15} \) in the case of a 16 x 16 matrix. For an 8 x 8 matrix the number of multiplying operations is 8. These operations involve products of real numbers (as opposed to complex numbers). The number of operations is independent of the decomposition of the Modal matrix into the product of 2 or more matrices but it may still be simpler to use a fast fourier technique such as the Cooley - Tukey technique.

The Cooley - Tukey Method. -The Cooley - Takey Method \[^4\] establishes efficient rules by which the components of the \( G \) vector can be calculated without actually writing down the matrices involved. It is the purpose of this section to state these rules after some preliminary remarks concerning the case involving a 4 x 4 modal matrix.
The basic vector-matrix equation is \( G = MF \) where \( F \) is known or measured. When \( M \) is 4 x 4 this can be written explicitly as:

\[
\begin{array}{c}
G_0 \\
G_1 \\
G_2 \\
G_3
\end{array}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & a & a^2 & a^3 \\
1 & a^2 & 1 & a^2 \\
1 & a^3 & a^2 & a^1
\end{bmatrix}
\begin{array}{c}
F_0 \\
F_1 \\
F_2 \\
F_3
\end{array}
\]

By direct multiplication it can be verified that this equation can be written:

\[
\begin{array}{c}
G_0 \\
G_2 \\
G_1 \\
G_3
\end{array}
= \begin{bmatrix}
1 & a^0 & 0 & 0 \\
1 & a^2 & 0 & 0 \\
0 & 0 & 1 & a^1 \\
0 & 0 & 1 & a^3
\end{bmatrix}
\begin{array}{c}
1 & 0 & a^0 & 0 \\
0 & 1 & 0 & a^1 \\
1 & 0 & a^2 & 0 \\
1 & 0 & a^3 & 0
\end{array}
\begin{array}{c}
F_0 \\
F_1 \\
F_2 \\
F_3
\end{array}
\]

where it is to be noted that \( G_1 \) and \( G_2 \) have been interchanged, \( a^0 = 1 \) and \( N = 4 = 2^\gamma, \gamma = 2 \). Operating on \( F \) with the second matrix one has

\[
\begin{array}{c}
G_0 \\
G_2 \\
G_1 \\
G_3
\end{array}
= \begin{bmatrix}
1 & a^0 & 0 & 0 \\
1 & a^2 & 0 & 0 \\
0 & 0 & 1 & a^1 \\
0 & 0 & 1 & a^3
\end{bmatrix}
\begin{array}{c}
F_0 + a^0 F_1 \\
F_1 + a^0 F_3 \\
F_0 + a^2 F_2 \\
F_1 + a^2 F_3
\end{array}
\]

\[
\begin{bmatrix}
1 & a^0 & 0 \\
1 & a^2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
K_0 \\
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\]

where for convenience we have denoted the intermediate state of \( F \) as \( K \).
In the Cooley-Tukey method the order of the components of the G-vector is determined as follows: The subscripts of these components are written as binary numbers. These numbers are inverted and the inverted binary numbers are converted to decimal numbers. The result is the new order for the G-vector components. The following examples illustrate method for $N = 4$ and $N = 8$.

### $N = 4$

<table>
<thead>
<tr>
<th>Original Subscript</th>
<th>Binary Form $\gamma$ - bits</th>
<th>Inverted Form</th>
<th>Final Subscript</th>
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</thead>
<tbody>
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<td>00</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>01</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>01</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>

### $N = 8 = 2^3$

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<th>Binary Form $\gamma$ - bits</th>
<th>Inverted Form</th>
<th>Final Subscript</th>
</tr>
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<tbody>
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</tr>
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<td>100</td>
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</tr>
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<td>010</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
<td>110</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
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<td>111</td>
<td>7</td>
</tr>
</tbody>
</table>

Multiplications and additions performed in obtaining the elements of K and G can be obtained directly from a suitably constructed logic tree diagram. The following tree diagram indicates the operations to be performed to obtain the K and G vectors for the 4 x 4 case discussed previously. The original F matrix is placed on the left and the G matrix appears on the right.
Solid lines represent multiplications, and dashed lines represent additions. The numbers in the small circles are the powers of a. For example, \( K_2 \) is obtained by multiplying \( F_2 \) by \( a^2 \) and adding \( F_0 \) to the product. The logic of determining the powers of a in the circles, and the multiplications and additions is as follows:

1. Assume \( N \) sampled values \( F_0, F_1, F_2, \ldots F_{N-1} \) where \( N = 2^\gamma \). The integer \( \gamma \) equals the number of bits in the binary representation of \( N \).

2. Arrange \( F_0, F_1, F_2, \ldots F_{N-1} \) in \( N \) rows and label these as column 0.

3. Draw \( \ell \) columns to the right of \( F \) and label these as 1, 2, \ldots, \( \gamma \). The \( \gamma \) column is \( G \), the remainder are intermediate columns.

4. Let the number in a circle which represents the power of “a” be denoted by \( p \). To find \( p \) for row \( k \) and column \( \ell \), write the binary number \( k \), slide this number \( \gamma - \ell \) places to the right filling the newly opened locations with zeros, reverse the order of the resulting binary bits to obtain \( p \).

5. The solid lines in the tree diagram denote multiplication. The rule to obtain these is given as follows: Represent \( k \) in binary form as \( k = k_{\gamma-1}, k_{\gamma-2}, \ldots, k_0 \) where \( k_0, k_1, \ldots, k_{\gamma-1} \) are each either 0 or 1. In column \( \ell \), node \( k \) has a solid line drawn to it from a node in the \((\ell - 1)\) column having a one in the \( k_{\gamma-\ell} \) bit location but otherwise having the same binary representation as \( k \). For example, for \( \gamma = 4, \ell = 3 \), row zero \((k = 0 = 000)\) will have a solid line drawn to it from the node in column 2 for which \( k = 0010 = 2 \).

6. The dashed lines in the tree diagram indicate addition. Node \((k, \ell)\) is connected by a dashed line from a node in the \( \ell - 1 \) column having the binary representation as \( k \) except that the \( k_{\gamma-\ell} \) digit is replaced by a zero.

The tree for \( N = 8 \) is shown below.
The Inverse Fourier Transform  The equation \( G = MF \) for a given \( F \) permits calculation of \( G \). The components of \( G \) occur as complex conjugate pairs which can be written in complex notation and substituted back into the discrete fourier transform pair to obtain the inverse Fourier transform \( F \). This can be a useful check on the calculated results for \( G \).

Assuming the function is known in the frequency domain the problem of converting from a set of discrete samples to the time domain is of interest. In this problem \( N \) samples of \( G(t) \) are known and \( F = M^{-1}G = \frac{1}{N}M^*G \) is to be found.

In the evaluation of \( G = MF \) where \( N \) discrete equispaced samples of \( F(t) \), \( 0 \leq t \leq T \), are known, we have shown that the known values of \( F \) may be used to form a circulant matrix

\[
C = \begin{bmatrix}
F_0 & F_1 & F_2 & \cdots & F_{N-1} \\
F_{N-1} & F_0 & F_1 & \cdots & F_{N-2} \\
\vdots & & & & \ddots \\
F_1 & F_2 & F_3 & \cdots & F_0
\end{bmatrix} = F_0I + F_1P + \ldots + F_{N-1}P^{N-1}
\]

The transform of \( F \) consists of \( N \) discrete samples of \( G \). These are given by the \( N \) spectral values of \( C \) which in turn can be represented as the direct product of \( M \), the modal matrix of the permutation matrix \( P \), and a vector.
where it is assumed that the samples of \(F(t)\) are ordered with respect to the time interval \((0, T)\).

To evaluate \(F = M^{-1} G\) where \(N\) equispaced samples of \(G(w)\) are known we form a circulant matrix and take its transpose.

The transform of \(G\) consists of \(N\) discrete samples of \(F\). These are given by the \(N\) spectral values of \(K'\) which in turn can be represented as the direct product of \(M\), the modal matrix of \(P\) and a vector.
the subscripts of the components of $G$ imply a set of ordered samples in the frequency interval $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$. The inverted vector $G$ can then be operated on in accordance with the Cooley - Tukey algorithm or with any other desired approach.

The above description of the evaluation of $F$ and $G$ indicates that where a positive sense for the independent variable is unknown only $F$ or $F^*$ (and $G$ or $G^*$) can be calculated. However, the observables $\langle F, F^* \rangle$ and $\langle G, G^* \rangle$ can be determined in any case.

**Conclusions** This paper provides an elementary description of the Fast Fourier Transform and its connection with the circulant and permutation matrices. The paper includes the transform and its inverse of a discrete set of sample values. The Cooley - Tukey technique applies specifically to $N = K = n^\uparrow$. This paper applies to the case where $N = K$ and shows that, since the modal matrix of $P$ gives only one bit of information (viz., an angle between two directions) the evaluation of the discrete Fourier transform involves considerably fewer than $N^2$ multiplications. The Cooley - Tukey algorithm mechanizes the evaluation of the transform to take advantage of this fact.

**References**

