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by

Noah Bareket

A Dissertation Submitted to the Faculty of the
COMMITTEE ON OPTICAL SCIENCES (GRADUATE)
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

1979
I hereby recommend that this dissertation prepared under my direction
by Noah Bareket
entitled Image Parameters for Fast Evaluation of
Adaptive Optical Systems
be accepted as fulfilling the dissertation requirement for the Degree
of Doctor of Philosophy.

Dissertation Director 11-8-79

As members of the Final Examination Committee, we certify that we have
read this dissertation and agree that it may be presented for final
defense.

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RUTH
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ABSTRACT

Adaptive optical systems generate images with time-varying irradiance distributions. Traditional methods of evaluating the optical performance of these systems are much too slow for real-time analysis, where the required rate can be as high as $10^3$ Hz. In some applications this speed is necessary to control the feedback loop, thereby improving the optical performance. An image quality parameter for such an application serves as a sharpness function that converges to a maximum value as the system quality improves. In other applications, a summary measure of the image quality is necessary, but the convergence property of the quality parameter is not required.

In this work a set of image parameters is defined in terms of the first and second moments of the irradiance distribution. The set includes the centroid, variance, eccentricity, and orientation of the principal axes. Analog and digital techniques of measuring these parameters are investigated. The relationship between the second moment of the point spread function (PSF) and the optical transfer function (OTF) is established through the moment theorem. It is found that the second moment is the sum of a contribution of the pupil amplitude distribution, and a term that is directly related to the wavefront aberrations. In practice it is possible to redefine the second moment such that a converging quality parameter, identical to one successfully used in lens design, is obtained.
One way to measure the above parameters is to sample the PSF by an array of detectors and use the digitized data to calculate the moments of the distribution. There is a tradeoff between the measurement accuracy and the measurement time. Theoretical analysis and computer simulation, show that it is possible to accurately measure the moments with considerable undersampling. The measurement error is found by applying the moment theorem to the sampled distributions. Sampling with small elements is compared to sampling with extended elements, and it is shown that the latter is always more accurate. Error calculations for a square aperture and for a Gaussian beam are presented. For the Gaussian the sampling interval can be as large as $2.3\sigma$ ($\sigma$ being the variance) with a 5% error. Simulation of atmospherically aberrated images show that although the form of the irradiance distribution changes considerably, the measurement errors decrease due to the increase of the effective sampling rate, which is given by the ratio between the spot size and the sampling interval.

An alternate way of measuring the image parameters is by means of an analog measurement system. Such a system is described here. Its operation is based on chopping the image by a series of slits or knife-edge apertures. Algorithms for performing the calculations are developed and functional block diagrams of electronic circuitry which perform these calculations are presented. A demonstration unit has been built and qualitative analysis of its performance confirms the applicability of the technique.
CHAPTER 1

INTRODUCTION

Evaluating the performance of an optical system is usually a long, time-consuming process. However, with the development of adaptive optical systems, the need has arisen to evaluate the optical quality at fast rates (Hardy, 1978; Pearson, Freeman, and Reynolds, 1978). One principle measure of the optical quality is the irradiance distribution of the image of a point object, the point spread function (PSF). The PSF tells essentially all that there is to know about the quality of the image, and there are various ways of measuring it (Shannon, 1965; Levi, 1968). However, these methods involve many sampling points and time-consuming data-processing and are incapable of producing information at the required rate, which can be as high as $10^3$Hz (Muller and Buffington, 1974; Shannon and Wolfe, 1977). It is necessary, therefore, to reduce the information content of the PSF into a small set of parameters that are linked to the image quality and are measurable at the required rate.

A related problem arises when it is necessary to characterize the shape of the irradiance distribution so it can be reproduced at a later time. This requirement arises in simulators, for example: in simulating the effects of illuminating a target with a light source. Here it is necessary to measure the irradiance distribution with a resolution which is equivalent to the resolution of the simulation device, and no more. Consequently the PSF might have much redundant information.
Both of these problems emerged during the design of an optical system for testing and evaluating the performance of adaptive-optics trackers. These trackers transmit a laser beam which illuminates the target. The returned radiation is then analyzed to correct for atmospheric phase-distortion effects (Pearson, Freeman, and Reynolds, 1978). The test system simulates a target and various atmospheric effects. It is supposed to analyze the laser beam, which is transmitted by the tracker and is distorted by the simulated atmosphere. The analysis should supply information both for the purpose of generating a similar irradiance distribution on the simulated target, and for performance evaluation of the tested tracker (Shannon and Wolfe, 1977).

It was necessary, therefore, to define a set of image parameters suitable for this task. These parameters should be a measure of the image quality, give information about the shape of the image and, finally, be measurable to a satisfactory accuracy at a rate of $10^3\text{Hz}$. Such a set can be defined in terms of the first and second moments of the irradiance distribution.

The fundamental task in this dissertation was to show the relationship of these parameters to established, well-used image quality parameters and to find a way of measuring them at the required rates. The first problem is treated in Chapter 2. Commonly used image parameters are reviewed and their adequacy for our specific task is evaluated. Next, the qualities of the second moment of the PSF are investigated. By the moment theorem, this parameter is related to the second derivative, at the origin, of the optical transfer function (OTF). It is
shown that the second moment is the sum of a contribution of the aperture amplitude distribution, which yields a sometimes infinite term, and a term which is directly related to the wavefront aberrations. This last term is identical to the value of the second moment of an image as calculated by geometrical ray-tracing. This establishes a link between a commonly used merit function in automatic lens design (Brixner, 1978; Robb, 1978) and its diffraction equivalent. Finally, a set of parameters suitable for simulation purposes are defined: the centroid, variance, eccentricity and angular orientation of the principal axes of the PSF distribution.

In Chapter 3 we carry out a theoretical investigation of the properties of the moments of sampled functions. It is shown that it is possible to accurately measure the second moment of the image with considerable undersampling. The measurement error is found by applying the moment theorem to the sample distributions. Sampling with small detector elements is compared to sampling with extended elements and it is shown that the latter is always more accurate. Error calculations for a Gaussian beam and for a square aperture are presented.

Computer simulation of the measurements of these parameters of atmospherically aberrated images is presented in Chapter 5. In the simulation, truncated Gaussian beams were used as well as uniformly illuminated circular apertures. Random wavefront distributions were generated to match the power spectral density of atmospheric turbulence and superimposed on the aperture. A fast Fourier transform routine was used to calculate the far-field distribution which was analyzed by a simulated
The main conclusion of the simulation is that the relative error in determining the second moment of the image distribution drops rapidly as the image aberrations increase, due to the increase in the effective sampling rate. As a rule, it is then sufficient to determine the sampling rate that gives a required accuracy for the unaberrated image, which is a well-defined function, and be sure that the measurement error will not exceed the accuracy requirement.

In Chapter 5 we present an analog method of determining the image parameters. It is based on scanning the image by a set of three knife-edge slits on a rotating wheel. The mathematical basis of the measurement algorithms is developed, and block diagrams of the processing electronics are presented. A demonstration unit has been built and tested.

There are two possible solutions, therefore, to the problem of fast measurement of the image parameters. The first is to reduce the amount of data processed by means of undersampling, and the second is to use analog methods which are inherently faster than digital computer processed, measurements.
CHAPTER 2

IMAGE QUALITY PARAMETERS

The need for a single parameter, or a small set of parameters to describe the quality of the image of a point object (the PSF) arises in various applications. In automatic lens design, a "merit function" is used to evaluate the optical quality of the lens and the design process aims at minimizing that function. Similarly, the concept of "sharpness function" is introduced in connection with adaptive optical systems which are designed to compensate and correct for the effects of atmospherically induced aberrations (Muller and Buffington, 1974). In both of these applications it is required that such a merit function will effectively converge and lead to an optimal system. However, in other applications the convergence is not necessary: all that is needed is a summary measure of the performance of the optical system. For example, in the system which initiated this study, it is necessary to measure the irradiance distribution of an incoming beam and simulate a similar distribution at a specific position on a simulated target.

In this chapter, commonly used parameters of irradiance distribution are described and their applicability in each of the above categories evaluated. Special attention is given to the specific requirements of adaptive optical systems: accurate measurability at high rates. The parameters considered include the Strehl ratio, the energy falling within a circle of specified radius, and two spot size measures (one of which
is given by the irradiance fall-off to some specific value and the other by the variance of the distribution). These parameters are actually special cases of more general sharpness functions, as discussed by Muller and Buffington (1974). We start our discussion by reviewing the concept of sharpness functions and then evaluate the applicability of each of the above parameters. The second part of this chapter deals with the properties of the second moment of the irradiance distribution of the image of a point object. The exact relationships between the moments of the irradiance distribution and the wavefront aberrations are derived. In addition, we suggest that the variance of the distribution in two orthogonal directions, the eccentricity and the orientation of the principal axes are a set of quality and descriptive parameters that can be measured at high rates and is useful both as a sharpness function and for the purpose of simulation.

**Image Sharpness Functions**

The concept of using a single parameter to describe the quality of an image or the quality of an optical system dates back to the nineteenth century, when the Rayleigh quarter wavelength criterion was introduced (Born and Wolf, 1975, p. 468). With the advent of automatic lens design the concept of merit functions has been further developed (Maréchal, 1947; Linfoot, 1959; Hopkins, 1966). Muller and Buffington (1974) introduced the concept of sharpness functions in relation to adaptive optical systems. In their work they defined several functions of the irradiance distribution $E(x,y)$ and analyzed the convergence properties of some of them, either analytically or by computer simulation. They
assumed isoplanatic optical systems with near-field, turbulence induced, aberrations.

The definitions and the results of their analysis are tabulated here (Table 1). When an analytical proof of convergence to a maximum exists, it is indicated so. It is also indicated whether or not the function resulted in satisfactory performance of a simulated adaptive optical system.

Most of the commonly used image quality parameters can be identified with the definitions of Table 1. The Strehl ratio criterion is derived from $S_2$ with $(x_0,y_0)$ being on the optical axis. $S_3$ gives the encircled energy criterion when the mask function $M(x,y)$ is a cylinder function centered at the origin. $S_6$, which is a special case of $S_3$, is the second moment of the irradiance distribution, from which the rms spot size criterion (another common name: radius of gyration) is derived. $S_7$ is a measure of the entropy of the image. Quantization of $E$ (by counting photons) gives meaning to $E!$.

The Strehl Ratio

The Strehl ratio (Born and Wolf, 1975, pp. 461-464) is defined as the ratio between the peak irradiance (at the diffraction focus) of the aberrated image and that of the diffraction-limited one. For small aberrations, it is related to the wavefront aberration variance, as described by Maréchal (1947):

$$\frac{E_{\text{max}}}{E_{0, \text{max}}} = 1 - \left(\frac{2\pi}{\lambda}\right)^2 \bar{w}^2$$
Table 1. Image Sharpness Functions.

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<th>Definition</th>
<th>Analytical Proof</th>
<th>Satisfactory Simulation</th>
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<tr>
<td>$S_1 = \iiint E^2 \mathrm{d}x \mathrm{d}y$</td>
<td>exists</td>
<td>yes</td>
</tr>
<tr>
<td>$S_2 = E(x_0, y_0)$</td>
<td>not proved</td>
<td>only for a bright source</td>
</tr>
<tr>
<td>$S_3 = \iiint ME \mathrm{d}x \mathrm{d}y$</td>
<td>only for $M=E_0=$diffraction limited image</td>
<td>satisfactory for $M=cy$ function</td>
</tr>
<tr>
<td>$S_4 = \iiint \left</td>
<td>a \frac{m+n}{m} E(x, y) \right</td>
<td>^2 \mathrm{d}x \mathrm{d}y$</td>
</tr>
<tr>
<td>$S_5 = \iiint E^n \mathrm{d}x \mathrm{d}y$</td>
<td>exists</td>
<td>yes for $n=2,3,4$</td>
</tr>
<tr>
<td>$n \geq 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_6 = -\iiint (x^2+y^2)E \mathrm{d}x \mathrm{d}y$</td>
<td>exists</td>
<td>yes, but poor convergence</td>
</tr>
<tr>
<td>$S_7 = -\iiint \ln(E!) \mathrm{d}x \mathrm{d}y$</td>
<td>exists</td>
<td>yes</td>
</tr>
<tr>
<td>$S_8 = -\iiint</td>
<td>E-E_0</td>
<td>^n \mathrm{d}x \mathrm{d}y$</td>
</tr>
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Note: $E!$ is meaningful when the irradiance is measured by a photon counter, in terms of photons per second.
where $\tilde{w}^2$ is the variance of the wavefront wave aberrations. The latter is a proven and commonly used optical quality criterion, and the Strehl ratio is used widely for the same purpose.

As a quality parameter, the Strehl ratio is of limited usefulness when large aberrations are considered. A Strehl ratio of 0.8 corresponds to a rms phase aberration of $\lambda/14$. This corresponds approximately to the Rayleigh quarter-wavelength criterion (Welford, 1974). However, as the aberrations increase, the above approximation becomes less accurate. While it is easy to calculate this parameter accurately from ray trace data on a computer, it is not as easy a task to perform when real-time measurements of actual images are concerned. The measured parameter is highly sensitive to radiometric errors and to radiance fluctuations of the source.

Encircled Energy

For systems which aim at concentrating maximum energy on a target area, this parameter is a relevant one. Consequently, it is widely used in evaluating adaptive optical systems (Wang, 1978). Although not proved analytically, its convergence as a sharpness function was proved by computer simulation when the radius of the circle was chosen as the radius of the Airy disk. The encircled energy parameter has no direct functional relationship to wavefront based quality criteria. Moreover, it is not uniquely defined since the diameter of the "bucket" and the centroid of the distribution might vary in an undetermined way.
Irradiance Falloff

Not as useful as a sharpness function, this parameter is sometimes used to describe the distribution. It is implicitly assumed that the peak irradiance is on the optical axis and further assumed that the irradiance is monotonically decreasing off-axis. This measure therefore is useful only when images of known distributions (i.e., Gaussian beams) and small aberrations are considered.

Second Moment of the Irradiance Distribution

The properties of the second moment of the point spread function (PSF) of optical systems as image quality criteria have been investigated for many years (Linfoot, 1955, Hopkins, 1966). The main interest was in the applicability of a criterion based on spot-size to lens design (Kreitzer, 1976; Brixner, 1978). Also, it is one of the sharpness functions investigated by Muller and Buffington (1974) for active optics applications.

There is a distinct difference between a quality criterion for the purpose of lens design and one used in adaptive optical systems. While lens design programs trace a finite number of rays to generate a geometrical PSF, active optics systems, whether real or simulated, deal with a diffraction PSF. Linfoot (1959) pointed out the relationship between geometrical PSF, diffraction PSF, and the corresponding optical transfer functions (OTF); this analysis was based on a second order expansion of the OTF around the origin. Linfoot derived the already known relationship between the second moment of the PSF and the second
derivative of the OTF at the origin. He pointed out the divergence of the second moment of the diffraction PSF of sharp edged apertures, but concluded that the increase in the central second moment of the diffraction PSF, caused by aberrations, is equal to the increase in the corresponding parameter of the geometrical PSF.

We derive here exact relationships among the second moment of the diffraction PSF, the optical transfer function, and the wavefront aberrations, for an aperture of both phase and amplitude variations. The use of Fourier transform relationships leads to simple yet rigorous calculations. The results are comparable to those obtained by Linfoot but are more general. In particular, the equivalency of image quality criteria that are based on the second moments of diffraction and geometrical point spread functions is demonstrated.

Consider the image of a point object with an irradiance distribution \( E(x,y) \). Assuming that the total flux in the image is normalized to unity, the second moment of the distribution is given by

\[
m_2 = \iint (x^2 + y^2)E(x,y)\,dx\,dy. \tag{2.1}
\]

The two-dimensional problem can be reduced to one dimension by separating the integral into two parts. We define one part as

\[
m_{x,2} = \iint x^2E(x,y)\,dx\,dy = \int x^2\ell(x)\,dx \tag{2.2}
\]

where \( \ell(x) = \int E(x,y)\,dy \) is the line spread function (LSF) along the x axis. The second moment is given by \( m_2 = m_{x,2} + m_{y,2} \) and is independent of the orientation of the coordinate system. The OTF is given by the autocorrelation of the pupil function.
\[ \text{OTF}(\xi, \eta) = \gamma[P(\xi, \eta)] \quad (2.3) \]

where \( \xi, \eta \) are normalized spatial frequencies (Hopkins, 1966) and \( \gamma \) is the autocorrelation operator. When properly normalized the OTF is equal to the Fourier transform of the PSF. The moment theorem (Gaskill, 1978), as applied to Fourier transform pairs, states that

\[ m_k(f) = \left( \frac{1}{j2\pi} \right)^k \frac{d^k F(\xi)}{d\xi^k} \bigg| \xi = 0 \]

where \( m_k(f) \) is the \( k \text{th} \) moment of the function \( f(x) \) and \( F(\xi) \) is the Fourier transform of \( f(x) \). Thus, the second moment of the LSF in the \( x \) direction is given by

\[ m_{x,2} = - \frac{1}{4\pi^2} \frac{\partial^2}{\partial \xi^2} [\text{OTF}(0,0)] \quad (2.4) \]

and similarly, the partial derivative with respect to \( \eta \) at the origin gives an expression for \( m_{y,2} \).

A general pupil function can be written as

\[ P(\xi, \eta) = A(\xi, \eta) \exp[jkw(\xi, \eta)] \quad (2.5) \]

where \( A(\xi, \eta) \) is a real aperture function and \( w(\xi, \eta) \) is the wavefront aberration function. Using a property common to convolutions, namely that \( (f*g)' = f'*g' \), we can express the second derivative of the OTF as the autocorrelation of the first derivative of the pupil function. (The derivatives, indicated by primes, are taken only with respect to the first variable. Convolutions are denoted by asterisks.)

\[ \text{OTF}''(\xi, \eta) = \gamma[P'(\xi, \eta)] \quad (2.6) \]
Since
\[ P'(\xi, \eta) = A'(\xi, \eta)e^{jkw(\xi, \eta)} + jkw'(\xi, \eta)P(\xi, \eta) \]  
we have
\[ OTF''(\xi, \eta) = -\gamma[A'(\xi, \eta)e^{jkw(\xi, \eta)}] - k^2\gamma[w'(\xi, \eta)P(\xi, \eta)] \\
- jk[w'(\xi, \eta)P(\xi, \eta) \ast A'(\xi, \eta)e^{-jkw(\xi, \eta)}] \\
- A'(\xi, \eta)e^{jkw(\xi, \eta)} \ast w'(\xi, \eta)P(\xi, \eta)] . \tag{2.8} \]

The last term in Eq. (2.8) vanishes at the origin, where each cross-correlation term is equal to \( \int A(\xi, \eta)A'(\xi, \eta)w'(\xi, \eta)d\xi d\eta \) and the expression for the second moment takes the form
\[ m_{x_2} = \frac{1}{4\pi^2} \left\{ \int[A'(\xi, \eta)]^2d\xi d\eta + k^2\int[A(\xi, \eta)w'(\xi, \eta)]^2d\xi d\eta \right\} . \tag{2.9} \]

Equation (2.9) separates the effects of the aperture amplitude distribution and wavefront aberrations. It is more general than the limit obtained by Linfoot. The first term in Eq. (2.9) gives rise to the divergence of \( m_2 \) whenever the aperture has sharp boundaries. It is most clearly demonstrated for a square aperture. Let \( A(\xi, \eta) = \text{rect} \left( \frac{\xi}{2a} \right) \). Then \( A'(\xi, \eta) = [\delta(\xi+a) - \delta(\xi-a)]\text{rect} \left( \frac{\xi}{2a} \right) \) and \( \int[A'(\xi, \eta)]^2d\xi d\eta = 2a \int[\delta(\xi+a) - \delta(\xi-a)]^2d\xi \). This integral can be evaluated as follows:
\[ \int[\delta(\xi+a) - \delta(\xi-a)]^2d\xi = \int[\delta(\xi+a) - \delta(\xi-a)][\delta(\xi+a) - \delta(\xi-a)]d\xi \\
= [\delta(\xi+a) - \delta(\xi-a)] \bigg|_{\xi=-a} - [\delta(\xi+a) - \delta(\xi-a)] \bigg|_{\xi=a} \\
= \delta(\xi+a) \bigg|_{\xi=-a} + \delta(\xi-a) \bigg|_{\xi=a} = \infty, \ a \neq 0. \]
The second term in Eq. (2.9) is the contribution of the wavefront aberrations to \( m_2 \) and is identical to the expression obtained in the geometrical PSF limit (Linfoot, 1959).
The relationships developed so far make implicit use of the assumption that Fourier transform relationships (as expressed by the Fraunhofer integral) between the pupil and image field distributions hold over the entire image plane which, in theory, is infinite in size. However, the standard Rayleigh-Sommerfeld scalar theory of diffraction is valid only near the optical axis. Thus, in a strict sense, the OTF as calculated from the pupil function autocorrelation is not correct. This is especially manifested near the origin where the first derivative of the OTF is discontinuous. This discontinuity causes the divergence of the second moment of sharp-boundary apertures. The invalidity of the diffraction theory at large diffraction angles does not render our results useless. In actual applications the measured diffraction PSF is always truncated. The OTF, as calculated directly from the PSF, is therefore convolved with a smoothing function that eliminates the discontinuity at the origin. Thus, the first term in Eq. (2.9) is bounded and constant. The effect of truncation is most easily demonstrated for square aperture and image plane, since the integrals are separable. The first derivative of such an aperture is a pair of \( \delta \)-functions. If the PSF is truncated by a square of width \( D \), the smoothing function is \( D^2 \text{sinc}(D\xi, D\eta) \), and the measured second moment of the truncated and unaberrated PSF takes the form

\[
m_2 = \frac{D}{2} \int_{-D/2}^{D/2} \int_{-D/2}^{D/2} \text{sinc}^2\left(\frac{\xi}{w}, \frac{\eta}{w}\right) d\xi d\eta
\]

\[
= \frac{Dw}{\pi^2} \left[1 - \text{sinc} \left(\frac{D}{w}\right)\right]
\]  

(2.10)
where \( w \) is the first zero of the PSF. For a Gaussian amplitude distribution with a 1/e width of \( 2w \) and total flux normalized to unity, the first term in Eq. (2.9) is

\[
m_{x,2} = \frac{1}{w^2}.
\]

(2.11)

Here no truncation in the pupil plane or in the image plane is assumed. When such truncation is taken into account, a term that is proportional to the truncation width in both planes, [as in Eq. (2.10)] is added to the expression for \( m_{x,2} \). This term, however, is weighted by the wavefront amplitude at the aperture boundary and normally is negligible.

The properties of the rms spot-size as a merit function in automatic lens design were studied by Kreitzer (1976). The spot size can be expressed as a quadratic form in the wave aberration coefficients as calculated by the second term in Eq. (2.9). The resulting expression can be orthonormalized so that the cross-terms between the aberration coefficients are eliminated, and the expression takes the form of a sum of squared terms each of which is a linear expression in the aberration coefficients. Minimization of such an expression is obtained by having each linear term equal to zero, yielding an optimal balancing of the aberrations. A similar treatment of the expression of the wave aberration variance yields another set of optimally balanced coefficients.

Kreitzer (1976) compared the OTF of aberrated systems which were optimized according to the above procedures. He found that at low spatial frequencies (high OTF values) the spot-size criterion gives better results, but at high spatial frequencies, when the OTF has values
of about 0.6 or less, the wavefront variance criteria gives better results, even with a high level of aberrations.

The rms spot-size criterion finds practical applications in existing lens optimization programs. Its efficiency has been demonstrated in many cases, both for convergence to an optimal solution (Brixner, 1978) and for speed of convergence (Robb, 1978). In view of our results, which show the equivalency of the second moment of diffraction based PSF and the geometrical PSF, it is reasonable to assume that an rms spot-size quality criterion can be successfully applied to active optical systems.

Unlike the other image parameters which were reviewed here, the second moment is a normalized parameter, and not an absolute measure of irradiance; consequently it is much less sensitive to radiometric errors or to fluctuations in the source output.

A major drawback in applying this parameter to active optical systems has been the time-consuming data gathering and processing which is involved in measuring it. This work shows how this problem can be overcome.

**Image Parameters for Shape Simulation**

The adaptive optics test system, described in the introduction, tries to simulate the effects of illuminating a target with a laser beam. First, the incoming laser radiation is analyzed in the image plane. Then, the irradiance distribution is attempted to be duplicated on the simulated target by means of a focused laser beam which writes the desired distribution on it. The problem of analyzing the image for the purpose of simulating its shape poses the same tradeoff between accuracy
and speed of measurement as encountered with measuring the quality parameters.

A detailed description of the image of a point source (i.e., the point spread function) for the purpose of shape simulation involves a large amount of data, high processing rates and is not easy to interpret. Furthermore, it might be too detailed, since the resolution of the system which detects the simulated image is usually not better than that of the illuminating system which produces the image; this applies to the aforementioned simulator. On the other hand, a single descriptive parameter does not give enough spatial information for adequate simulation. The Strehl ratio is effective only for an almost perfect image. The encircled energy and irradiance fall-off are not general enough to include different functional forms of the image distribution. The second moment is better than the other parameters, but is limited in the sense that radial symmetry has to be assumed, as with any other single parameter description.

We suggest here a set of parameters that give a second order description of the shape of the PSF and are based on the first and second moments of the distribution. These parameters are: variance in two orthogonal directions, covariance, the angle of the principal axes and the eccentricity, in addition to the image centroid. These parameters have two-fold symmetry and can adequately detect and describe astigmatic images. Some active optical systems (Buffington et al., 1977) use a two-axis orthogonal correction scheme. The suggested set of parameters give independent information along any two orthogonal axes so it is possible to use independent correction algorithms for these axes. To calculate
these parameters we treat here the image of a point source which has an irradiance distribution \( E(x,y) \).

Assume a distribution \( E(x,y) \) in an arbitrary coordinate frame that is normalized such that \( \iint E(x,y) \, dx \, dy = 1 \). The variance in each direction is defined as:

\[
\sigma_x^2 = \iint E(x,y) \, x^2 \, dx \, dy - \bar{x}^2 \\
\sigma_y^2 = \iint E(x,y) \, y^2 \, dx \, dy - \bar{y}^2 .
\]  

(2.12)

The spot-size radius \( \rho \) can be defined as

\[
\rho = \sqrt{\sigma_x^2 + \sigma_y^2} .
\]  

(2.13)

Note that \( \rho \) is independent of the choice of the orientation of the coordinate system. The covariance is defined as:

\[
\sigma_{xy} = \iint E(x,y) \, xy \, dx \, dy - \bar{x}\bar{y} .
\]  

(2.14)

We can find a coordinate system \((x',y')\) in which \( \sigma_{xy} \) is equal to zero. These axes are the principal axes of the distribution. If \( \sigma_{x'} > \sigma_{y'} \), the eccentricity \( \varepsilon \) is given by

\[
\varepsilon = \sqrt{\frac{\sigma_{x'}^2 - \sigma_{y'}^2}{\sigma_x^2}}
\]  

(2.15)

Figure 2.1 defines the geometry of the principal axes. The coordinates are transformed by:

\[
x' = x \cos \theta + y \sin \theta \\
y' = -x \sin \theta + y \cos \theta .
\]
Fig. 2.1. Geometry of Principal Axes.
The covariance $\sigma_{x'y'}$ can be calculated by multiplying the irradiance $E(x,y)$ by the distance to the $x'$ axis, the distance to the $y'$ axis, and integrating: (for ease of calculation we assume that the centroid is at the origin. This assumption does not change the end results.) Thus,

$$\sigma_{x'y'} = \iint E(x,y)x'y' \, dx \, dy.$$ 

Since the principal axes are defined by $\sigma_{x'y'} = 0$, we have:

$$\iint E(x,y)(x \cos \theta + y \sin \theta)(-x \sin \theta + y \cos \theta) \, dx \, dy = 0$$

or:

$$\frac{1}{2} \sin 2 \theta \iint E(x,y)(y^2-x^2) \, dx \, dy + \cos 2 \theta \iint E(x,y) \, xy \, dx \, dy = 0. \quad (2.16)$$

The angle of the principal axes is given by:

$$\tan 2 \theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2}. \quad (2.17)$$

Let $\alpha = \sigma_x^2 - \sigma_y^2$ and $\beta = 2\sigma_{xy}$, then

$$\cos 2 \theta = \pm \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \quad (2.18a)$$

$$\sin 2 \theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}. \quad (2.18b)$$

We can assume that $\frac{\pi}{2} \geq \theta \geq 0$, and the sign of $\cos 2 \theta$ can be determined accordingly.

To find the eccentricity we have from Eqs. (2.15) and (2.18):

$$\varepsilon^2 = \frac{\sigma_{x'-y'}^2 - \sigma_{y'}^2}{\sigma_{x'}^2}. \quad (2.19)$$
Further: 
\[ \sigma_x'^2 - \sigma_y'^2 = \iint E(x,y)(x'^2 - y'^2) \, dx \, dy \]

\[ = \cos \theta \iint E(x,y)(x^2 - y^2) \, dx \, dy + 2 \sin \theta \]

\[ \cdot \iint E(x,y)xy \, dx \, dy \]

\[ = a \cos \theta + \beta \sin \theta \]

\[ = \sqrt{\beta^2 + \alpha^2} \quad (2.19) \]

Again, the sign is determined by defining \( \frac{\pi}{2} \geq \theta \geq 0 \).

Let us further assume that \( \frac{\pi}{2} \geq 2\theta \geq 0 \). Then \( \tan 2\theta \geq 0 \) and

\[ \sigma_x'^2 - \sigma_y'^2 = \sqrt{\beta^2 + \alpha^2} \]. Then we have from Eqs. (2.13), (2.15) and (2.19):

\[ \varepsilon^2 = \frac{2\sqrt{\beta^2 + \alpha^2}}{\rho^2 + \sqrt{\beta^2 + \alpha^2}} \quad (2.20) \]

when \( \sigma_x'^2 < \sigma_y'^2 \), \( a \cos 2\theta + \beta \sin 2\theta < 0 \) and then

\[ \varepsilon^2 = \frac{2\sqrt{\beta^2 + \alpha^2}}{\rho^2 - \sqrt{\beta^2 + \alpha^2}} \quad (2.21) \]

if \( 2\theta > \frac{\pi}{2} \), \( \alpha^2 \) appears with a minus sign in those equations.

In the following chapters an investigation of the qualities of these parameters is presented. In particular, measurement errors due to sampling are calculated, and an analog method of measuring the parameters is developed.
CHAPTER 3

DIGITAL MEASUREMENT OF THE IMAGE PARAMETERS

The set of image parameters which was defined here can be calculated digitally in a straightforward way. However, problems arise when the measurement rate increases, due to the time-consuming calculations which are needed to perform the numerical integrations. One obvious way to increase the speed of measurement is by decreasing the number of samples taken. This decreases the accuracy of the measurement and thus it is necessary to determine the largest sampling interval which still gives the required accuracy.

Insight into the problem of optimizing the sampling interval can be gained through the moment theorem (Gaskill, 1978), which states that the moments of a function are given by the Fourier transform and its derivatives at the origin. It suggests that even with considerable undersampling (i.e., sampling at frequencies lower than the Nyquist frequency) the moments of the function can be recovered. In this chapter we test this assumption for two types of sampling schemes. The first involves δ-type sampling, i.e., the sampling element is much smaller than the sampling interval, and the second uses sampling elements of the same size as the sampling interval.

The Moments of Sampled Functions

The Fourier transform of a sampled function is given by an infinite sum of shifted versions of the transform of the original function.
When the sampling frequency is less than the Nyquist frequency there exists aliasing, i.e., the shifted transforms overlap and distort the original transform. While aliasing might distort the measured distribution considerably, its effect on the moments of the distribution should be small, since the moments depend on the value of the transform and its derivatives at the origin. Figure 3.1 shows the Fourier transform of a Gaussian distribution as compared to the transform of a sampled version of this function. The sampling interval used is \(1.8\sigma\), \(\sigma^2\) being the variance. Note that the shape of the transform is not changed at the origin, while it is totally distorted away from it.

To illustrate the use of the moment theorem in analyzing the sampling errors we treat here one-dimensional distributions which are even about the centroid. Let \(f(x)\) be an even function with a Fourier transform \(F(\xi)\), and \(g(x)\) is a shifted version of \(f(x)\), i.e., \(g(x)=f(x-x_0)\), with a transform \(G(\xi)=e^{-j2\pi x_0\xi}F(\xi)\). The first and second moments of \(g(x)\) can be derived from \(G(\xi)\) and its derivatives using the moment theorem:

\[
m_0 = G(0) \quad (3.1)
\]
\[
m_1 = -G^{(1)}(0)/j2\pi G(0) \quad (3.2)
\]
\[
m_2 = -G^{(2)}(0)/4\pi^2 G(0) \quad (3.3)
\]

The first moment gives the centroid \(\bar{x}\) and the one-dimensional spot size is

\[
\rho = (m_2 - m_1^2)^{1/2}.
\]

When sampled by an infinite \(\delta\)-type array, the sampled function is given by \(g_s(x)=x_s \sum_{n=\text{-}\infty}^{\infty} g(nx_s)\delta(x-nx_s)\) where \(x_s\) is the sampling interval. The sampled function, as defined here, includes the factor \(x_s\), so that the
Fig. 3.1. Fourier Transform of a Gaussian.

Dashed line: transform of original function. Solid line: transform of sampled function. Sampling interval is 1.8\omega.
first moment of the sampled function approximates the first moment of the
original function, i.e., \( \int_{-\infty}^{\infty} g_S(x) = \sum_{n=-\infty}^{\infty} x g(nx_S) \). The moments of \( g_S(x) \)
in terms of the function \( F(\xi) \) and its derivatives are:

\[
\hat{m}_0 = F(0) + 2 \sum_{n=1}^{\infty} F(n\xi_S) \cos(2\pi n\xi_S x_o) \quad (3.4)
\]

\[
\hat{m}_1 = x_o + \sum_{n=1}^{\infty} \frac{F^{(1)}(n\xi_S)}{n} \sin(2\pi n\xi_S x_o)/\pi \hat{m}_0 \quad (3.5)
\]

\[
\hat{m}_2 = \rho^2 + x_o^2 + 2x_o \sum_{n=1}^{\infty} \frac{F^{(1)}(n\xi_S)}{n} \sin(2\pi n\xi_S x_o)/\pi \hat{m}_0 - \sum_{n=1}^{\infty} \frac{F^{(2)}(n\xi_S)}{2\pi^2 \hat{m}_0} \cos(2\pi n\xi_S x_o). \quad (3.6)
\]

In deriving the above expressions we took advantage of the symmetry prop­
erties of the function \( F(\xi) \) and its derivatives. The sampling frequency
\( \frac{1}{x_S} \) is designated by \( \xi_S \). Note that by calculating the moments in the
Fourier domain, one obtains general expressions for the sampling errors,
as infinite sums, and can immediately recognize some qualities. For
example, the error in determining the centroid is always zero for func­
tions centered on the origin, as expected from the symmetry properties.
For such functions the error in determining the second moment is only
the last term in Eq. (3.6). The periodicity of the error terms as a
function of the centroid position is also evident as the errors are given
by a Fourier series in \( x_o \) with the period \( \xi_S \). The details of this deriva­
tion are given in Appendix A.

Assume now that the sampling array has an element size \( \alpha \). The
value of the function at the sample point \( nx_S \) is given by the local
average of the function over the interval \( (nx_S - \frac{\alpha}{2}, nx_S + \frac{\alpha}{2}) \). The
sampled function is given by the convolution of \( \text{rect} \left( \frac{x}{a} \right) \) and a \( \delta \)-type sampling function:

\[
g_s(x) = \frac{x}{a} \sum_{n=-\infty}^{\infty} g(x) \ast \text{rect} \left( \frac{x}{a} \delta(x-nx_s) \right)
\]

and its Fourier transform is

\[
G_s(\xi) = \sum_{n=-\infty}^{\infty} G(\xi-n\xi_s) \text{sinc}[\alpha(\xi-n\xi_s)] .
\]

The values of the transform and its derivatives at the origin are:

\[
G_s(0) = \sum_{n=-\infty}^{\infty} G(-n\xi_s) \text{sinc}(n\alpha\xi_s)
\]

\[
G_s^{(1)}(0) = \sum_{n=-\infty}^{\infty} G^{(1)}(-n\xi_s) \text{sinc}(n\alpha\xi_s) - G(-n\xi_s) [\cos(n\pi\alpha\xi_s)
\]

\[
- \text{sinc}(n\alpha\xi_s)]/n\xi_s
\]

\[
G_s^{(2)}(0) = \sum_{n=-\infty}^{\infty} G^{(2)}(-n\xi_s) \text{sinc}(n\alpha\xi_s) - 2G^{(1)}(-n\xi_s)
\]

\[
[\cos(n\pi\alpha\xi_s) - \text{sinc}(n\alpha\xi_s)]/n\xi_s + G(-n\xi_s)
\]

\[
\{2[\text{sinc}(n\alpha\xi_s) - \cos(n\pi\alpha\xi_s)]/n^2\xi_s^2 - \pi^2\alpha^2\text{sinc}(n\alpha\xi_s)\}
\]

(These equations are derived in Appendix A.) The above expressions can be simplified by assuming that \( \alpha\xi_s = 1 \). This corresponds to sampling with element size equal to the sampling interval. In this case the sinc function vanishes for all \( n \neq 0 \) and \( \cos(n\pi\alpha\xi_s) = (-1)^n \). As before, we use the symmetry properties of \( F(\xi) \) and its derivatives to find the moments of the sampled function:

\[
\hat{m}_0 = F(0)
\]

\[
\hat{m}_1 = x_o + \sum_{n=1}^{\infty} (-1)^n F(n\xi_s) \sin(2\pi n\xi_s x_o)/\pi n\xi_s \hat{m}_0
\]
\[ \hat{m}_2 = \rho^2 + x_o^2 + a^2/12 + 2x_o \sum_{n=1}^{\infty} (-1)^n F(n\xi_s) \sin(2\pi n\xi_s x_o) / \pi n \xi_s \hat{\rho} \]

\[ - \sum_{n=1}^{\infty} (-1)^n F^{(1)}(n\xi_s) \cos(2\pi n\xi_s x_o) / \pi^2 n^2 \xi_s \hat{m}_0 + \sum_{n=1}^{\infty} (-1)^n F(n\xi_s) \cos(2\pi n\xi_s x_o) / \pi^2 n^2 \xi_s^2 \hat{m}_0 . \] 

While the nature of the infinite sums in Eq. (3.13) and (3.14) are not immediately transparent for general functions \( F(\xi) \), it can be shown that they converge faster than the error sums in Eq. (3.5) and (3.6). In fact, numerical calculations show that the dominant error term in Eq. (3.14) is the constant term \( a^2/12 \). This term is the normalized variance of the function \( rect(\xi/a) \) which is used in the convolution (3.7). It can be shown by direct integration of the convolution of two functions that this result is general, i.e., the variance of the convolution equals the sum of the variances, if either of the two functions has a zero first moment (Bracewell, 1965).

**Application to a Gaussian LSF**

We examine now a Gaussian profile, centered at \( x_o \) with variance \( \sigma^2 \). The distribution is normalized such that the total flux is unity (i.e., \( m_0 = 1 \)). Thus the function takes the form

\[ g(x) = \exp\left[-(x-x_o)^2/2\sigma^2\right]/(2\pi\sigma^2)^{1/2} . \] 

(3.15)

Using Eqs. (3.4) to (3.6) we get the following expressions for the moments of the \( \delta \)-type sampled function:

\[ \hat{m}_0 = 1 + 2 \sum_{n=1}^{\infty} \exp(-2\pi^2 n^2 \sigma^2 \xi_s) \cos(2\pi n \xi_s x_o) \] 

(3.16)
\[ \hat{m}_1 = x_o - 4\pi \xi_s \sigma^2 \sum_{n=1}^{\infty} n \exp(-2\pi^2 n^2 \sigma^2 \xi_s^2) \sin(2\pi n \xi_s x_o)/\hat{m}_o \]  
\(3.17\)

\[ \hat{m}_2 = \sigma^2 + x_o^2 - 8\pi \xi_s \sigma^2 x_o \sum_{n=1}^{\infty} n \exp(-2\pi^2 n^2 \sigma^2 \xi_s^2) \sin(2\pi n \xi_s x_o)/\hat{m}_o \]
\[-8\pi^2 \sigma^4 \xi_s^2 \sum_{n=1}^{\infty} n^2 \exp(-2\pi^2 n^2 \sigma^2 \xi_s^2) \cos(2\pi n \xi_s x_o)/\hat{m}_o. \]  
\(3.18\)

We can rewrite Eq. (3.18) in the following way:

\[ \hat{\hat{m}}_2 = \sigma^2 + x_o^2 + 2x_o \xi_1 + \xi_2, \]  
\(3.19\)

where \(\xi_1\) is the centroid error and \(\xi_2\) is the variance error for \(x_o = 0\).

The estimated spot width is

\[ \rho^2 = \hat{\hat{m}}_2 - \hat{m}_1^2 \]  
\(3.20\)

or

\[ \rho^2 = \sigma^2 + \xi_2 - \xi_1^2. \]  
\(3.21\)

We use Eqs. (3.12) through (3.14) to find the same parameters for the case of extended sampling element. The zero moment is simply the total measured flux which is normalized to unity. Subtrating the constant term in Eq. (3.14) we get:

\[ \hat{\hat{m}}_1 = x_o + \sum_{n=1}^{\infty} (-1)^n \exp(-2\pi^2 \sigma^2 n^2 \xi_s^2) \sin(2\pi n \xi_s x_o)/\pi n \xi_s \]  
\(3.22\)

\[ \hat{\hat{m}}_2 = \sigma^2 + x_o^2 + 2x_o \sum_{n=1}^{\infty} (-1)^n \exp(-2\pi^2 \sigma^2 n^2 \xi_s^2) \sin(2\pi n \xi_s x_o)/\pi n \xi_s \]
\[+ 4\sigma^2 \sum_{n=1}^{\infty} (-1)^n \exp(-2\pi^2 \sigma^2 n^2 \xi_s^2) \cos(2\pi n \xi_s x_o) + \sum_{n=1}^{\infty} (-1)^n \]
\[\exp(-2\pi^2 \sigma^2 n^2 \xi_s^2) \cos(2\pi n \xi_s x_o)/\pi^2 n^2 \xi_s^2 \]  
\(3.23\)

Again, we can express the estimated spot size as in Eq. (3.21).
We plot here (Fig. 3.2) the spot size error as a function of sampling frequency for the two cases. The sampling frequency is expressed in units of $\sigma^{-1}$. The strong dependence on $\xi_0$ is evident at low sampling frequencies ($\xi_s = 1/2\sigma$). Since the centroid position may vary at random, the worst case spot size error should be considered when the sampling methods are compared. The extended element sampling gives better results: a smaller error and less sensitivity to centroid position. In order to have a spot size error of less than 5% we have to use a sampling interval which is smaller than $1.9\sigma$ ($\xi_s = .53$) when $\delta$-type sampling is used. When extended element sampling is used, the sampling interval can be as large as $2.3\sigma$, and still be under 5% error. In this case the error can be kept under 1% by sampling at a rate of $1.9\sigma$. Of course, correction has to be made for the constant contribution of the sampling interval variance.

Figure 3.3 demonstrates the periodic behavior of the sampling error. Both the amplitude and the period of the error fluctuations are dependent on $\xi_s$, as evident from the plot.

**Application to a $\text{sinc}^2(x)$ Distribution**

This is an example of a distribution that does not have a second moment since the moment integral does not converge. In practice, however, the function is defined and sampled over a finite field, and a second moment of the truncated function exists. The truncated function is given by

$$g(x) = \text{sinc}^2\left(\frac{x-x_0}{w}\right) \text{rect}\left(\frac{x}{D}\right).$$

(3.24)
Fig. 3.2. Spot Size Error ($\Delta \rho / \rho$) as a Function of Sampling Frequency for the Function $\exp[-(x-x_0)^2/2\sigma^2]/(2\pi\sigma^2)^{1/2}$.

(a) $\delta$-type sampling; (b) extended sampling element. The curves are given for various positions of the centroid $x_0$. 
Fig. 3.3. Relative Spot Size Error for Various Sampling Frequencies \( \xi \) as a Function of Centroid Location.
This function has zeroes at $x_0 - w$ and $x_0 + w$. By direct integration we find the variance:

$$
\sigma^2 = \frac{wD}{2\pi^2} \left[ 1 - \text{sinc}\left(\frac{D}{w}\right) \right].
\tag{3.25}
$$

Since the variance is a function of the field width $D$ it is necessary to define another suitable measure of the spot size. A common measure is the separation between the first zeroes, and assuming that $D \gg w$ we can find the spot size from the variance:

$$
2w = 4\pi^2 \sigma^2 / D.
\tag{3.26}
$$

The Fourier transform $F(\xi)$ which is used in Eqs. (3.4) - (3.6) and (3.12) - (3.14) is given by the convolution:

$$
F(\xi) = wD\text{tri}(w\xi)\ast\text{sinc}(D\xi).
\tag{3.27}
$$

The transform is not bounded and involves integration. It is easier in this case to calculate the error in determining the spot size from the sampled function itself. The results are given in Fig. 3.4a for two positions of the centroid. The strong dependence on the centroid location is evident. The sampling frequency that gives an error of less than 5% for all $x_0$ positions is quite large, about $2.7/w$, in comparison to the Nyquist frequency of the nontruncated function, which is $2/w$.

Consider now sampling with an extended element. Here it is simpler to use Eq. (3.14) to find the sampling error, since the sums converge rapidly. We define the integral:

$$
I(a,b) = \int_a^b \text{sinc}(Du)du.
$$
Fig. 3.4. Spot Size Error as a Function of Sampling Frequency for a \( \text{sinc}^2(x) \) Distribution.

(a) \( \delta \)-type sampling; (b) extended element sampling.
The elements of the sums in Eq. (3.14) can be written in terms of this integral:

\[ F(n_x) = D[I(n_x - 1, n_x + 1) - n_x[I(n_x - 1, n_x + 1)] - I(n_x, n_x + 1)] + \frac{2}{\pi^2 D} \cos(n \pi D \xi_s)[\cos(\pi D) - 1] \]  

(3.28)

\[ F'(n_x) = -D[I(n_x - 1, n_x + 1) - I(n_x, n_x + 1)]. \]  

(3.29)

The sampling error was calculated for \( D = 10w \), and is plotted in Fig. (3.4b). The superiority of extended element sampling is clearly evident. The error diminishes rapidly when the sampling frequency exceeds \( 1/w \), half the Nyquist frequency. The \( 5\% \) error point is at \( \xi_s = 1.04/w \). (An element size of little less than half the spot size guarantees a measuring error of less than \( 5\% \).)
CHAPTER 4

COMPUTER SIMULATION OF ATMOSPHERICALLY ABERRATED IMAGES

In order to test the theoretical results of Chapter 3, a computer simulation of the measurement of the quality parameters of atmospherically aberrated images was carried out. Two computer programs were developed; the first generates a random irradiance distribution with the statistics of atmospherically aberrated images and the second program calculates the parameters with variable sampling intervals.

In this chapter we discuss the method of generating aberrated wavefronts and present the simulation results.

Generating the Wavefronts

The statistical properties of atmospherically aberrated wavefronts has been investigated by many authors. The space is too short to thoroughly review the theory of atmospheric propagation and only the main results are stated here. A recent review, which includes many references appears in the Infrared Handbook (Hufnagel, 1978).

The function which is most widely used to describe the phase distribution of the aberrated wavefront is the "wave structure function" $D_w(r)$. The structure function is related to the variance of the phase fluctuation $\sigma_\phi^2$ and the phase autocorrelation $C_\phi(r)$ by:

$$D_w(r) = 2[\sigma_\phi^2 - C_\phi(r)] , \quad (4.1)$$

where isotropy of the turbulence is assumed.
Kolmogorov (1941) developed a mathematical model for turbulence which predicts, under simplifying conditions of isotropy, that the temperature fluctuations of the atmosphere obey a two-thirds power law in r. He showed that the structure function of the temperature is:

$$D_T(r) = C_T^2 r^{2/3}.$$  \hfill (4.2)

This equation is valid over the "inertial subrange", $\ell_o << r << L_o$, where $\ell_o$ and $L_o$ are called, respectively, the inner scale and outer scale of turbulence. It is assumed that within the inertial subrange the kinetic energy associated with large eddies is redistributed without loss to successively smaller and smaller eddies until finally dissipated by viscosity. Estimates of $\ell_o$ and $L_o$ range from less than one centimeter for $\ell_o$ near the ground to values for $L_o$ of approximately half the distance to the ground when this distance exceeds a few meters.

Since the changes in the index of refraction of the atmosphere are caused almost solely by temperature variations, the wave structure function can be directly related to the temperature structure function. It can be shown (Fried, 1966) that the wave structure function obeys a five-thirds power law in r:

$$D_w(r) = 6.88(r/r_o)^{5/3}$$ \hfill (4.3)

where $r_o$ is a characteristic length which depends on the wavelength, the distance of propagation and on $C_T^2$.

Finally, it is necessary to find the probability law of the random phase fluctuations. By the Law of Large Numbers it is shown that the phase assumes a Gaussian distribution. Thus, specifying the variance and
the correlation function is sufficient to characterize the joint probability function of the phase fluctuations.

The phase autocorrelation function is determined directly from the wave structure function [Eqs. (4.1), (4.3)]. Its Fourier transform is the power spectral density (Wiener spectrum) of the random phase. When the proper (three-dimensional) Fourier transform is carried out (Frieden, 1978) one gets the so-called "Kolmogorov spectrum" which obeys an eleven-thirds power law:

$$\phi_n(\rho;s) = 8.16 C_n^2(s) \rho^{-11/3}. \quad (4.4)$$

Here $\phi_n$ is the PSD of the index variations as a function of altitude ($s$) and spatial frequency ($\rho$). $C_n^2$ is the refraction index structure parameter. It is clear that the Kolmogorov spectrum depends very much on $C_n^2$, and indeed, it is the single most important parameter in describing atmospheric turbulence.

In our simulation we assumed that $C_n^2$ is constant within the inertial sub range ($\ell_o < r < L_o$). We further assumed that the PSD has a value of 1 for $\rho < 1/L_o$ and 0 for $\rho > 1/L_o$. These simplifying assumptions yield a truncated PSD which approximates the more complex "true" PSD to a level which is sufficient for our simulation purposes.

The mathematical basis for the simulation algorithm is developed in Appendix B. It is shown that if a random function which is $\delta$-correlated (has an infinitely narrow autocorrelation function) is convolved with another function $S$, the resulting function has an autocorrelation which is proportional to the autocorrelation of $S$. Thus, in the Fourier domain, the square root of the PSD can be multiplied by a
δ-correlated function and transformed back to give a realization of a random phase distribution with the required correlation. Once the random wavefront was generated it was multiplied by an amplitude function which was either a circular pupil of varying diameter or a Gaussian of varying variance. Further options of the program included addition of third order aberrations, defocus, and the scaling of the wavefront aberrations to a specific variance. The final step in generating the random images was Fourier transforming the wavefront and squaring to get the far-field irradiance distribution.

Both the wavefront and the image were given in terms of a 64x64 matrix. A schematic description of the computer program is given in Fig. 4.1.

Some images which were produced by this program are presented here (Fig. 4.2a-4.2d). Both a 3D plot and a contour plot are given. We used a circular pupil with a diameter of 32 resolution points, which is one-half of the number of resolution points in the pupil space. We assumed plane wave illumination of unit amplitude. The strength of the turbulence-induced aberrations varied from an rms w of zero to one wavelength. Since the 3D plots are normalized, direct comparison of absolute irradiance is not possible but the distribution of the energy in the images can be compared. Note that the major effect of the turbulence under our simulation parameters is the introduction of various amounts of tilt, as is especially clear in the case of \( \bar{w} = \lambda \) (Fig. 4.2d). This suggests that we can apply, approximately, the theoretical analysis of undersampling even when the images are distorted by the atmosphere, since
Fig. 4.1. Schematic Diagram of Program Which Generates Atmospherically Aberrated Images.
Fig. 4.2. Atmospherically Aberrated PSF (Computer Simulation).

(a) Unaberrated PSF.
Fig. 4.2. Atmospherically Aberrated PSF (Computer Simulation), Continued.

(b) with rms wavefront aberrations of \( \bar{\omega} = 0.5\lambda \).
Fig. 4.2. Atmospherically Aberrated PSF (Computer Simulation), Continued.

(c) with rms wavefront aberrations of $\tilde{\omega}=0.7\lambda$. 
Fig. 4.2. Atmospherically Aberrated PSF (Computer Simulation), Continued.

(d) with rms wavefront aberrations of $\bar{\omega} = 1\lambda$. 
the functional form (in this case an Airy distribution) is not much distorted.

Simulation of the Measurement of the Image Parameters

The simulated images were presented as a 64x64 matrix of irradiance levels at the focal plane. We simulated an array detector with the number of resolution elements ranging from 1x1 to 20x20. The first step was the centering of the image on the measurement field. This was done by determining the true centroid of the distribution and relabeling the matrix such that the centroid occupies the center element (32,32) of the matrix. (The algorithms of determining the parameters will be discussed later.) The tilt reduction simulates an actual feature of the proposed design of the beam analysis system which initiated this work, where the tilt is removed by a pair of fast-moving positioning mirrors.

The next step was the determination of the power level on each detector element. This was done by expanding the array to a size which is the smallest common denominator of 64 and the number of detector elements. By reproducing the original energy in each image "cell" into the expanded cells we could now divide the expanded matrix into an N x N matrix, where N^2 is the number of elements in the simulated, two-dimensional, detector array. This method of interpolation conserved the initial field area and kept it unchanged when the detector element size changed. The simpler method of division, which is also less time-consuming, of throwing out the margin of the image so an integral multiple of n is obtained proved to introduce an error in the parameter
calculation. While the introduced error was relatively small, it was comparable to the changes in the measurement error due to variations in the detector element size and had to be avoided.

The algorithms which were used to calculate the image parameters were adopted, in a straightforward way, from the definitions of the moments. It was assumed that all the power falling on a detector element is concentrated at the center of the element. The size of the detector array was normalized to unity, and the origin of the coordinate system fixed at the center of the 2D array.

Designating the value of the \((I,J)\) matrix element by \(E(I,J)\), the following algorithms result:

\[
\bar{I} = \frac{\sum_{I=1}^{N} I E(I,J)}{\sum_{I=1}^{N} E(I,J)} \quad (4.5)
\]

\[
\sigma_{I}^2 = \frac{\sum_{I=1}^{N} I^2 E(I,J)}{\sum_{I=1}^{N} E(I,J)} - \bar{I}^2 \quad (4.6)
\]

\[
\bar{x} = \frac{1}{N} (\bar{I} - 1/2) - 1/2 \quad (4.7)
\]

\[
\sigma_{x}^2 = \frac{1}{N^2} \sigma_{I}^2 . \quad (4.8)
\]

Here Eqs. (4.5) and (4.6) give the non-normalized values and Eqs. (4.7) and (4.8) give the normalized values of the centroid and variance along the \(x\)-axis. Similar algorithms were used to find these parameters along the \(y\)-axis, and the covariance. The spot size was estimated by:
\[ \hat{\rho} = \left( \sigma_x^2 + \sigma_y^2 - \frac{\sigma_x^2}{6} \right)^{\frac{1}{2}} \]  

(4.9)

where \( \alpha = \frac{1}{N} \) is the normalized detector element size, and the \( \frac{\sigma_x^2}{6} \) correction factor is explained by Eq. (3.14).

In order to calculate the eccentricity and orientation of the principal axes Eqs. (2.17) through (2.21) can be used. However, these equations are given in a form which is sensitive to errors. For example, Eq. (2.17) which gives \( \tan 2\theta \) (\( \theta \) is the angle of the principal axis) is very sensitive to errors when \( \sigma_x^2 = \sigma_y^2 \) and \( \sigma_{xy} = 0 \). A better form to use is:

\[ \cot 2\theta = \frac{1}{2} \left( \frac{\sigma_x^2}{\sigma_{xy}} - \frac{\sigma_y^2}{\sigma_{xy}} \right) \]  

(4.10)

If \( \left| \frac{\sigma_x^2}{\sigma_{xy}} \right| + \left| \frac{\sigma_y^2}{\sigma_{xy}} \right| >> 1 \) we have either of the following cases:

(a) \( \sigma_x^2 = \sigma_y^2 \) which means that the distribution is radially symmetric, the eccentricity is close to zero and naturally no angular orientation can be specified, and (b) \( \sigma_x^2 \neq \sigma_y^2 \) in which case \( \theta=0 \) or \( \theta=90^\circ \), according to the sign of \( \sigma_x^2 - \sigma_y^2 \). Exactly when it can be assumed that the distribution is radially symmetric depends, of course, on the level of precision desired.

Once the angle of orientation was determined (i.e., the distribution is not radially symmetric), the eccentricity can be calculated in a straightforward way from Eq. (2.20) or Eq. (2.21).

Results of the Simulation

We present here simulation results for atmospherically aberrated images which have no other (inherent) aberrations. Both plane wave illumination and Gaussian beams are treated.
A truncated Gaussian wavefront was used, the aperture diameter being eight times the 1/e width of the unaberrated Gaussian beam. The image field covered by the simulated detector array was 16σ, σ² being the variance of the diffraction limited PSF. Thus, the sampling frequency can be expressed as \( \xi_s = n/16 \) in units of \( \sigma^{-1} \), where \( n \) is the number of the array elements.

The results for various distributions with rms wavefront variations ranging from zero to one wavelength are presented in Figs. 4.3 and 4.4. Figure 4.3 plots the relative spot-size error as a function of the normalized sampling frequency \( \hat{\xi}_s \), which is in units of reciprocal spot size. The dashed line represents the simulation results for an unaberrated wavefront. The error curve is similar to the calculated one (Fig. 3.26) but shifted to higher sampling frequencies. The sampling interval for an error of less than 5% is 1.7σ, in comparison to 2.3σ as calculated theoretically. At lower sampling frequencies the simulation gives considerably larger errors than the theory predicts. This discrepancy is caused mainly by the truncation both in the aperture and image and by the limited accuracy of the FFT routine (~ 3%). As expected the aberrated distributions depart even more from the ideal case, since the effects of truncation are more severe.

In Fig. 4.4 we plot the relative error in determining the spot size vs. the spot size for the various distributions. The sampling frequency is kept constant and is given in terms of \( \sigma^{-1} \) of the unaberrated PSF. This corresponds to the practical situation of fixed array
Fig. 4.3. Relative Errors in Determining the Spot Size as Calculated by Computer Simulation.

The dashed line corresponds to an unaberrated wavefront (Gaussian beam).
Fig. 4.4. Relative Error as a Function of Spot Size for $\xi_s = 0.48, 0.54,$ and $0.60$ (Computer Simulation. (Gaussian beam.)
dimensions. As expected, the relative measurement error decreases as
the spot size (and the normalized sampling rate) increases, and the fall-
off is rather rapid, faster than 1/x decline. While the increased wave-
front aberration increases the measurement error for the same normalized
sampling frequency, the increase in \( \rho \) due to image spread (and the resul-
tant increase in the normalized frequency) reduces the sampling error
sharply.

Plane Wave Illumination

The simulation of plane wave illumination included also a varia-
tion of the pupil size which, in theory, does not have any effect on the
error terms. In practice, however, we produced various sized spots rela-
tive to the 64x64 field of the output matrix. The Airy-discs so produced
had a first-zero width ranging from 18.4 matrix elements to 27.8 matrix
elements. The calculated spot-size errors for the unaberrated images of
different size (as a function of the effective sampling rate) agreed to
within 1%. This points out the good accuracy of the fast Fourier trans-
form routine and the adequacy of a 64x64 sampling space.

The cumulative results of the simulation are presented in Fig.
4.5 and Fig. 4.6. When compared to the results for the Gaussian beam,
the most striking difference is that for the plane wave illumination the
aberrated images have a lower relative error (at the same effective
sampling rate) than the diffraction-limited one: It is the opposite
with Gaussian beams (Fig. 4.3). This can be correlated to the smoothness
of the pupil function: A pure Gaussian is smoother than any aberrated
Fig. 4.5. Relative Error in Measuring the Spot Size of Aberrated Images as a Function of Sampling Frequency.

(Circular aperture, plane wave illumination.)
Fig. 4.6. Relative Error as a Function of Spot Size.

(Circular aperture, plane wave illumination.)
distribution, while a sharp-edged pupil, illuminated by a plane wave becomes smoother when aberrated, decreasing the sharp features in the image distribution.

Qualitatively, Fig. 4.5 can be compared to the theoretical calculations for a sinc$^2(x)$ distribution (Fig. 3.4b). The similarity between the plots for unaberrated images is evident. There is, of course, a difference between the LSF of an Airy distribution and a sinc$^2(x)$ distribution, and the plots are not identical. Also, it can be seen that the relative errors are larger than the errors for a Gaussian beam, as predicted.

Figure 4.6 shows how the relative error drops as the image size increases (due to atmospherically induced aberrations) when the detector size is being kept constant. The drop is much faster than for Gaussian beams, in agreement with Fig. 4.5.

To conclude, we note that the 5% error point corresponds to a sampling frequency of $\xi_s \approx 1.2\sigma^{-1}$, which translates to a sampling interval of 0.28 of the Airy disc diameter (i.e., 4 detector elements per Airy disc diameter).

Conclusion

Two valuable conclusions can be drawn from this simulation: The first one is that it is enough to consider only the diffraction-limited image when it is necessary to determine the optimal number of detector elements. The change in image parameters (shape, size) do not increase the measurement error, but indeed decrease it. This is explained by the increase in image spread due to atmospheric effects,
which increases the effective sampling rate. The second conclusion is that the theoretical prediction of being able to give an accurate measure of the image distribution moments with considerable undersampling is indeed correct. The various plots, both theoretical and simulated, give a handy measure of the tradeoff between sampling frequency (i.e., detector element size) and the measurement error.
CHAPTER 5

IMAGE CHOPPING TECHNIQUES

Analog measurement schemes have several advantages over digital measurement: They can be faster and cheaper, saving the cost of expensive data acquisition and manipulation systems (i.e., computers). However, analog systems are less accurate, and an analog measurement and calculation system cannot always be devised.

In this chapter an analog method to measure the beam parameters is described. The measurement technique is based on chopping the image in three directions with a knife-edge or a slit and collecting the transmitted energy on a detector; the output signals are processed by a conceptually simple circuit. The principle of using rotating slit or knife-edge choppers to measure various image parameters is not new. Such devices have been used for dynamic alignment of optical instruments and for detecting the centroids of spot images. Other applications of rotating reticles include filtering and background noise suppression in conjunction with tracking systems (Hudson, 1969; Legault, 1978). These techniques were basically based on timing the appearance of a pulse, or pulses generated by the chopper. Here, a rigorous mathematical development is presented which shows an exact analog way to measure the image moments.
Analysis of Knife-edge Chopper

Let the total energy of an image be collected by a single detector while the image is scanned by a knife-edge moving in the +x direction as in Fig. 5.1. The detector output \( V(t) \) is thus proportional to the integrated irradiance:

\[
V(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x,y) \, dx \, dy \quad (5.1)
\]

the differentiated output is

\[
V'(t) = \frac{dx(t)}{dt} \int_{-\infty}^{\infty} E[x(t),y] \, dy \quad (5.2)
\]

assume that the knife-edge moves at a constant speed so that \( x = x_0 + vt \) and

\[
\int E[x(t),y] \, dy = \frac{V'(t)}{v} \quad (5.3)
\]

Expressions can be found for both the centroid and the variance in terms of \( V(t) \):

\[
\bar{x} = \int \int E(x,y)x \, dx \, dy = \int x \, dx \int E(x,y) \, dy \quad (5.4)
\]

where it is assumed that the total power on the detector is normalized to unity. Thus

\[
\bar{x} = \int_{0}^{T} x(t) \frac{V'(t)}{v} \, d(vt) = v \int_{0}^{T} tV'(t) \, dt + x_0 V(t). \quad (5.5)
\]

At time \( T \) all the image is unmasked. For simplicity it can be assumed that \( x_0 = 0 \) and the second integral vanishes. Similarly, the variance is

\[
\sigma_x^2 = \int \int E(x,y)x^2 \, dx \, dy - \bar{x}^2 = v^2 \int_{0}^{T} t^2 \, V'(t) \, dt - \bar{x}^2 \quad (5.6)
\]
Fig. 5.1. Geometry of Image Chopping.
where \( \bar{x} \) is given by Eq. (5.5) and \( x_0 = 0 \). If the image is chopped along the y-axis \( \sigma_y^2 \) is found. The covariance \( \sigma_{xy} \) can be found by chopping at 45°, as in Fig. 5.2. The quantity which is measured is \( \sigma_{x'}^2 \), the variance along the \( x' \) axis. In terms of the original coordinates we have:

\[
\sigma_x^2 = \iint E(x,y) \frac{1}{2} (x^2 + y^2 + 2xy) dx dy - \frac{1}{2} (x+y)^2
\]

\[
= \frac{1}{2} (\sigma_x^2 + \sigma_y^2) + \sigma_{xy}.
\]

(5.7)

If the same image (or multiple images of the same distribution) is chopped in the three directions defined above, the time differentiated outputs will give

\[
V'_{x}(t) \longrightarrow \sigma_x^2
\]

\[
V'_{y}(t) \longrightarrow \sigma_y^2
\]

\[
V'_{45}(t) \longrightarrow \frac{1}{2} (\sigma_x^2 + \sigma_y^2) + \sigma_{xy}
\]

and from here all the image parameters defined in Chapter 2 can be calculated.

If a narrow slit is used to scan the image the detector output is proportional to \( V'(t) \). The operations of Eq. (5.5) and Eq. (5.6) can be performed by an analog circuit with a functional block diagram as in Fig. 5.3. A set of three slits in a rotating disc can be placed as shown in Fig. 5.4. To understand the scan direction of the 45° slits, consider Fig. 5.5. The direction of motion of the slits is a vector sum of a motion along the slit axis which has no effect on the detector output and a motion component perpendicular to the slit axis which determines the
Fig. 5.2. Geometry of Covariance Measurements.
Fig. 5.3. Block Diagram for Analog Calculation of $\sigma^2$ Using Slit Apertures.
Fig. 5.4. Chopper Disc, Showing Slit Orientation.
Fig. 5.5. Effective Scan Direction.
measurement axis. Note that the scan velocity $v$ is slower for the $45^\circ$ tilted slits by a factor of $\frac{\sqrt{2}}{2}$. The effects of changes in $v$ due to rotational scanning are treated below.

The expressions for the centroid [Eq. (5.5)] and variance [Eq. (5.6)] can be simplified by integrating by parts. Since $V(T)=1$ by Eq. (5.1), we get

$$\bar{x} = vT - v \int_0^T V(t) dt$$

$$\sigma^2 = (vT)^2 - 2v^2 \int_0^T tV(t) dt - \bar{x}^2.$$  

A block diagram of an analog circuit which calculates $\sigma^2$ is shown in Fig. 5.6. There is an advantage to the knife-edge scanning method since the measurement is less sensitive to the quality of the slits and the electronic processing is somewhat simpler.

**Error Due to Rotational Scanning**

When a chopper wheel is used to scan the image, error is introduced into the measurement due to the radial variation of the tangential slit velocity. We show here that the error is negligible when the image centroid is known and when the maximum image dimension is much smaller than the disc radius.

Consider the geometry of Fig. 5.7. The time derivative of the detector output is proportional to the line integral along the knife-edge boundary:

$$V'(t) = \frac{dx(t)}{dt} \int E[x(t) + \Delta x, y] dy.$$  

(5.10)
Fig. 5.6. Block Diagram for Calculation of $\sigma^2$ with Knife-edge Apertures.
Fig. 5.7. Geometry of Rotational Scanning.
Typically the maximum dimension of the image, \( r \), is much smaller than the chopping disc radius, \( R \).

We approximate:

\[
\Delta x \approx y \theta
\]

\[
\frac{dx}{dt} \approx v
\]

and \( E(x + \Delta x, y) = E(x, y) + \Delta x \frac{\partial E(x, y)}{\partial x} \) \hfill (5.11)

The relative error due to rotational chopping is given by:

\[
\delta = \frac{\Delta V'(t)}{V'(t)} = \frac{\int y \frac{\partial E(x, y)}{\partial x} \, dy}{\int E(x, y) \, dy} \hfill (5.12)
\]

An upper boundary for the relative error is given by:

\[
\delta_{\text{max}} = \theta_{\text{max}} \cdot y_{\text{max}} \cdot \frac{\langle \frac{\partial E(x, y)}{\partial x} \rangle}{\langle E(x, y) \rangle}
\]

\[
= \frac{r^2}{R} \cdot \frac{\langle \frac{\partial E(x, y)}{\partial x} \rangle}{\langle E(x, y) \rangle} \hfill (5.13)
\]

The averages that appear in the expression are taken along the knife-edge. While \( E(x, y) \) is always positive, the partial derivative is oscillating and most probably will average to a very small value. It is clearly seen that as long as \( \frac{r}{R} \ll 1 \), the relative error in the detector's output is extremely small.

The error becomes meaningful when the image centroid can wander over an area which is much larger than the spot size. In this case the slit velocity cannot be determined \textit{a priori} and a first order correction has to be introduced \textit{a posteriori}. To calculate the error we assume that
the image dimensions are much smaller than the image field (i.e., the
field over which we measure the image parameters). Thus, the image can
be considered as a δ-function, located at the centroid. Consider Fig.
5.8 for the geometry of the problem. The circle of radius $r$ represents
the image field and the image is at $(x_o, y_o)$. The mean radius of the disc
is $R$, where the center of the knife-edge is located. We assume that
$r = \bar{R}$. The following parameters are defined:

- $\omega$ - angular velocity of the disc
- $\bar{v} = \omega R$ - average tangential velocity of the knife-edge
- $\theta_{\max} = \tan^{-1}\left(\frac{R}{R}\right)$
- $\frac{R}{R}$ - maximum angular deviation of the knife-edge
- $T = \frac{R}{\bar{v}}$ - scan time of half the image field
- $\theta(x_o, y_o) = \tan^{-1}\left(\frac{x_o}{R+y_o}\right)$
- $\frac{x_o}{R+y_o}$ - angular position of the image
- $V(t)$ - the detector's output, which is

$$V(t) = 0 \text{ when } t < T \quad \frac{(x_o, y_o)}{\theta_{\max}} = \bar{R} \quad \frac{x_o}{(R+y_o)} \quad T$$

$$= 1 \text{ when } t > \frac{\bar{R} x_o}{r(R+y_o)} \quad T.$$  

Using Eq. (5.8) we estimate the centroid $x_o$: 
Fig. 5.8. Rotational Scanning.

Field of view is much larger than the image.
Once the image centroid position is estimated using the scan velocity $\bar{v}$, a first order correction of a form derived from Eq. (5.14) can be applied. Also, the scan velocity should be corrected by the same factor and used to scale the second moment measurements. The residual relative error in determining the centroid is equal to $\frac{x_0y_0}{(R+x_0)(R+y_0)}$, which takes a maximum value of approximately $\left(\frac{R}{x_0}\right)^2$.

It can be concluded that the main source of error due to rotational scanning arises from the variation of the tangential scanning velocity of the knife-edge but it can be accounted for by a first-order correction. The error due to the image spread is negligible, at least as long as the image is small relative to the field of view.

**Demonstration Unit**

A demonstration unit which included a light source and imaging system, a chopper wheel, detector and an analog processor was built and operated. The output of an Argon laser was imaged by an afocal telescope in front of a silicon photo-diode detector system. A chopper wheel, which included a set of three knife-edge slits, with a geometry of Fig. 5.4, was placed at the position of the beam's waist. An infrared LED-detector pair was placed across the chopper wheel to supply a synchronization pulse.
The detector's output was fed into a linear variable gain amplifier so that a standard amplitude signal could be generated, independent of detector output variations, so that light sources and detectors could be varied. There was also a provision for delaying the sync-pulse which starts the measurement process and varying the duration T of the integration. This corresponds to a variable field of view and image position.

No effort has been made to reach the ultimate in accuracy, dynamic range, etc., and standard off-the-shelf analog units were used. The integrators and the time-ramp generator were of classical operational-amplifier-and-capacitor design. The multipliers, dividers and squarer were all based on an IC unit (Analog Devices AD 535) which had limited dynamic range. As a result, no attempt has been made to calibrate the instrument and measure its accuracy or long-time stability. The intention was to identify weak points in the design and to determine those components that are most critical in limiting the accuracy of the instrument.

Qualitative observations demonstrated that the instrument operates as predicted. The most convincing experiment involved moving the image along an axis parallel to the orientation of one of the knife-edge slits. With a beam waist diameter of about 1.5 mm and a chopper-wheel radius of about 75 mm it was expected that the error due to rotational scanning will be negligible. The following was observed: of the three output pulses, with height proportional to the position of the image in a direction perpendicular to the corresponding knife-edge, one pulse increased in amplitude, another decreased and the third did
not vary. This, of course, agrees with the position changes relative to the three slits. Finally, it was observed that when the beam waist diameter has been changed by adjusting the beam expanding telescope, all three output pulses varied in amplitude in an identical manner, as expected.

The most limiting characteristic of the circuit was the small dynamic range. The integrators and the multiply/divide units were approximately linear only over a 3db range. Also, mainly due to thermal drift, the circuit had to be trimmed frequently. It is clear that any useful instrument will have to be free of these problems. Several solutions are suggested. Better integrators can be built using voltage-to-frequency conversion (Goodenough, 1977) or based on tapped analog delay (TAD) units (Weckler, 1978). Fast integration with good precision and large dynamic range can be thus achieved. The other weak component, the multiply/divide units, can be improved by using logarithmic amplifiers so that the multiplications are converted into summations. An alternative is to use multiplying digital-to-analog converters: units which multiply one analog input by a digital (clock) input very accurately.
APPENDIX A

DERIVATION OF THE MOMENTS OF SAMPLED FUNCTIONS

We derive here the expressions for the first and second moments of one-dimensional functions which are sampled by δ-type or extended-element arrays.

δ-type Sampling

Let \( g(x) \) be a function symmetrical about the point \( x_0 \). We can represent \( g(x) \) as a shifted version of the even function \( f(x) \):

\[
g(x) = f(x-x_0) . \tag{A.1}
\]

When sampled by an infinite, δ-type array, the sampled function is given by:

\[
g_s(x) = x_s \sum_{n=-\infty}^{\infty} g(nx_s) \delta(x-nx_s) \tag{A.2}
\]

where \( x_s \) is the sampling interval. The Fourier transform of \( g_s(x) \) is

\[
G_s(\xi) = \sum_{n=-\infty}^{\infty} G(\xi-n\xi_s) = \sum_{n=-\infty}^{\infty} e^{-j2\pi x_o (\xi-n\xi_s)} F(\xi-n\xi_s) \tag{A.3}
\]

where \( \xi_s = \frac{1}{x_s} \) is the sampling frequency, and Fourier transforms of the functions are denoted by the corresponding capital letter. The first and second derivatives of \( G_s(\xi) \) are given by

\[
G_s^{(1)}(\xi) = \sum_{n=-\infty}^{\infty} \left[ -j2\pi x_o F(\xi-n\xi_s) + F^{(1)}(\xi-n\xi_s) \right] e^{-j2\pi x_o (\xi-n\xi_s)} \tag{A.4}
\]
Using Eqs. (3.1) through (3.3) we find the moments:

\[ \hat{m}_0 = G_s(0) = \sum_{n=-\infty}^{\infty} F(-n \xi_s) e^{j2\pi x_o n \xi_s} = F(0) + \sum_{n=1}^{\infty} \left( e^{j2\pi x_o n \xi_s} - e^{-j2\pi x_o n \xi_s} \right) \]

\[ = F(0) + 2 \sum_{n=1}^{\infty} F(n \xi_s) \cos(2\pi n \xi_s x_o) . \]  
(A.6)

Here advantage has been taken of the symmetry properties of the Fourier transform of an even function; i.e., \( F(-n \xi_s) = F(n \xi_s) \). Similarly,

\[ \hat{m}_1 = -\frac{1}{j2\pi x_o} \sum_{n=-\infty}^{\infty} -j2\pi x_o F(-n \xi_s) e^{j2\pi x_o n \xi_s} = -\frac{1}{j2\pi} \sum_{n=-\infty}^{\infty} F(1)(-n \xi_s) e^{j2\pi x_o n \xi_s} \]

\[ = x_o + \frac{1}{\pi x_o} \sum_{n=1}^{\infty} F(1)(n \xi_s) \sin(2\pi n \xi_s x_o) . \]  
(A.7)

\[ \hat{m}_2 = -\frac{1}{4\pi x_o} \sum_{n=-\infty}^{\infty} \left[ -4\pi^2 x_o^2 F(n \xi_s) - j4\pi x_o F(1)(n \xi_s) + F(2)(-n \xi_s) \right] \]

\[ e^{j2\pi n \xi_s x_o} \]

\[ = \rho^2 + x_o^2 + \frac{2x_o}{\pi x_o} \sum_{n=1}^{\infty} F(1)(n \xi_s) \sin(2\pi n \xi_s x_o) - \frac{1}{2\pi^2 \hat{m}_0} \]

\[ \sum_{n=1}^{\infty} F(2)(n \xi_s) \cos(2\pi n \xi_s x_o) . \]  
(A.8)
Again, use has been made of the symmetry properties of the function \( F(\xi) \) and its derivatives.

**Extended Element Sampling**

The value of the distribution at the sample point \( nx_s \) is given by the local average of the function over the interval \( (nx_s - \frac{a}{2}, nx_s + \frac{a}{2}) \) where \( a \) is the detector element size. The sampled function is described by the convolution of \( \text{rect}(\frac{x}{\alpha}) \) and a \( \delta \)-type sampling function:

\[
g_s(x) = \frac{x}{\alpha} \sum_{n=-\infty}^{\infty} g(x) \ast \text{rect}(\frac{x}{\alpha}) \delta(x-nx_s) . \tag{A.9}
\]

The Fourier transform is

\[
G_s(\xi) = \sum_{n=-\infty}^{\infty} G(\xi-n\xi_s) \text{sinc}[\alpha(\xi-n\xi_s)] . \tag{A.10}
\]

We calculate now the first and second derivatives of the transform. Note that the derivatives of the sinc function are given by

\[
\text{sinc}^{(1)}(x) = \frac{1}{x} [\cos(\pi x) - \text{sinc}(x)]
\]

\[
\text{sinc}^{(2)}(x) = \frac{2}{x^2} [\cos(\pi x) - \text{sinc}(x)] - \pi^2 \text{sinc}(x)
\]

We have:

\[
G_s^{(1)}(\xi) = \sum_{n=-\infty}^{\infty} G^{(1)}(\xi-n\xi_s) \text{sinc}[\alpha(\xi-n\xi_s)] + G(\xi-n\xi_s)
\]

\[
\{\cos[\pi \alpha(\xi-n\xi_s)] - \text{sinc}[\alpha(\xi-n\xi_s)]\}/(\xi-n\xi_s) \tag{A.11}
\]
\[ G_s^{(2)}(\xi) = \sum_{n=-\infty}^{\infty} G^{(2)}(\xi-n\xi_s) \text{sinc}(\xi-n\xi_s) - 2G^{(1)}(\xi-n\xi_s) \]

\[ \left\{ \cos[\pi\alpha(\xi-n\xi_s)] - \text{sinc}[\alpha(\xi-n\xi_s)] \right\}/(\xi-n\xi_s) - \]

\[ G(\xi-n\xi_s) \left[ 2\left\{ \cos[\pi\alpha(\xi-n\xi_s)] - \text{sinc}[\alpha(\xi-n\xi_s)] \right\}/(\xi-n\xi_s)^2 + \right. \]

\[ \pi^2\alpha^2 \text{sinc}[\alpha(\xi-n\xi_s)] \left. \right\} . \quad (A.12) \]

Since both the \text{sinc} and \text{cos} functions are even about the origin, the transform and its derivatives at the origin can be expressed as in Eqs. (3.9) through (3.11). To simplify these expressions we take advantage of the following limits:

\[ \lim_{x \to 0} \frac{\cos(\pi x) - \text{sinc}(x)}{x} = 0 \]

\[ \lim_{x \to 0} \frac{\cos(\pi x) - \text{sinc}(x)}{x} = \frac{1}{3} \]

and the moments can be expressed as follows:

\[ \hat{m}_0 = F(0) + 2 \sum_{n=1}^{\infty} F(n\xi_s)\cos(2\pi n\xi_s x_0)\text{sinc}(n\alpha\xi_s) \quad (A.13) \]

\[ \hat{m}_1 = x_0 + \sum_{n=1}^{\infty} F^{(1)}(n\xi_s)\text{sinc}(n\alpha\xi_s)\sin(2\pi n\xi_s x_0)/\pi n\xi_s + \]

\[ \sum_{n=1}^{\infty} F(n\xi_s)\sin(2\pi n\xi_s x_0)\left[ \cos(n\pi\alpha\xi_s) - \text{sinc}(n\alpha\xi_s) \right]/n\pi\xi_s \hat{m}_0 \quad (A.14) \]
\[ \hat{\omega}_2 = \rho^2 + x_o + \frac{a^2}{12m_o} + \frac{1}{m_o} \sum_{n=1}^{\infty} \left[ 2x_o F(n\xi_s) \cos(2\pi n\xi_s x_o) \right. \]

\[ \left. - \frac{2x_o}{\pi} F(1)(n\xi_s) \sin(2\pi n\xi_s x_o) + \frac{1}{2\pi^2} F(2)(n\xi_s) \cos(2\pi n\xi_s x_o) \right] \text{sinc}(n\alpha\xi_s) \]

\[ + \frac{1}{m_o n\xi_s} \sum_{n=1}^{\infty} \left[ \frac{2x_o}{\pi^2} F(n\xi_s) \sin(2\pi n\xi_s x_o) - \frac{1}{\pi^2} F(1)(n\xi_s) \cos(2\pi n\xi_s x_o) \right] \text{sinc}(n\alpha\xi_s) \]

\[ \left. \cos(2\pi n\xi_s x_o) \right] \left[ \cos(\pi n\alpha\xi_s) - \text{sinc}(\pi n\xi_s) \right] + \frac{1}{2\pi^2 m_o} \]

\[ \sum_{n=1}^{\infty} F(n\xi_s) \cos(2\pi n\xi_s x_o) \left\{ 2 \left[ \cos(\pi n\alpha\xi_s) - \text{sinc}(\pi n\xi_s) \right] / n^2 \xi_s^2 \right\} \]

\[ + \pi^2 a^2 \text{sinc}(n\alpha\xi_s) \} \right) \]  \hspace{1cm} (A.15)

The above expressions are considerably simplified when \( \alpha = \frac{1}{\xi_s} \). This corresponds to sampling with a detector element of the size of the sampling interval. In this case \( \text{sinc}(n\alpha\xi_s) = 0 \) for \( n \neq 0 \) and \( \cos(\pi n\alpha\xi_s) = (-1)^n \). The expressions for the moments of this sampling scheme are given by Eqs. (3.12) through (3.14).
APPENDIX B

REALIZATION OF A RANDOM PROCESS

In this appendix we lay out the mathematical basis for the technique we used to generate a realization of a random process with a given autocorrelation function. The treatment is in one dimension, but the extension to two dimensions is straightforward.

Let $H(x)$ be a stationary, ergodic random process with zero mean. If we denote the probability density function of the random variable $H$ by $p(h;x)$ the following properties exist (Beckman, 1967):

- **stationarity**
  
  \[ p(h;x) = p(h) \]

- **zero mean**
  
  \[ \int_{-\infty}^{\infty} h \, p(h) \, dh = 0 \]

and in addition we request that $H$ is $\delta$-correlated, i.e., the autocorrelation function is:

\[ \gamma_H(x_1, x_2) = \langle H(x_1)H(x_2) \rangle \]

\[ = \iint_{-\infty}^{\infty} h_1h_2 \, p(h_1,h_2; x_1,x_2) \, dh_1 \, dh_2 \]

\[ = \sigma^2 \delta(x_1-x_2) \]  

(B.1)

where $\sigma^2$ is the second moment (variance) of $H$. 

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Let $S(x)$ be a positive and even function. We define the random variable $G(x)$ as the convolution of $H(x)$ and $S(x)$:

$$G(x) = \int_{-\infty}^{\infty} H(\alpha)S(x-\alpha)d\alpha. \quad (B.2)$$

The mean of $G(x)$ is

$$\langle G(x) \rangle = \langle \int_{-\infty}^{\infty} H(\alpha)S(x-\alpha)d\alpha \rangle$$

since $H$ is stationary we can write

$$\langle G(x) \rangle = \int_{-\infty}^{\infty} \langle H(\alpha) \rangle S(x-\alpha)d\alpha = 0. \quad (B.3)$$

The autocorrelation of $G$ can be written as

$$\langle G(x_1)G(x_2) \rangle = \langle \int_{-\infty}^{\infty} H(\alpha)S(x_1-\alpha)d\alpha \int_{-\infty}^{\infty} H(\beta)S(x_2-\beta)d\beta \rangle$$

$$= \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} S(x_1-\alpha)S(x_2-\beta)\langle H(\alpha)H(\beta) \rangle d\alpha d\beta. \quad (B.4)$$

Making use of the properties of the autocorrelation of $H(x)$ as expressed in Eq. (B.1) we have:

$$\gamma_G(x_1, x_2) = \sigma^2 \int_{-\infty}^{\infty} S(x_1-\alpha)d\alpha \int_{-\infty}^{\infty} S(x_2-\beta)\delta(\alpha-\beta)d\beta$$

$$= \sigma^2 \int_{-\infty}^{\infty} S(x_1-\alpha)S(x_2-\alpha)d\alpha$$

$$= \sigma^2 S(\Delta x) * S(\Delta x) \quad (B.5)$$

where $\Delta x = x_1-x_2$.

The power spectral density (PSD) of $G$ is given by

$$\Phi_G(\xi) = \mathcal{F}\{\gamma_G(\Delta x)\} = \sigma^2 \hat{S}(\xi) \quad (B.6)$$

where $\hat{S}(\xi) = \mathcal{F}\{S(x)\}$. 
The PSD of $G$ is thus proportional to the Fourier transform of $S$, squared, and the probability law for $G$ is the same as for $H$. This is independent of the particular realization of $H$ used. In order to generate a realization of a random process with a specified PSD and known probability law, the following steps are taken:

1. Generate a random variable $H$ of the same probability law which is also $\delta$-correlated.
2. Take the Fourier transform of $H$.
3. Multiply $H$ by the square root of the given PSD.
4. Take the inverse Fourier transform.

Steps 1 and 2 can be combined into a single step, if a random function $\tilde{H}$ can be generated such that it's inverse Fourier transform has the right probability density and is $\delta$-correlated. A $\delta$-correlated process has a constant PSD, thus we have (Beckman, 1967, p. 214):

$$\phi_H(\xi) = \frac{1}{2} |H(\xi)|^2 = \text{constant},$$

if $\tilde{H}(x)$ is ergodic. Also, since $H(x)$ is real, $\tilde{H}(\xi)$ must be hermitian.

Suppose that $\tilde{H}(\xi)$ is a random, uniformly distributed phasor (UDP) of unit amplitude (Beckman, 1967, p. 115).

$$\tilde{H}(\xi) = e^{i\phi(\xi)}$$

where $\phi(\xi)$ is uniformly distributed between 0 and $2\pi$. $\tilde{H}(\xi)$ can be conveniently generated on the computer by existing random number generating routines, and made hermitian by defining $\tilde{H}(-\xi) = \tilde{H}^*(\xi)$. We shall calculate now the statistics of $H(x)$. Since $\tilde{H}(\xi)$ is hermitian, we have:
\[ \hat{H}(x) = \int_{-\infty}^{\infty} \tilde{H}(\xi) \cos(2\pi \xi x) d\xi. \]

The mean of \( H(x) \) is:

\[ \langle H(x) \rangle = \int_{-\infty}^{\infty} \langle \tilde{H}(\xi) \rangle \cos(2\pi \xi x) d\xi = 0 \]

since the mean of a UDT is zero. Also, \( |\tilde{H}(\xi)|^2 = 1 \) and according to Eq. (B.7), \( H(x) \) is \( \delta \)-correlated. \( H \) is then of the correct form and is suitable to generate a realization of the random process.

Finally, it is necessary to show that \( H(x) \) is normally distributed. This can be done by extending the concept of a Rayleigh phasor (Beckman, 1967, p. 118) to Fourier transform. A Rayleigh phasor is a random variable which is defined as

\[ Q = R e^{i\theta} = \sum_{i=1}^{n} A_i e^{i\phi_i} \]

where the terms are independent UDP phasors and the \( A_i \) are all distributed identically. If \( n \) is large, by the Law of Large Numbers both the real and imaginary parts of \( Q \) are normally distributed. This concept can be extended to include integrals:

\[ Q = \int_{-\infty}^{\infty} A(\alpha) e^{i\phi(\alpha)} d\alpha \]

and \( A(\alpha) \) can be the Fourier kernel. Thus, \( H(x) \) is normally distributed with zero mean and so is \( G(x) \) as defined by Eq. (B.2).

By extending the above derivations to 2-D, the mathematical basis is laid for algorithms which generate normally distributed, zero mean, wavefront phase variations.
LIST OF REFERENCES


