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A single-fluid, self-consistent formulation of particle transport and fluid dynamics

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A SINGLE-FLUID, SELF-CONSISTENT FORMULATION OF PARTICLE TRANSPORT AND FLUID DYNAMICS

by

Lance Lee Williams

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Lance Lee Williams

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DEDICATION

I dedicate this work to my parents, Ron and Patty Williams. Although neither had the luxury of going to college, they worked hard to give me that opportunity. And when I was choosing my path in life, they gave me the freedom to make my own choice, and the support to follow that path, a path which lead to this dissertation.

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ABSTRACT

I present a formulation of fluid dynamics that is consistent with particle transport and acceleration. This formulation consists of two parts: a transport equation that describes the evolution of a particle distribution function in terms of a fluid velocity in which the distribution is embedded, and an equation for the fluid velocity that involves integrals of the distribution function. The motivation of this work is to provide a formalism for calculating the effect of particle acceleration on the flows of typical astrophysical plasmas.

It is shown that the equation to be solved simultaneously with the transport equation is just the momentum equation for the system, and that the number and energy equations are implicit in the transport equation. There is no restriction on the energies of particles constituting such systems. Connections are made to the cosmic-ray transport equation, two-fluid models of cosmic-ray–thermal gas interaction, and self-consistent Monte Carlo models of particle acceleration at parallel shocks.

The formalism is developed for non-relativistic flow speeds. It is assumed that particle distributions are nearly isotropic in the fluid frame, an assumption that is generally valid in space plasmas. It is assumed that particle scattering mean-free-paths are much less than the length scales associated with changes in the fluid velocity or particle distribution.

1. Notation

Let me introduce the mathematical notation I'll be using. This notation is chosen over the alternatives for its clarity of meaning and its elegance of expression.

This work will involve vector and tensor quantities. I will use small roman subscripts or superscripts to denote the spatial cartesian components of the vector or tensor. There is no distinction between subscripts and superscripts, as one encounters in general relativity, since all calculations are on flat spacetime. For example, the vector S^a has three components: (S_x, S_y, S_z). The tensor Π_{ab} has $3x3 = 9$ free components: Π_{xx}, Π_{xy}, Π_{xz}, etc. The small roman letters x, y, and z will be reserved to denote actual cartesian components.

The length of a vector will be denoted simply by the name of the vector without the free index. For example, the length of S^a is denoted S. I'll also use the summation convention, which means summation over repeated indices is implied. For example, $p^a U^a = p_x U_x + p_y U_y + p_z U_z$. A quantity like $\Pi_{ab} B^b$ is a vector quantity: it has one free index, with summation implied on the other two indices.

I will denote spatial variables simply as x^a. Momentum variables will be denoted p^a. As a further streamlining, spatial derivatives written in the text will be denoted ∂_a: the partial derivative with respect to the ath component of the spatial coordinate. Also, ∂_p denotes the partial derivative with respect to the magnitude of the momentum; ∂_t denotes the partial derivative with respect to time. When I write out equations, I will write out the derivative. Here are some vector operations written in the usual boldface notation and in the index notation:

$$\text{dot product} \qquad \mathbf{A} \cdot \mathbf{B} = A^a B^a = \delta_{ab} A^a B^b$$

$$S = (S^a S^a)^{1/2} = (\mathbf{S} \cdot \mathbf{S})^{1/2}$$

$$\text{divergence} \qquad \nabla \cdot \mathbf{S} = \frac{\partial S^a}{\partial x^a} = \partial_a S^a = \delta_{ab}\frac{\partial S^a}{\partial x^b}$$

$$\text{gradient} \qquad \nabla \mathbf{f} = \frac{\partial f}{\partial x^a} = \partial_a f$$

$$\text{cross product} \qquad \mathbf{S} \times \mathbf{B} = \epsilon_{abc} S^b B^c$$

$$\text{curl} \qquad \nabla \times \mathbf{B} = \epsilon_{abc}\frac{\partial B^c}{\partial x^b}$$

A couple of useful tensors were introduced above. One is sometimes known as the 'Kronnecker delta': δ_{ab}. This tensor has the value 1 when $a = b$ and zero otherwise. Its trace $\delta_{aa} = 3$. The other is sometimes known as the 'Levi-Civita' tensor: ϵ_{abc}. This tensor has the value 1 for cyclic permutations of the coordinates: xyz, yzx, zxy; -1 for anti-cyclic permutations: xzy, yxz, zyx; and zero otherwise. Note that $\delta_{ab}\epsilon_{abc} = 0$. Also, the contraction of ϵ_{abc} into a symmetric tensor is zero. When $\Pi_{ab} = \Pi_{ba}$, $\Pi_{ab}\epsilon_{abc} = 0$.

In referring to equations, I will call the left hand side LHS and the right hand side RHS.

2. History of Cosmic-Ray Discovery

This section is intended to be accessible to the layman. My references are Rossi (1964) and Friedlander (1990).

The study of cosmic rays is truly a twentieth-century science. The understanding of the nature of the 'cosmic radiation' has evolved in parallel with the development of the cornerstones of modern physics: special relativity and quantum mechanics. Since our understanding of cosmic rays depends on these disciplines, this understanding could never progress without these modern revolutions of physics. In fact, the study of cosmic rays contributed to these revolutions.

The first experimental observations of the effects of cosmic rays employed a simple device called an electroscope. An electroscope is a metal case with some arbitrary gas sealed inside. Insulated from the case is a metal rod which penetrates the case. At the end of the rod inside the electroscope are two metal 'leaves'; strips of flexible metal not unlike aluminum foil. The leaves are attached at the same point, and so are in close contact with one another hanging inside the electroscope. There is a glass window in the electroscope through which the leaves may be viewed.

Everyone is familiar with the experience of walking across a carpet and then touching a metal object and getting a small shock. This is because the shoes moving over the rug can actually rub some of the charged particles that constitute all matter off of the rug. Now the body has an excess charge. When an uncharged metal object is touched, the charge from the body spreads out into the object, and this flow of charge is experienced as a shock, often with an audible 'pop'.

Imagine, then, that in this charged state one were to touch the rod of an electroscope. What happens? The rod is insulated from the rest of the electroscope, so the charge that was on the body now spreads itself uniformly between the body and the electroscope rod: the 'potential difference' between the body and the rod is removed and both have the same charge. Since the rod is insulated from the box of the electroscope, the charge can only spread itself over the rod, which it does. This causes the charged leaves of the electroscope to separate, because objects with like charge repel each other. Looking through the window of the electroscope, one sees the leaves in an inverted 'V'.

If the insulation between the rod and the box is perfect, the leaves should remain separated indefinitely because there is no place for the charge to go. The gas inside the box is neutral and will not conduct electricity. However, it is observed that the leaves always fall back together. It was this simple observation that indicated that something was going on that was not accounted for by the understanding of nature prevalent at that time.

At the turn of the century, the electrical nature of matter was fairly well-understood. It was known that atoms consist of equal amounts of opposite charge. The discharge of the electroscope could be understood if some force were ionizing the gas in the electroscope; that is, separating the charge in some of the atoms. Once separated, the charge opposite to that on the rod was attracted to the rod, and neutralized the charge on the rod. The leaves fell back together. The leftover opposite charge distributed itself uniformly over the metal case. In this way, it was understood how the charged rod could be neutralized if only the ionizing force were known.

At the turn of the century, radioactivity was a new discovery, just being explored. It was known that radioactive substances also ionized gases, so it was possible that the ionizing force observed with the electroscope was due to radioactive materials in the earth's crust. It this were so, then the effect of ionization should decrease with distance above the earth. It was for this reason that a man named Hess took an electroscope up in a balloon in 1912. What he found is that the ionizing effect *increased* with height above the earth. This was the evidence that the ionizing agent comes from outside of the earth. It was Millikan who gave the name cosmic rays to this radiation.

But what were the cosmic rays? In studies of radioactive substances, Rutherford observed three types of radiation which he called alpha, beta and gamma. Alpha and beta rays are both charged particles; alphas are helium nuclei and betas are electrons. The gamma rays are high-energy electromagnetic waves, photons, and are uncharged. It was observed that no amount of shielding of the electroscope could completely eliminate the ionizing radiation. Whatever it was, it must be very energetic. By analogy with radioactivity studies, it was believed in the years following 1912 that the cosmic rays were high-energy gamma rays.

In 1928, Millikan proposed that the cosmic rays, which he presumed to consist of gamma rays, arose from spontaneous fusion occurring in interstellar space. In 1929, the Geiger counter was invented. In that same year, Bothe and Kohlhorster observed simultaneous discharge in sequential Geiger counters and concluded that cosmic rays were probably not gamma rays; instead, they proposed, they are likely charged particles. This is because charged particles are much more strongly-interacting with matter than are the neutral gamma rays. Since cosmic rays were observed to traverse large amounts of matter, a meter of lead for example, the energies of these particles were deduced to be greater than 10^9 eV. An eV is a measure of energy useful for atoms and molecules. For comparison, the energy of an air molecule at room temperature is about 1/40 eV.

Since cosmic rays are charged, they interact with the earth's magnetic field. Depending on the sign of a particle's charge and its direction of approach, the earth's magnetic field may deflect the approaching particle into the earth's surface or away from the earth. Around 1933 it was established that cosmic rays tend to arrive from the western portion of the sky. This east-west asymmetry fixed the sign of cosmic-ray charge; they were predominantly positive.

Also about this time, Anderson discovered the positron, the anti-particle for the electron. This discovery confirmed a prediction by Dirac, that the electron must have an antiparticle. His prediction was based on reconciling quantum mechanics with special relativity, which lead to the 'Dirac equation' to describe the evolution of relativistic spin-1/2 particles. The observation of the production of equal numbers of electrons and positrons in cosmic-ray-induced air showers also served to vindicate Dirac's theory.

In 1937 a new particle, the muon, was identified. It is negatively charged like the electron but is much more massive. These particles are unstable and decay to electrons in something like a millionth of a second. In fact, the muons served to test one of the strangest predictions of special relativity; the effect of time dilation. It is known that muons are produced high in the atmosphere when a cosmic ray strikes the upper atmosphere. Even travelling at the limiting speed for a material particle, the speed of light, one would think that the muon can't live long enough to make it to the ground. Yet muons are observed at the surface of the earth. It is the effect of time dilation that accounts for this. Since the muon is moving at relativistic speeds with respect to the earth, the muon is aging more slowly than would an identical particle at rest on the surface of the earth. In the frame of reference of the muon, it still decays in a millionth of a second, but observers at rest with respect to the muon perceive that time runs slower in the moving muon. Observers see that time is 'dilated' in the moving muon.

With the discovery of the muon, the nature of the cosmic radiation at the surface of the earth could be established. Practically all cosmic rays observed at sea level are muons, electrons and gamma rays. Muons are the dominant species, and are the most penetrating. These particles are all created in the earth's atmosphere when a cosmic ray from space strikes the upper atmosphere. The cosmic ray has so much energy that the impact with the atmosphere is able to create many new particles and still give *them* enourmous energies. These new particles then propagate downward through the atmosphere, striking molecules in the atmosphere, and creating still more particles. This cascade of particles is known as an air shower. These particles are collectively known as secondary cosmic rays. The flux of particles at sea level is about one particle per square centimeter per minute. So if you hold out your hand palm up, you see nothing or feel nothing. Yet even so, a sub-atomic particle with a lifetime of a millionth of a second moving at the speed of light, is crossing your palm every second.

The particles that strike the upper atmosphere in the first place, coming from outer space, are known as primary cosmic rays. The primaries are the ones that are mainly positively charged. Their interaction with the magnetic field is responsible for the east-west asymmetry, and gives rise to air showers that occur mainly from the west. The minimum primary energy to produce a shower at sea-level is 10^{14} eV. This is one hundred times more energy than will be produced in the superconducting supercollider. In fact, the highest-energy particles can only be observed by looking for the air showers that they initiate. This is done with large arrays of light detectors that look for the light produced in these showers. Computers then invert the data to figure out what energy in a particle would have been necessary to initiate the shower, and what direction it came from. This technique has been used successfully to observe particles up to 10^{21} eV. At 10^{21} eV, a single sub-atomic particle carries the same amount of energy as a baseball thrown at 100 miles per hour. And this energy is no upper limit on the energies of particles that exist; it's just the highest energy so far measured.

In the early 1940's, balloon experiments determined that the primary cosmic rays were mainly protons. In 1948 heavy nuclei were detected in the primary radiation. In 1961 electrons were detected in the primary radiation. The primary radiation is about 93% protons, 6% helium nuclei, and less than 1% heavy nuclei, electrons, and everything else. In fact, the composition of cosmic rays does not follow the cosmic abundances of the elements observed in cold matter and in the sun and stars. Heavy elements are enhanced in the cosmic-ray population relative to their abundances in neutral matter. This is a clue to the acceleration of cosmic rays.

The subject of this thesis is a unified theoretical description of the primary cosmic radiation and the background thermal space plasmas that constitute most of the universe. A plasma is a gas that it completely ionized; the charge in all the atoms has been separated. The preponderance of protons in the primary radiation is part of the reason why the considerations presented here include only protons. The space plasmas that fill the universe are neutral, being composed of equal numbers of protons and electrons. I can treat the plasmas as being composed only of protons because the protons are 2000 times more massive than the electrons; the electrons are dynamically insignificant (although they may be important when strong electric fields exist). In my treatment, the electrons don't really participate in the dynamics; they just insure neutrality.

In the next sections I'll introduce the reader to the state of cosmic-ray physics before this thesis, and try and motivate the present work. I'll introduce the cosmic-ray transport equation, diffusive shock acceleration, cosmic-ray viscosity, and two-fluid models of the interaction between cosmic rays and the thermal space plasma. I'll not be interested in tracing the history of the development of these ideas; only in presenting the state of cosmic-ray physics.

3. Background in Cosmic-Ray Physics

3.1) Properties of Space Plasmas

Space plasmas constitute by far the most expansive and pervasive state of matter in the universe. Taking the Milky Way as a typical example of a galaxy, the volume of a galaxy is $\sim 10^{62} m^3$ (cubic meters). There are $\sim 10^{11}$ stars in a galaxy, occupying a total volume of $\sim 10^{39} m^3$. Except for the volume occupied by stars, giant molecular clouds, and a few paltry chunks of rock and ice, the entire volume of a galaxy is occupied by space plasmas, dominating all these other forms of matter by some twenty orders of magnitude. And that's just within galaxies. Intergalactic space likely contains nothing but space plasma.

Space plasmas are also important by mass. Typical number densities are one particle (a proton) per cubic centimeter, or $10^6/m^3$ (see Boyd & Sanderson, 1969, for typical interstellar parameters). This implies a total galactic mass of $\sim 10^{68}$ protons. At $\sim 10^{57}$ protons per star, there are also $\sim 10^{68}$ protons in the stars. Space plasmas, although extremely tenuous, are as important as the stars in terms of mass.

Typical temperatures of the space plasmas are $\sim 10^5$ Kelvin. The plasmas are generally in motion. Inside the heliosphere, flow speeds average about 400 km/s. This is faster than waves can propagate in the plasma frame (about 40 km/s), so these speeds are supersonic; shocks in the space plasmas are common wherever the plasma meets an obstacle or another flow. Flow speeds outside the heliosphere are inferred to be about 20 km/s. This is only our local interstellar flow speed; values of the plasma flow speed throughout interstellar space may vary widely.

The scale of variation of the flow in the galaxy is at least an AU (10^{11} m), and ranges up to the size of the galaxy itself. On the scale of the flow, the scale that the space plasma can be described as a fluid, it is electrically neutral. Furthermore, the plasma is highly conducting so that the electric field approximately vanishes in the plasma frame. Space plasmas may carry embedded magnetic fields, with typical interstellar strengths $\sim 10^{-6}$ gauss.

A range of particle energies exist in the space plasmas. There is a thermal population with the peak around $\sim 10^5$ Kelvin ~ 10 eV. But particle energies go as high as can be measured, now about 10^{21} eV. This is for a single particle, presumably a proton but possibly a nucleus; Fe for example. These are enormous energies for a single sub-atomic particle. It is observed that the particle populations of space plasmas are approximately isotropic in the frame of the flow. This means the distribution does not depend on the direction of particle momentum, but only its magnitude. Collision mean free paths are about 1000 AU, so interparticle interactions are negligible. Thus, the space plasmas are known as 'collisionless'. The isotropization comes about by particles scattering off the magnetic field. Although the magnetic field is ordered on the scale of the flow, fluctuations occur on smaller scales. It is interaction with these random fluctuations that makes the particle distribution isotropic.

The particle motion is free along magnetic field lines, but circular moving perpendicular to the field. The net motion is a spiral along the field lines. The radius of the spiral is called the gyro-radius. The gyro-radius of an MeV proton in a 10^{-6} G field is $\sim 10^{-3}$ AU. An MeV is well beyond the thermal peak, yet the gyro-radius is still much smaller than the flow scales. It is interaction on these scales, the scale of the gyro-radius, that allows for a smoothing of the distribution on fluid scales. In the fluid description of space plasmas, the role played by the mean free path in descriptions of neutral gases is played by the gyro-radius. This is the micro-scale in the system.

3.2) Cosmic-Ray Transport Equation

The centerpiece of theoretical cosmic-ray physics is the cosmic-ray transport equation, first proposed by Parker (1965) (see also Gleeson & Axford, 1967; Jokipii, 1969; Webb & Gleeson, 1979). Here it is:

$$\frac{\partial f}{\partial t} + U^a \frac{\partial f}{\partial x^a} = \frac{\partial}{\partial x^a}\left(\kappa_{ab}\frac{\partial f}{\partial x^b}\right) + \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f}{\partial p} \tag{3.1}$$

The cosmic-ray population is described by the distribution function $f(x^a, p, t)$, which represents the number of particles per unit volume in phase space. The phase space is composed of the positions x^a of the particles, and the magnitude of the momentum p of the particles. That f depends only on the magnitude of particle momentum reflects the fact that the distribution of cosmic-rays is observed to be isotropic in momentum space. The velocity of the space plasma in which the cosmic rays are embedded is $U^a(x^a, t)$. The quantity actually measured by spacecraft is $4\pi p^2 f$.

In derivations of (3.1), the momentum coordinate is typically measured with respect to the frame moving with velocity U^a, but if $p \gg mU$, as is usual for cosmic rays, then the momenta can be measured with respect to the inertial frame. f is the isotropic part of the total cosmic-ray distribution function. The plasma flow velocity U^a is presumed to be much less than the speed of light. For a completely relativistic treatment, see Webb (1985).

The interpretation of the cosmic-ray transport equation is simple. The second term on the LHS represents change of f due to convection of particles with velocity U^a. The first term on the RHS represents diffusion of particles, and is characterized by a diffusion tensor κ_{ab}. When an average magnetic field is present in the space plasma, the antisymmetric part of κ_{ab} will contain the gradient and curvature drifts of particles through the magnetic field. The final term on the RHS represents adiabatic compression or expansion of the distribution due to a divergence of the fluid velocity. The adiabatic compression term accounts for the exchange of energy between the cosmic rays and the space plasma. It is just the 'PdV' term of the first law of thermodynamics. To see this, it is instructive to write the transport equation in conservation-law form:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x^a}\left(fU^a - \kappa_{ab}\frac{\partial f}{\partial x^a}\right) = \frac{\partial U^a}{\partial x^a}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f}{3}\right) \tag{3.2}$$

The current in (3.2) is a sum of convection with the fluid, and diffusion that is proportional to the gradient of f. The quantity $p^2 f/3$ is the partial pressure of cosmic rays at momentum p. The term on the RHS acts as a source in the conservation equation for f. It is this source term that describes the transfer of energy from the fluid to the particles. It behaves as a source in the conservation equation because it can move particles into values of p, and therefore points in phase space, that were not previously occupied. The adiabatic term also describes an energy *loss* for the particles if the fluid is expanding.

3.3) Extended Cosmic-Ray Transport Equation

Just in the past five years the transport equation for cosmic rays has been extended to include new and smaller effects (Earl, Jokipii and Morfill, 1988; Webb, 1989; Williams and Jokipii, 1991). The importance of the extensions is that they describe how an *arbitrary* flow configuration affects the cosmic-ray distribution f. The old transport equation (3.1) only describes an energy exchange between f and the flow if the flow has a divergence. The divergence of a vector field is not sufficient to specify the field. One may also imagine flow configurations in which the divergence of the flow velocity vanishes, but the flow is still non-uniform. For example, a shear in the flow. Does the omission of shear-flows from the transport equation mean that such flows have no effect on f? The answer is no. Such flows *do* have an effect on f. In fact, they serve to accelerate particles just as compressive flows do. Unlike the compressive flows which accelerate only adiabatically, acceleration in shear flows is non-adiabatic. That is, the energy change occurring in shear flows described by the extended transport equation is not reversible. It turns out that the behavior of this new energy-changing term has exactly the behavior of a shear viscosity, and that is exactly what it is. Thus the term 'cosmic-ray viscosity'. Here is the extended transport equation, written in conservation-form:

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{\partial}{\partial x^a}\left(fU^a + \frac{S^a}{m}\right) = &- m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}\left(pm\kappa_{ab}\frac{\partial f}{\partial x^b}\right) \\ &+ \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f}{3}\delta_{ab} + p\Pi_{ab}\right)\end{aligned} \tag{3.3}$$

$$S^a \equiv -m\kappa_{ab}\left[\frac{\partial f}{\partial x^b} - \frac{m^2}{p}\frac{\partial f}{\partial p}\frac{dU^b}{dt}\right]$$

$$\Pi_{ab} \equiv \frac{p^3\tau}{15}\frac{\partial f}{\partial p}\Upsilon_{abcd}\left(\frac{\partial U^c}{\partial x^d} + \frac{\partial U^d}{\partial x^c} - \frac{2}{3}\frac{\partial U^e}{\partial x^e}\delta_{cd}\right)$$

The tensor Υ_{abcd} is an ugly thing that I derive later on, but I won't bother with it here. The tensors κ_{ab} and Υ_{abcd} appear because magnetic fields are generally present in the space plasmas in which cosmic rays propagate. Thus there is a preferred direction that manifests itself in the transport coefficients.

The quantity τ is a scattering time. It is essentially an unknown parameter *not* fixed by the theory. This is also true for κ_{ab} as well, even in the old equation. In (3.3), κ_{ab} is a function of τ. Thus some knowledge of the scattering is assumed in the cosmic-ray transport equation.

The cosmic-ray viscosity is contained in Π_{ab}. In the conservation form of (3.3), it is apparent how the adiabatic energy-change and viscosity terms enter on an equal footing. The viscous term serves to generalize the flow configurations addressed by the transport equation. Together, $(p^2 f/3)\delta_{ab} + \Pi_{ab}$ constitute the partial pressure of cosmic rays. If this quantity were divided by a mass and integrated over all $p^2 dp$, one would obtain the total integrated pressure contributed by cosmic rays to the fluid in which they are embedded. The acceleration of particles through the mechanism of cosmic-ray viscosity is no different than what one encounters in ordinary fluid dynamics. Viscosity acts to dissipate ordered kinetic energy of flow into random internal kinetic energy. That is also true for cosmic-ray viscosity.

The other new terms in the extended transport equation, the correction to S^a and the first term on the RHS of (3.3), have been called the 'inertial' terms because they are proportional to the acceleration vector of the fluid. Only one of the inertial terms, the first one on the RHS, serves to accelerate particles. Whereas all the terms in the old transport equation were pretty much of the same size, the new terms of the extended transport equation are smaller. Thus the old transport equation can be viewed as a zeroth-order equation, and the extended equation as a giving the first-order corrections. In the derivation of the extended equation, the ordering parameter is the ratio of the scattering time to the timescale for a change in the fluid flow. Thus the ordering depends on the scattering that one assumes for the system.

3.4) Diffusive Shock Acceleration

Although physicists have been aware of cosmic rays for 75 years or so, it is only in the past 15 years that an understanding of how they are accelerated has developed. The mechanism of diffusive shock acceleration was first suggested in the late seventies, and was based on a simple analysis of the transport equation, although its core idea goes back to Fermi (1949). See Drury (1983) and Jones & Ellison (1991) for reviews. Many other acceleration mechanisms have been suggested over the decades, but only diffusive shock acceleration has direct experimental evidence in its favor. This evidence is through observations made at the earth's bowshock, the boundary between the region dominated by earth's magnetic field and the solar wind (Ellison, Möbius and Paschmann, 1990).

As the name suggests, the site of diffusive shock acceleration is shock waves in the space plasma. These shocks are known as 'collisionless shocks'. The microphysics of the environment of the collisionless shock is different than that for conventional gas shocks. However, one can apply the usual conservation equations to collisionless shocks and find equations relating the upstream and downstream properties, the Rankine-Hugoniot equation for example. An excellent introduction to the physics of collisionless shocks is the recent article by Sagdeev and Kennel (1991). One may refer to Boyd and Sanderson (1969) for the taxonomy of these shocks, and to learn about the important role played by the magnetic field.

The cosmic-ray spectrum shows a powerlaw over many decades of energy (see Appendix F). Diffusive shock acceleration predicts just such a powerlaw, and the prediction is fairly robust. Although the transport equation depends on the assumed scattering through the diffusion coefficient, this assumption does not enter in the predicted value for the spectral index of the powerlaw. The spectral index depends only on the compression ratio of the shock at which acceleration occurs. The observed spectral index is consistent with realistic values for the compression ratios of astrophysical shocks.

The derivation of the powerlaw follows quite simply from the transport equation, and I'll give it here. Assume a one-dimensional parallel shock (the magnetic field is parallel to the shock normal). Then the diffusion coefficient along the flow is unaffected by the magnetic field. Look for steady-state solutions. With these assumptions, (3.2) takes the form:

$$\frac{\partial}{\partial x}\left(fU - \kappa\frac{\partial f}{\partial x}\right) = \frac{\partial U}{\partial x}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f}{3}\right) \tag{3.4}$$

The velocity of the fluid is assumed to be a step function $\Theta(x)$ at $x = 0$ so that the derivative is a delta function at $x = 0$:

$$U(x) = U_1 + (U_2 - U_1)\Theta(x) \Longrightarrow \frac{\partial U}{\partial x} = (U_2 - U_1)\delta(x)$$

The fluid streams in from $x = -\infty$ with velocity U_1 and undergoes a discontinuous transition at $x = 0$, streaming away to $x = +\infty$ with velocity U_2; $U_1 > U_2$. Assume $f(x < 0) = f_1$ and $f(x \geq 0) = f_2$. Then integrate (3.4) from $-\infty$ to $+\infty$, and assume all gradients vanish there. The result is:

$$f_2U_2 - f_1U_1 = (U_2 - U_1)\frac{1}{p^2}\frac{\partial}{\partial p}(p^3 f_2/3) \tag{3.5}$$

The solution to this equation is:

$$f_2(p) = p^{-a}\int_0^p a f_1(x) x^{a-1}dx + Cp^{-a} \tag{3.6}$$

$$a \equiv \frac{3U_1}{U_1 - U_2}$$

Here C is a constant. So the downstream spectrum is a powerlaw, obtained by convolving the upstream spectrum with a powerlaw. The spectral index depends only on the compression ratio. For a strong shock $U_1/U_2 \sim 4$ and $a \sim 4$. In fact, the observed spectral index is about 4.6 (see Appendix F). A compression ratio of 2.9 would yield the observed spectral index.

However, if cosmic rays are accelerated in this way, they are probably accelerated sequentially at more than one shock. So the spectrum is not indicating any one shock, but rather some sort of mean of all the shocks in the galaxy. The conventional wisdom is that the acceleration site for particles up to about 10^{16} eV is in the shocks associated with supernova remnants.

The physical picture of this acceleration mechanism is that particles can diffuse back and forth across the shock. They get a small 'kick' at each crossing of the shock, a kick in momentum of about $m(U_1 - U_2)$. This is a small increment and it takes many crossings to boost a particle to 10^{15} eV. Fewer and fewer particles make the more and more crossings that are necessary to get to ever higher energies. Thus the negative slope of the spectrum. It is reflecting the low probability of a particle making the necessary number of crossings to get to high energy. And here lies the limiting factor for this mechanism. The large number of crossings necessary for a particle to get to high energies means that the acceleration process is not instantaneous, but gradual. Assuming a particle stays in a single acceleration site, the shock itself must live long enough for particles to make the many necessary crossings it takes to move to high energies. This is why this mechanism as applied to supernova remnants can only explain the particles up to about 10^{16} eV. The remnants don't last long enough to get particles beyond this energy. Therefore to explain the acceleration of particles beyond this energy, some look for longer-lived structures in the universe. The transport equation allows one to understand how particles, interacting in a common astrophysical environment, may be accelerated into a powerlaw distribution. However, the transport equation is a 'test-particle' equation because the fluid velocity that enters this equation is independent of the cosmic-ray distribution. It turns out that diffusive shock acceleration is quite efficient and can deposit a large amount of the incoming fluid kinetic energy into the energy of cosmic rays (Drury, Markiewicz and Volk, 1989; Ellison, Möbius and Paschmann, 1990; Jones and Kang, 1990). Therefore the flow, losing significant amounts of energy to cosmic rays, will be modified. This modified flow will in turn affect the acceleration process.

It is to better understand the self-consistent interaction between space plasmas and superthermal particles that I have worked on developing a set of equations that treats the entire system as a single fluid. One can obtain spectral information for the entire system, including cosmic rays, as well as calculate the dynamics of the system, when a significant fraction of the internal energy is contained in a very small fraction of the particles.

Earlier attempts along these same lines have lead to two-fluid descriptions of this interaction, which I outline below.

3.5) Two-Fluid Models

The development of the two-fluid models was an attempt to account for the effect that the acceleration of particles would have on a shock in the thermal plasma. This lead to calculations of the modifications of 'shock structure' which resulted from the influence of the cosmic rays (Drury & Volk, 1981; Axford, Leer and McKenzie, 1982; Drury, Axford and Summers, 1982; Drury, 1983). The equations for the two-fluid system are as follows.

There is a continuity equation for the thermal plasma with mass density ρ_{th}:

$$\frac{\partial \rho_{th}}{\partial t} + \frac{\partial}{\partial x^a}(\rho_{th} U^a) = 0 \tag{3.7}$$

It is assumed that the cosmic-ray mass density is negligible.

A term for the cosmic-ray contribution to the pressure, P_{cr}, is inserted into the momentum equation for the thermal plasma:

$$\rho_{th}\frac{dU^a}{dt} = -\frac{\partial}{\partial x^a}(P_{th} + P_{cr}) \tag{3.8}$$

An equation for the internal energy of the thermal plasma, T_{th}, is written down that is completely decoupled from the cosmic-ray energy:

$$\frac{\partial T_{th}}{\partial t} + \frac{\partial}{\partial x^a}(T_{th}U^a) + P_{th}\frac{\partial U^a}{\partial x^a} = 0 \tag{3.9}$$

An energy equation for the internal energy of the cosmic-rays, T_{cr}, is obtained by multiplying the transport equation (3.1) by the kinetic energy per particle and integrating over all momenta.

$$\frac{\partial T_{cr}}{\partial t} + \frac{\partial}{\partial x^a}(T_{cr}U^a) + P_{cr}\frac{\partial U^a}{\partial x^a} = \frac{\partial}{\partial x^a}\left(\overline{\kappa}\frac{\partial T_{cr}}{\partial x^a}\right) \tag{3.10}$$

$$4\pi\int(mc^2 - m_0c^2)\kappa\frac{\partial f_{cr}}{\partial x^a}p^2dp \equiv \overline{\kappa}4\pi\int(mc^2 - m_0c^2)\frac{\partial f_{cr}}{\partial x^a}p^2dp = \overline{\kappa}\frac{\partial T_{cr}}{\partial x^a}$$

This last expression defines $\overline{\kappa}$. This system ignores the magnetic field in the dynamics. To close the equations, a couple of equations of state are assumed:

$$P_{th} = (\gamma_{th} - 1)T_{th}$$

$$P_{cr} = (\gamma_{cr} - 1)T_{cr}$$

Although $\overline{\kappa}$ and γ_{cr} depend on the cosmic-ray distribution, values for these parameters are simply assumed at the outset. Thus the two-fluid system is partly 'linearized'.

The two-fluid model as presented above is the differential-equation approach to investigating the back-reaction of cosmic rays on the shock in which they are accelerated. The basic result of calculations in the context of two-fluid models is that the shock is smoothed by the cosmic-ray pressure. The accelerated particles are able to diffuse from downstream into the upstream region where their contribution to the pressure is able to slow the incoming gas. Thus the velocity profile goes from discontinuous to smooth.

3.6) Self-Consistent Numerical Models

The non-linearities of the cosmic-ray/thermal plasma system has lead to research into their interaction by developing numerical models of the system. Notable is the work of Ellison who has developed self-consistent models that make no artificial divisions between the thermal and superthermal particles (see Ellison & Reynolds, 1991, for the latest version). The models assume a system of protons, all scattering under a single scattering law which is assumed to be completely elastic. Elastic scattering is thought to represent the nature of the scattering in space plasmas, because the particles scatter off irregularities in the magnetic field.

This thesis may be likened to a differential-equation version of Ellison's models because I attempt to show how a transport equation and fluid equations can be obtained from a single starting point. The transport equation and fluid equations apply to particles of all energies. We both assume a single scattering law for all particles, and treat the system as consisting wholly of protons. Ellison has concentrated on using his model to study processes at quasi-parallel steady-state shocks. Ellison finds that a thermal distribution upstream leads naturally to a powerlaw downstream. There is no other process in his models than simple random elastic scattering. Conservation of momentum and energy is mandated to within 10%. Starting with an initial discontinuous velocity profile, he runs his code until it converges to a steady state that satisfies the conservation requirement. He has compared his results to observations at the earth's bowshock and found good agreement (Ellison, Möbius and Paschmann, 1990). He also finds that the incoming flow is slowed somewhat by diffusion of accelerated particles from downstream, as the two-fluid models predict. However, he also predicts that there is always a 'sub-shock' in the system. Although the incoming speed is diminished somewhat, it still makes a (smaller) discontinuous drop to the downstream speed. This is in opposition to the two-fluid models, which predict that completely smooth transitions exist.

One conclusion of these numerical models is that no seed population of semi-energetic particles is necessary to get the downstream powerlaw. It was thought for several years that since the cosmic-ray transport equation (3.1) applied only to energetic particles, that the solution (3.6) was only correct for energetic particles as well, and that f_1 must be a population of particles already slightly energized. Thus if particles could make it to, say, 10^6 eV or so, one can understand how they are moved to higher energies. But the question remained, how are particles extracted from the thermal pool in the first place? Ellison's numerical models confirm what had been predicted earlier by Eichler (1980), that no seed population was necessary. The solution (3.6) is qualitatively correct even if f_1 is thermal.

The view of Eichler and Ellison is that particle acceleration is a self-regulating process (Eichler, 1979). At an initially discontinuous transition in the flow, even the thermal particles have scattering mean free paths long enough for them to scatter across the velocity transition, and some of the thermal particles begin to be accelerated. As more particles are accelerated, the downstream pressure of these accelerated particles begins to build and to diffuse upstream. The incoming flow senses this pressure increase before it gets to the transition, and slows somewhat. Now the transition is smoothed and so the distance across the transition between the full upstream and downstream velocities has increased. Thus some particles may no longer have mean free paths long enough to sample the entire velocity change in a single scattering, and it becomes more difficult for these particles to become accelerated. However, the particles that were able to be accelerated from the thermal pool while the shock was still completely discontinuous may now have a mean free path large enough to allow them to sample the entire velocity difference in a single scattering, and they can be accelerated further. This assumes that the mean free path increases with energy. Now as more particles are accelerated to higher energies, the downstream pressure sends a stronger precursor into the upstream region, which slows the incoming plasma further upstream, and further smooths the transition. Thus even larger mean free paths are necessary to sample the entire transition, and the energy that a particle must have to participate in the acceleration process increases even more. Thus the acceleration process regulates itself in that acceleration of thermal particles is increasingly suppressed as more particles are accelerated.

The results of Ellison's work has a certain elegant appeal because it unifies the description of the cosmic-ray and thermal particles. But it is based on numerical models of billiard ball collisions, and it may not always be clear what underlying physical mechanism is responsible for a particular aspect of a result. Analytic equations have the advantage that the physics is exposed as a series of separate effects. Furthermore, the Monte Carlo models require large amounts of computing time on the fastest computers, whereas the PDEs describing a system can be solved, even numerically, with much less effort. I believe the unification of description is appropriate and I demonstrate in this work that the usual cosmic-ray transport theory and the dynamic equations for the thermal space plasma can be obtained from a single starting point. Furthermore, one can generalize the transport equation and the dynamical equations to include particles of *all* energies, and such equations are developed herein. Thus the equations presented here describe the spectral and dynamical evolution of the entire system of particles of all energies.

4. Central Idea

The key idea of this thesis is that a modified version of the usual diffusive-convective equation of cosmic-ray transport is valid not only for the high-energy particle population, but for the thermal population of the background fluid medium as well. The transport equation which has been so successful in understanding diffusive shock acceleration and cosmic-ray modulation can be extended to contain the spectral information for the entire system. Furthermore, fluid equations for the entire system of particles can be obtained from the same starting point that leads one to the general transport equation. In this way, a unified description of the system of cosmic rays and thermal plasma is obtained. This description contains acceleration of thermal particles to cosmic-ray energies, and self-consistently describes how this acceleration will affect the dynamics of the flow of the bulk plasma. The *cosmic-ray* transport equation is restricted to particles whose speeds are much greater than the fluid flow speed. This is not a necessary assumption to derive a transport equation, and I will relax the restriction on particle speeds. With no restriction on particle speeds, the background thermal plasma may be included in all results.

I want to show that starting from a kinetic equation, which describes the phase-space evolution of a system of particles, one can derive a general transport equation, treating particles of all energies, as well as the usual fluid equations of magnetohydrodynamics. There is a single pressure which includes contributions from particles of all energies. The fluid velocity is defined as an average over the velocities of particles of all energies. There are assumptions necessary for this derivation. The primary assumption is that the distribution function of the particles be nearly isotropic in momentum, with the momentum referred to the fluid frame. As is typical in fluid descriptions, this is equivalent to assuming that particle mean free paths are much less than fluid lengthscales. This is all that is necessary to connect the fluid equations with the transport equation. I will further assume that the fluid velocity is non-relativistic, for computational (and intuitional) simplicity.

I think it is useful for the reader to see the end results of the following derivations before embarking on them. The following equations are meant to describe the co–evolution of the distribution function f_0, and the fluid velocity of the scattering centers U^a (as well as pressure, density and other fluid quantities) of a neutral space plasma composed only of protons (the electrons are ignored). The particle energies are arbitrary. It is assumed that the fluid velocity is much less than the speed of light. The momentum dependence of the distribution is presumed to be nearly isotropic referred to the frame moving with the fluid velocity. A scattering is presumed and characterized by τ, the timescale for the distribution to relax to isotropy. The hydromagnetic condition is presumed to hold, and an average magnetic field B^a may exist in the system.

Here are the key equations to be derived in this work, with the equation numbers referring to where they occur in the following derivations. I will not trouble with defining all quantities because that is done following, but the notation is standard.

$$\frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} + \frac{1}{m}\frac{\partial S^a}{\partial x^a} = \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} + m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab}) \tag{7.11}$$

$$S^a = -m\kappa_{ab}\left[\frac{\partial f_0}{\partial x^b} - \frac{m^2}{p}\frac{\partial f_0}{\partial p}\frac{dU^b}{dt}\right] \tag{9.2}$$

$$\Pi_{ab} = \Upsilon_{abcd}\frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{cd} \tag{10.5}$$

This relates the evolution of the isotropic part of the particle distribution function to the fluid velocity of the scattering centers. Since the fluid velocity depends on the particle distribution, (7.11) is non-linear. The diffusion tensor κ_{ab} and the viscosity tensor Υ_{abcd} describe the anisotropic transport that occurs in the presence of a magnetic field.

The transport equation (7.11) is consistent with a set of fluid equations describing the conservation of particle number, momentum and energy.

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^a}(nU^a + G^a) = 0 \tag{8.2}$$

$$\frac{\partial}{\partial t}(\rho U^a + F^a) + \frac{\partial}{\partial x^b}(\rho U^a U^b + P_{ab} + F^a U^b + F^b U^a + P^B{}_{ab}) = 0 \tag{8.12}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\rho U^2}{2} + T + F^a U^a + \frac{B^2}{8\pi}\right) \\ &+ \frac{\partial}{\partial x^a}\left(\frac{\rho U^2 U^a}{2} + P_{ab}U^b + TU^a + Q^a + \frac{1}{2}F^a U^2 + U^a U^b F^b + P^B{}_{ab}U^b\right) = 0 \end{aligned} \tag{8.13}$$

$$P^B{}_{ab} \equiv \frac{1}{4\pi}\left\{\frac{1}{2}B^2\delta_{ab} - B^a B^b\right\}$$

These are supplemented by the induction equation for the evolution of the magnetic field.

$$\frac{\partial B^a}{\partial t} = \frac{\partial}{\partial x^b}(U^a B^b - U^b B^a) \tag{8.14}$$

The fluid equations are all defined in terms of integrals over the distribution function f_0 which appears in (7.11). These integrals are defined in (6.2). The term G^a appears in the number equation because the fluid velocity is chosen to be the frame of the scattering centers. Typical formulations of fluid dynamics choose the fluid velocity such that $G^a \equiv 0$. The difference between these two choices is actually a small quantity, as is shown later.

I present a prescription for performing self-consistent calculations of both f_0 and U^a. It turns out that the equations to be solved simulataneously are the transport equation (7.11) for f_0, and the momentum equation (8.11) for U^a. These four equations determine the self-consistent evolution of the particle distribution and fluid velocity. The number and energy equations are implicit in (7.11) and (8.11). The momentum equation can be used to replace U^a in the general transport equation (7.11) with other fluid quantities, such as the number density. Then the transport equation is explicitly a non-linear integro-differential equation in f_0. This is the non-linearity necessary to account for the effect of particle acceleration on fluid dynamics. These fluid equations and the transport equation are all derived from the same starting point under the same assumptions. They are consistent. The equation of cosmic-ray transport follows when (7.11) is restricted to particles with velocities much greater than the fluid velocity, but there is no separation of particles based on their energy. The distribution function and all fluid quantities are for particles of arbitrary energy.

5. The Kinetic Equation

The starting point is what I will call the kinetic equation; some may call it the Boltzmann equation. It describes the evolution of a distribution of particles in phase space that is a function of time, space, and momentum: $f(t, x^a, \widetilde{p}^a)$. The momentum of the particles is fully relativistic. A general kinetic equation for a single species of particles may be written:

$$\frac{\partial f}{\partial t} + \frac{\widetilde{p}^a}{\widetilde{m}}\frac{\partial f}{\partial x^a} + F^a \frac{\partial f}{\partial \widetilde{p}^a} = \left(\frac{\delta f}{\delta t}\right)_{scatter} \tag{5.1}$$

The term on the RHS represents scattering that will change the distribution on some timescale different than the timescales of the LHS. For a justification of this equation, see Appendix A. The quantity $\widetilde{m}$ is the mass of the particles, which is a function of their momentum: $\widetilde{m}(\widetilde{p}) = (1 + \widetilde{p}^2/{m_0}^2c^2)^{1/2}$. Their rest mass is denoted by m_0, the speed of light by c. The term F^a is an external force acting on the particles. t, x^a, and $\widetilde{p}^a$ are all referenced to the same inertial frame. The tilda anticipates a transformation of the momentum coordinate.

Space plasmas are highly conductive media, and the electric field is virtually zero in the fluid frame, $E^a \simeq 0$. Therefore, particle energies are conserved when referred to the fluid frame. For this reason, I will transform the momentum coordinate of equation (5.1) into the frame moving with fluid velocity U^a:

$$\widetilde{p}^a = m(p)U^a(x^b, t) + p^a \tag{5.2}$$

Now $m = (1 + p^2/{m_0}^2c^2)^{1/2}$. The transformation (5.2) is an approximation to the exact relativistic transformation, correct to order U/c. See Appendix B for details on the relativistic transformation of vectors. For a highly conducting fluid with embedded magnetic field B^a, the electric field $\widetilde{E}^a$ in the inertial (observer) frame is given by the hydromagnetic condition:

$$\widetilde{E^a} = \epsilon_{abc}\frac{U^c}{c}B^b \simeq \epsilon_{abc}\frac{U^c}{c}\widetilde{B^b} \tag{5.3}$$

To order U/c, B^a is the same in both frames. For a derivation of the hydromagnetic condition and transformation properties of the electromagetic field, see Appendix B.

When average magnetic and electric fields are present in the plasma, the external force F^a on the particles is the Lorentz force:

$$F^a = \frac{q}{\widetilde{m}c}\epsilon_{abc}\widetilde{p^b}\widetilde{B^c} + q\widetilde{E^a} \simeq \frac{q}{mc}\epsilon_{abc}p^b B^c \tag{5.4}$$

where q represents particle charge. Now it remains to transform (5.1) so that the momentum coordinate is referenced to the fluid frame U^a. This is done in Appendix C; the result is:

$$\frac{\partial f}{\partial t} + U^a\frac{\partial f}{\partial x^a} + \frac{p^a}{m}\frac{\partial f}{\partial x^a} + \left[F^a - m\frac{dU^a}{dt} - p^b\frac{\partial U^a}{\partial x^b}\right]\frac{\partial f}{\partial p^a} = \left(\frac{\delta f}{\delta t}\right)_{scatter} \tag{5.5}$$

where the convective derivative is defined:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U^b\frac{\partial}{\partial x^b}$$

In (5.5), terms of order U/c have been ignored, so it is restricted to non-relativistic flows.

Keep in mind that in (5.5) the time and space coordinates, as well as U^a, are referred to an inertial frame. The momentum coordinate is referred to the frame moving with velocity U^a. So this equation mixes reference frames. Although it sounds unusual, one also encounters such behavior in the cosmic-ray transport equation and the fluid equations.

6. The Distribution Function

The momentum dependence of the distribution function may be expressed generally as a sum of momentum-space spherical harmonics. The motivation for transforming the momentum dependence into a fluid frame is to allow for isotropy of the momentum dependence in the fluid frame. If the distribution is isotropic in the fluid frame, then the spherical harmonic series will be convergent.

I regard the distribution as being composed of the first three spherical harmonics. That this is adequate to approximate the total distribution will rely on the distribution being nearly isotropic so that the series of spherical harmonics converges. The reason for choosing the first three is that these three have familiar interpretations as the number density, momentum density and pressure. Mathematically, one may improve the approximation by truncating the series beyond the third term, but the new terms do not have easy physical interpretations. Furthermore, only the first three harmonics enter the generalized transport equation because the transport equation will turn out to be the zeroth moment of the kinetic equation.

The distribution may then be written:

$$f(x^a, p^a, t) = f_0(x^a, p, t) + \frac{3p^b}{p^2} S^b(x^a, p, t) + \frac{15p^a p^b}{2p^4} \Pi_{ab}(x^a, p, t) \qquad (6.1)$$

$$\Pi_{ab} = \Pi_{ba} \qquad \Pi_{aa} = 0$$

This yields the following moments of the distribution:

$$
\begin{aligned}
n &\equiv \int f d^3p = 4\pi \int f_0 p^2 dp \Longrightarrow \textit{number density} \\
\rho &\equiv \int m f d^3p = 4\pi \int m f_0 p^2 dp \Longrightarrow \textit{mass density} \\
P_{ab} &\equiv \int f \frac{p^a p^b}{m} d^3p = 4\pi \int \left(\frac{p^2 f_0}{3m} \delta_{ab} + \frac{\Pi_{ab}}{m} \right) p^2 dp \\
&\equiv P\delta_{ab} + 4\pi \int (\Pi_{ab}/m) p^2 dp \Longrightarrow \textit{pressure} \\
F^a &\equiv \int f p^a d^3p = 4\pi \int S^a p^2 dp \Longrightarrow \textit{momentum density} \\
G^a &\equiv \int f \frac{p^a}{m} d^3p = 4\pi \int \frac{S^a}{m} p^2 dp \Longrightarrow \textit{number flux} \\
T &\equiv \rho c^2 - n m_0 c^2 \Longrightarrow \textit{kinetic energy density} \\
Q^a &\equiv F^a c^2 - G^a m_0 c^2 \Longrightarrow \textit{kinetic energy flux}
\end{aligned}
\tag{6.2}
$$

For $d^3p = p^2 dp d\Omega$, one can also see that:

$$
\begin{aligned}
\frac{1}{4\pi} \int f d\Omega &= f_0 \\
\frac{1}{4\pi} \int f p^a d\Omega &= S^a \\
\frac{1}{4\pi} \int f p^a p^b d\Omega &= f_0 \frac{p^2}{3} \delta_{ab} + \Pi_{ab}
\end{aligned}
\tag{6.3}
$$

7. The Transport Equation

Next follows the development of a transport equation from the kinetic equation. This transport equation describes the evolution of the isotropic part of the distribution. When the fluid is nearly isotropic, the equation approximates the evolution of the total distribution.

These are the same qualities that the diffusive-convective equation of cosmic-ray transport (sometimes called the Parker equation) possesses. The cosmic-ray transport equation has been historically applied to particles whose speeds are much greater than the flow speed that appears in that equation. The transport equation about to be presented places no restriction on particle speeds, and describes particles of all energies, from thermal to cosmic-ray.

7.1) Scattering Term

To derive the transport equation the starting point is (5.5) but now a specific form is required for the scattering term. For the scattering, I will choose:

$$\frac{Df}{Dt} = \left(\frac{\delta f}{\delta t}\right)_{scatter} = \frac{f_0 - f}{\tau} \tag{7.1}$$

Here, f_0 is defined by (6.3), and $\tau = \tau(p)$. The LHS of (7.1) represents the LHS of (5.5). The RHS of (7.1) is known alternately as the relaxation-time approximation or the BGK operator (Bhatnagar, Gross & Krook, 1954). The physical interpretation is that in a time τ, particles are removed from phase space with their distribution f, and reappear with the isotropic distribution f_0. Thus the distribution relaxes to isotropy in a time τ. The advantage of (7.1) is the computational simplicity it allows. For typical space plasmas, τ will scale with the particle gyro-period in the local magnetic field. There is a lot of complex physics not yet understood that is embedded in the expression (7.1). This scattering term is isotropic in the fluid frame.

Choosing the relaxation operator to represent the scattering term will impose some conditions on the expansion (6.1) of the distribution. The assumption is made that the scattering law will conserve number, momentum and energy. In other words, the system does not gain or lose particles, momentum, or energy by virtue of its scattering. This is expressed:

$$\int \left(\frac{\delta f}{\delta t}\right)_s (1, \tilde{p}^a, \tilde{m}c^2) d^3\tilde{p} = 0 \tag{7.2}$$

With τ a function of the magnitude of momentum only, the relaxation operator satisfies the number and energy restrictions trivially. The condition that the scattering not contribute momentum to the system demands:

$$\int \frac{S^a}{\tau} p^2 dp = 0 \tag{7.3}$$

7.2) Transport Equation

The derivation of the transport equation is detailed in Appendix E. To get the transport equation from (5.5), one merely averages over all momentum-space solid angle. With the operator (7.1), the integral of (5.5) takes the form:

$$\frac{1}{4\pi} \int \frac{Df}{Dt} d\Omega = 0 \tag{7.4}$$

The transport equation that (7.4) implies is (see Appendix E):

$$\frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} - \frac{\partial U^a}{\partial x^a} \frac{p}{3} \frac{\partial f_0}{\partial p} + \frac{1}{m} \frac{\partial S^a}{\partial x^a} - m \frac{dU^a}{dt} \frac{1}{p^2} \frac{\partial}{\partial p}(pS^a) - \frac{\partial U^a}{\partial x^b} \frac{1}{p^2} \frac{\partial}{\partial p}(p\Pi_{ab}) = 0 \tag{7.5}$$

It is instructive to write (7.5) in a conservation form:

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial x^a}\left(f_0 U^a + \frac{S^a}{m}\right) = m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f_0}{3}\delta_{ab} + p\Pi_{ab}\right) \tag{7.6}$$

This equation, the zeroth moment of the kinetic equation, is an exact equation that does not assume the distribution is nearly isotropic, or that the series of spherical harmonics converges. Such approximations enter when expressions for S^a and Π_{ab} in terms of f_0 are chosen. When $\Pi_{ab} = 0$ and $S^a = m\kappa_{ab}\partial_b f_0$, (7.6) reduces to the usual diffusive-convective equation of cosmic-ray transport. The streaming flux S^a is then proportional to the gradient of the distribution and κ_{ab} is the diffusion tensor. When $\Pi_{ab} = -\xi(\partial_a U^b + \partial_b U^a - 2/3\partial_c U^c \delta_{ab})$ and $S^a = -m\kappa_{ab}\partial_b f_0 + \zeta dU^b/dt$, (7.6) reduces to the extended transport equation introduced by Earl, Jokipii and Morfill (1988). The coefficients ζ and ξ depend on the scattering and on f_0. But again, all these results assumed the transport equation applied only to fast particles, $U \ll p/m$. There is nothing in the derivation of (7.5,6) that restricts particle speed.

7.3) Streaming Flux

The transport equation contains two as-yet-undefined quantities: S^a and Π_{ab}. When the distribution is nearly isotropic, these may be solved for in terms of f_0. The equation that yields S^a is:

$$\frac{1}{4\pi}\int p^a \frac{Df}{Dt} d\Omega = -\frac{S^a}{\tau} \tag{7.7}$$

In Appendix E it is shown that:

$$-\frac{S^a}{\tau} \simeq \frac{p^2}{3m}\frac{\partial f_0}{\partial x^a} - \frac{mp}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p} - \frac{q}{mc}\epsilon_{abc}S^b B^c \tag{7.8}$$

7.4) Small Parameters, Lengthscales, Timescales

(7.8) is an approximation to the actual S^a, and the validity of the approximation may be expressed in terms of the small parameter λ/L, defined below.

The spatial scale associated with the derivatives on the LHS of (7.1) (contained in Df/Dt) is denoted L. Strictly speaking, f_0, S^a, Π_{ab}, and U^a all may have a different spatial scale, but I ignore this possibility. For the arguments that follow, one can assume that L is the smallest of all the spatial scales. The mean free path of the particles is denoted by λ, which is related to the scattering time $\tau(p)$: $\lambda \equiv p\tau/m$.

With an exception discussed below, I also assume a single characteristic timescale for Df/Dt, denoted t. Again, there may be several different timescales associated with the changes in f and U^a, but I assume one. This will imply a relation between L and t: $L = Ut$. Thus there exists the parameter τ/t, which is related to λ/L by $(\lambda/L)(mU/p) = \tau/t$. I take the fundamental small parameter as λ/L, and assume $\lambda/L \ll 1$. With no restriction on mU/p, $\tau/t \leq \lambda/L$. This notes that the thermal particles in space plasmas have speeds approximately that of the flow speed. Negligibly few particles have speeds much less than the flow speed (flow speeds are often hundreds of kilometers per second, temperatures are $\sim 10^5$ Kelvin).

Actually, there are at least two characteristic timescales for Df/Dt. The timescale associated with the Lorentz term is independent of the timescales associated with the derivatives in Df/Dt. I allow the timescale associated with the Lorentz term to be arbitrary. Indeed, the scattering time τ will generally be correlated with the particle gyro-frequency in space plasmas. In view of this, (7.8) implies that $S \sim (\lambda/L)pf_0$. Further, (7.8) is a solution to (7.7) that ignores terms of order λ/L smaller than the largest.

7.5) Particle Pressure

An equation for Π_{ab} may also be obtained from the kinetic equation:

$$\frac{1}{4\pi}\int p^a p^b \frac{Df}{Dt} d\Omega = -\frac{\Pi_{ab}}{\tau} \tag{7.9}$$

In Appendix E, an approximation to the solution to (7.9) is found:

$$\begin{aligned}
-\frac{\Pi_{ab}}{\tau} \simeq & -\frac{p^3}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} - \frac{p^2}{5m}\left[\frac{\partial S^a}{\partial x^b} + \frac{\partial S^b}{\partial x^a} - \frac{2}{3}\frac{\partial S^c}{\partial x^c}\delta_{ab}\right] \\
& + \frac{mp^3}{5}\left[\frac{dU^a}{dt}\frac{\partial}{\partial p}\left(\frac{S^b}{p^2}\right) + \frac{dU^b}{dt}\frac{\partial}{\partial p}\left(\frac{S^a}{p^2}\right) - \frac{2}{3}\frac{dU^c}{dt}\frac{\partial}{\partial p}\left(\frac{S^c}{p^2}\right)\delta_{ab}\right] \qquad (7.10) \\
& + \frac{q}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)
\end{aligned}$$

$$\Lambda_{ab} \equiv \frac{\partial U^a}{\partial x^b} + \frac{\partial U^b}{\partial x^a} - \frac{2}{3}\frac{\partial U^c}{\partial x^c}\delta_{ab}$$

(7.10) is an approximation to the solution to (7.9) that ignores terms of order λ/L smaller than the largest. From (7.10) note that the magnitude of $\Pi_{ab} \sim (\tau/t)p^2 f_0 \leq (\lambda/L)p^2 f_0$.

7.6) Building a Transport Equation Correct to Order λ/L

Putting the solutions to (7.7) and (7.9) into (7.6) will yield the particle transport equation. However, when $\lambda/L \ll 1$, approximations (7.8) and (7.10) to the solutions to (7.7) and (7.9) may be used. There are two terms in the transport equation (7.5) in S^a, and one in Π_{ab}. All the terms of (7.6) in f_0 have sizes $\sim f_0/t$. Using approximate expressions for S^a and Π_{ab} like those in (7.8) and (7.10), one may substitute back into (7.5) to yield a transport equation correct to some order in λ/L. That is what is done in this section.

Starting with the approximation for S^a, (7.8), which ignores terms of order λ/L, one can write the sizes of S^a relative to pf_0 implied by the first two terms on the RHS as: λ/L for the gradient term and $(\tau/t)(mU/p)$ for the acceleration term. The reason for not including the Lorentz term is that the timescale associated with that term may be as small as the scattering time. The size of the Lorentz term is independent of λ/L, and does not enter these ordering arguments.

Substituting these two different-sized approximations for S^a into the two terms in S^a in (7.5), there arise four terms. They are to be compared with the size of f_0/t. Therefore I display the sizes of these four terms relative to f_0/t in the following chart:

terms in (7.6) $\Rightarrow$	$\partial_a S^a/m$	$m(dU^a/dt)\partial_p(pS^a)/p^2$
$\frac{p^2}{3m}\frac{\partial f_0}{\partial x^a} \Longrightarrow$	$\frac{\lambda}{L}\frac{p}{mU}$	$\frac{\tau}{t}$
$\frac{mp}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p} \Longrightarrow$	$\frac{\tau}{t}$	$\frac{\tau}{t}\frac{m^2U^2}{p^2}$

From this one sees that since τ/t is linear in λ/L, all terms are of order λ/L smaller than f_0/t. One also sees that if the transport equation were to be restricted to particles with $p \gg mU$, that a single term would dominate the others by order m^2U^2/p^2. This term is the diffusion term in the convective-diffusive equation of cosmic-ray transport. If terms of order $(\lambda/L)^2$ had been kept in the approximation (7.8), then it would have introduced terms quadratic in λ/L into the transport equation.

Now examine (7.10), the approximation for Π_{ab}. The sizes of the first three terms on the RHS relative to $p^2 f_0$ are: τ/t, λ^2/L^2, and τ^2/t^2, respectively. Again, the size of the Lorentz term does not enter into the ordering arguments.

There is a single term in Π_{ab} in (7.5). When the three terms of the approximation (7.10), of sizes quoted above, are substituted into (7.5), there arise three terms of sizes: τ/t, λ^2/L^2, and τ^2/t^2 relative to f_0/t. Only the first is linear in λ/L, and only this term is kept.

Therefore, I write out the particle transport equation, correct to first order in λ/L and valid for particles of all energies.

$$\frac{\partial f_0}{\partial t} + U^a\frac{\partial f_0}{\partial x^a} + \frac{1}{m}\frac{\partial S^a}{\partial x^a} = \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} + m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab}) \tag{7.11}$$

$$S^a = -\frac{p^2\tau}{3m}\frac{\partial f_0}{\partial x^a} + \frac{mp\tau}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p} + \frac{q\tau}{mc}\epsilon_{abc}S^bB^c$$

$$\Pi_{ab} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} - \frac{q\tau}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)$$

If this equation were restricted to particles of $p \gg mU$, and terms of order m^2U^2/p^2 were eliminated, only the gradient part of S^a would survive, Π_{ab} would vanish, and it would reduce to the usual diffusive-convective equation of cosmic-ray transport.

7.7) Implications of the Scattering Constraint on S^a

Back in §3.3 I discussed how the scattering term must conserve momentum, number and energy. This constraint was expressed as a restriction on S^a/τ in (7.3). The expression (7.8) for S^a is correct to first order in τ, and calling the expression that appears in (7.8) S_1, we can say $S = S_1 + O(\tau^2)$ (I am suppressing the vector notation for now). Approximating S with S_1 is all that is necessary to obtain a transport equation that is correct to first order in τ. Indeed, I'll go on to formulate the fluid equations as well to only first order in τ. But the constraint (7.3), to first order in τ, will involve the correction S_2 to the approximation to S that is of order τ^2:

$$\int \frac{S}{\tau}p^2\,dp = \int\left(\frac{S_1}{\tau} + \frac{S_2}{\tau}\right)p^2\,dp + O(\tau^2) = 0 \qquad (7.12)$$

The quantity S_2 is never obtained in this work, because it is never needed. The point is that (7.12) implies that the integral of S_1 is not zero:

$$\int \frac{S_1}{\tau}p^2\,dp = O(\tau)$$

With this, we are ready to make a connection between expression (7.8) for S^a and the momentum equation:

$$4\pi\int \frac{{S_1}^a}{\tau}p^2\,dp = \frac{\partial P}{\partial x^a} + (\rho + P/c^2)\frac{dU^a}{dt} - \frac{q}{c}\epsilon_{abc}G^bB^c = O(\tau) \qquad (7.13)$$

This is just the inviscid momentum equation of MHD (with a relativistic correction), when the quantity qG^a is identified as the current generated by the fluid-frame number flux. The viscous corrections to the momentum equation enter at order τ. The full momentum equation, correct to order τ, is derived later. But this connection between the fluid-frame streaming flux and the momentum equation allows the following interpretation of S^a. Consider the expression (7.8) for S^a. The first term on the RHS represents a diffusive streaming of particles, in the fluid frame, due to a gradient in the distribution. This flux of particles carries momentum. The diffusive streaming of particles with respect to the fluid must be compensated by a recoil of the fluid. This is analagous to a mass sliding down an inclined plane. If the plane itself is free to move, it will recoil with enough momentum to compensate the momentum gained by the sliding mass. I interpret the second term on the RHS of (7.8) as the recoil due to diffusion of particles. The momentum associated with the recoil of all particles must balance the momentum associated with the diffusive streaming. This is the nature of the balance which is demanded by (7.3).

So the equation (7.11) is consistent with requiring a recoil of the fluid to balance the diffusive momentum density; and is also equivalent to assuming that the scattering term contributes no net momentum (as seen in the inertial frame) to the fluid. As noted in §7.6 above, the size of the acceleration term in (7.8) relative to the diffusion term is m^2U^2/p^2. At high values of momenta, the diffusive streaming dominates. At thermal values of momenta, the two are comparable. However, there are typically orders of magnitude more particles with thermal momenta than with very large (cosmic-ray) momenta. The net result is that the momentum associated with the diffusive streaming of cosmic rays through the fluid is exactly compensated by a recoil of the entire gas. Since there are many more thermal particles than cosmic-ray particles, one can approximately say that the momentum of streaming of high-energy particles is compensated by a recoil of the thermal population.

8. The Fluid Equations

8.1) The Conservation Equations

The equations for the conservation of particle number, momentum and kinetic energy are obtained by multiplying (5.5) by an appropriate factor and integrating over all momenta. Although the transport equation has been transformed to a new momentum coordinate, the integrals over all momenta still involve the inertial frame momentum $\tilde{p}^a$, because the kinetic equation is still referred to the inertial frame.

The conservation properties of the scattering law expressed in (7.2) will imply the equations:

$$\int \frac{Df}{Dt}(1, \tilde{p}^a, \tilde{m}c^2 - m_0c^2)d^3\tilde{p} \simeq \int \frac{Df}{Dt}(1, p^a + mU^a, mc^2 + p^aU^a - m_0c^2)d^3p = 0 \tag{8.1}$$

These are the conservation equations for the fluid. The approximation follows from the assumption that U/c is small. See Appendix B for transformation of the volume element.

Following are the conservation equations, represented by the integrals in (8.1). The evaluation of the integrals leading to these equations may be found in Appendix D. The first of the conservation equations indicated in (8.1) is conservation of number:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^a}(nU^a + G^a) = 0 \tag{8.2}$$

Next comes conservation of momentum:

$$\frac{\partial}{\partial t}(\rho U^a + F^a) + \frac{\partial}{\partial x^b}(\rho U^a U^b + P_{ab} + F^a U^b + F^b U^a) = \frac{q}{c}\epsilon_{abc}G^b B^c \tag{8.3}$$

The final equation indicated in (8.1) is conservation of kinetic energy:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2 + T + F^a U^a\right) - \frac{q}{c} U^a \epsilon_{abc} G^b B^c \\ + \frac{\partial}{\partial x^a}\left(\frac{1}{2}\rho U^2 U^a + \frac{1}{2} F^a U^2 + U^a U^b F^b + P_{ab} U^b + T U^a + Q^a\right) = 0\end{aligned} \tag{8.4}$$

It will also be useful to have the equation for the conservation of mass (total relativistic energy):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^a}(\rho U^a + F^a) + \frac{\partial U^a}{\partial x^b}\frac{P_{ab}}{c^2} = 0 \tag{8.5}$$

Since the particles may be relativistic, this is not necessarily proportional to the number equation.

One can use (8.3) and (8.5) to rewrite the momentum equation into the familiar 'F=mA' form.

$$\rho \frac{dU^a}{dt} = -\frac{\partial}{\partial x^b} P_{ab} - \frac{dF^a}{dt} - F^a \frac{\partial U^b}{\partial x^b} - F^b \frac{\partial U^a}{\partial x^b} + \frac{1}{c}\epsilon_{abc} G^b B^c \tag{8.6}$$

These equations hold for any scattering term that satisfies the conservation properties (7.2) and for any distribution that may be expanded as in (6.1). The form of these equations is familiar. T is the internal energy, and Q^a is a heat flux. The quantities F^a and G^a are new. They are analogous to Q^a; whereas Q^a represents an energy flux in the frame of the fluid, F^a is a mass flux in the fluid frame, and G^a is a number flux in that frame.

The fluid equations (8.2,3,4) differ from the usual equations of non-relativistic fluid dynamics because of the presence of F^a and G^a. Later I will show that the presence of G^a is because I chose the frame of the scattering centers to define the fluid velocity. The presence of F^a is required for consideration of gases with relativistic internal energies. F^a and G^a will follow from S^a. The solution to

$$S^a = -\frac{p^2\tau}{3m}\frac{\partial f_0}{\partial x^a} + \frac{mp\tau}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p} + \frac{q\tau}{mc}\epsilon_{abc}S^b B^c$$

is:

$$S^a = m\kappa_{ab}\left[\frac{m^2}{p}\frac{dU^b}{dt}\frac{\partial f_0}{\partial p} - \frac{\partial f_0}{\partial x^b}\right]$$

where the anisotropic diffusion tensor κ_{ab} is defined in §9. Therefore:

$$F^a = 4\pi \int m\kappa_{ab}\left[\frac{m^2}{p}\frac{dU^b}{dt}\frac{\partial f_0}{\partial p} - \frac{\partial f_0}{\partial x^b}\right] p^2 dp \tag{8.7}$$

$$G^a = 4\pi \int \kappa_{ab}\left[\frac{m^2}{p}\frac{dU^b}{dt}\frac{\partial f_0}{\partial p} - \frac{\partial f_0}{\partial x^b}\right] p^2 dp \tag{8.8}$$

The pressure P_{ab} is specified by the approximation for Π_{ab} used in the transport equation (7.11). In §10 on space-plasma viscosity, it is shown that the solution to

$$\Pi_{ab} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} - \frac{q\tau}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)$$

is:

$$\Pi_{ab} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Upsilon_{abcd}\Lambda_{cd}$$

The full pressure tensor may be written:

$$P_{ab} = P\delta_{ab} - \eta_{abcd}\Lambda_{cd} \tag{8.9}$$

$$\eta_{abcd} \equiv -4\pi \int \frac{p^3\tau}{15m}\frac{\partial f_0}{\partial p}\Upsilon_{abcd} p^2 dp$$

The minus sign is introduced to keep η_{abcd} positive for reasonable distributions.

F^a and G^a are similar to the more-familiar heat flux vector in that they are proportional to gradients of other fluid quantities. For example, the heat flux vector is usually taken to be proportional to the gradient of the energy density, with the coefficient of proportionality a transport coefficient to be determined empirically. Indeed, in typical formulations of fluid dynamics, the particle distribution function and scattering law is unknown. By analogy, we may define some as-yet-undetermined diffusion tensors ${\sigma_{ab}}^i$, scattering-time tensors ${\theta_{ab}}^i$, and a viscosity tensor η_{abcd}:

$$\begin{aligned} G^a &= -{\sigma_{ab}}^1 \frac{\partial n}{\partial x^b} - {\theta_{ab}}^1 n \frac{dU^b}{dt} \\ F^a &= -{\sigma_{ab}}^2 \frac{\partial \rho}{\partial x^b} - {\theta_{ab}}^2 \rho \frac{dU^b}{dt} \\ Q^a &= -{\sigma_{ab}}^3 \frac{\partial T}{\partial x^b} - {\theta_{ab}}^3 T \frac{dU^b}{dt} \\ P_{ab} &= P\delta_{ab} - \eta_{abcd}\Lambda_{cd} \end{aligned} \tag{8.10}$$

When one does not know the distribution function and scattering, the fluid dynamics is in terms of transport coefficients and an equation of state. With relations like (8.10), the fluid equations (8.2,3,4) constitute 5 equations in the 6 unknowns U^a, P, ρ, and n. An equation of state serves to close the system.

8.2) Including the Magnetic Field

Since the particles have charge q, then Ampere's Law will imply:

$$qG^a = \frac{c}{4\pi}\epsilon_{abc}\frac{\partial B^c}{\partial x^b} \tag{8.11}$$

To allow for the presence of an embedded magnetic field, the energy and momentum equations must be augmented in the usual way. Combining (8.11) with (8.3) yields the momentum equation:

$$\frac{\partial}{\partial t}(\rho U^a + F^a) + \frac{\partial}{\partial x^b}(\rho U^a U^b + P_{ab} + F^a U^b + F^b U^a + {P^B}_{ab}) = 0 \tag{8.12}$$

$${P^B}_{ab} \equiv \frac{1}{4\pi}\left(\frac{1}{2}B^2\delta_{ab} - B^a B^b\right)$$

This is consistent with the momentum equation of magnetohydrodynamics, ignoring the electric field. The electric field is ignored due to the hydromagnetic condition, which has the electric field order U/c smaller than the magnetic field. The momentum density of the electromagnetic field is:

$$\frac{1}{4\pi c}\epsilon_{abc}\widetilde{E^b}\widetilde{B^c} \simeq \frac{1}{4\pi c}\epsilon_{abc}\epsilon_{bde}\frac{U^e}{c}B^cB^d$$

The hydromagnetic condition was used to obtain the approximation. Thus the momentum density of the electromagnetic field is order U^2/c^2 smaller than the other term in the magnetic field, and is ignored.

The energy equation that includes the electromagnetic field is:

$$\frac{\partial}{\partial t}\left(\frac{\rho U^2}{2} + T + F^aU^a + \frac{B^2}{8\pi}\right) + \frac{\partial}{\partial x^a}\left(\frac{1}{2}\rho U^2U^a + \frac{1}{2}F^aU^2 + U^aU^bF^b + P_{ab}U^b + TU^a + Q^a + P^B{}_{ab}U^b\right) = 0 \tag{8.13}$$

This was obtained by combining (8.11) and (8.4). Within the hydromagnetic approximation, the energy density of the electromagnetic field is $B^2/8\pi$. This part I put in by hand.

The evolution of the magnetic field is determined by the induction equation, derived in Appendix B:

$$\frac{\partial B^a}{\partial t} = \frac{\partial}{\partial x^b}(U^aB^b - U^bB^a) \tag{8.14}$$

The magnetic field is thus specified given the fluid velocity. The electric field is given by the magnetic field and the fluid velocity through the hydromagnetic condition.

8.3) Relation to the Usual Fluid Equations: $G^a \equiv 0$

Typical formulations of fluid dynamics choose a fluid velocity V^a that is not exactly the frame of the scattering centers. The formalism may be applied to yield the equations formulated in terms of the fluid velocity V^a, defined:

$$\int f\frac{\widetilde{p}^a}{\widetilde{m}}d^3\widetilde{p} \equiv \widetilde{n}V^a \tag{8.15}$$

Then one uses the momentum transformation:

$$\widetilde{p}^a = p_v{}^a + m_v(p_v)V^a$$

Thus we have that:

$$\int f \frac{p_v{}^a}{m_v} d^3p_v \equiv G_v{}^a = 0 \tag{8.16}$$

The fluid equations referred to the standard frame V^a are obtained from the U^a-frame fluid equations by making the substitutions $U^a \to V^a$, $G^a \to 0$, $\rho \to \rho_v$, $n \to n_v$, $F^a \to F_v{}^a$, $T \to T_v$, $Q^a \to Q_v{}^a$, $P_{ab} \to P_{ab}{}^v$. In form, the equations referred to the two frames are similar, with only G^a vanishing. The relative sizes of some terms are different, though.

The relationships between the various fluid quantities referred to U^a and V^a follow from:

$$\frac{p_v{}^a}{m_v} + V^a = \frac{p^a}{m} + U^a \tag{8.17}$$

We assume that, like U, $V \ll c$. Then:

$$\begin{aligned} d^3\widetilde{p} &\simeq d^3p \simeq d^3p_v \\ \widetilde{m} &\simeq m \simeq m_v \\ \rho &\simeq \rho_v \\ n &\simeq n_v \end{aligned} \tag{8.18}$$

This implies:

$$n_v V^a \simeq nV^a = nU^a + G^a + O(\lambda^2/L^2) \tag{8.19}$$

The ordering in terms of λ/L follows because the expression for S^a, on which G^a is based, is correct to order λ/L. Thus we have that:

$$V^a - U^a \simeq O(\lambda/L) \tag{8.20}$$

The fluid dynamics being formulated here is only correct to order λ/L. So the relations between the other fluid quantities, correct to order λ/L, is:

$$\begin{aligned} {P_{ab}}^v &\simeq P_{ab} \\ T_v &\simeq T \\ {Q_v}^a &\simeq Q^a + T(U^a - V^a) + P_{ab}(U^b - V^b) \end{aligned} \tag{8.21}$$

There are also the identifications:

$$\begin{aligned} n_v V^a &= nU^a + G^a \\ \rho_v V^a + {F_v}^a &= \rho U^a + F^a \end{aligned} \tag{8.22}$$

8.4) Heat Flux is a First Order Moment !!

The reader may have been surprised to find that even in the formulation in terms of the fluid velocity V^a, the term F^a still survives in the fluid equations. The reason is that the gas is allowed to be relativistic, and so if one wants to allow for a diffusive energy flux Q^a, one must accept a diffusive mass flux F^a as well. We are used to considering the heat flux vector Q^a as formed from the trace of a third order moment $Q_{abc} \equiv \int f p^a p^b p^c d^3p$, so that $Q^a = Q_{abb}$. For a non-relativistic gas this is proportional to the flux of kinetic energy, with kinetic energy $\propto p^2$. For a relativistic gas the kinetic energy is not a simple power of p but is given by $mc^2 - m_0c^2$. When $p \ll m_0 c$, then $mc^2 - m_0c^2 \approx p^2/2m_0$. Otherwise Q^a must be as given by (6.2):

$$Q^a = \int (mc^2 - m_0c^2)\frac{p^a}{m} f d^3p$$

If one defines the order of a moment as that power of the momentum vector which enters the integral, then heat flux is a *first order* moment. If a gas with the fluid velocity defined as V^a has a heat flux, it must have a non-zero first moment (or else be non-relativistic):

$${Q_v}^a = \int (m_v c^2 - m_0 c^2) f \frac{{p_v}^a}{m_v} d^3p_v = \int f {p_v}^a c^2 \equiv {F_v}^a c^2 \tag{8.23}$$

Thus, even if we work in the usual fluid frame V^a, although the quantity G^a vanishes identically from the fluid equations the quantity F^a will remain.

8.5) The Equations for a Non-relativistic Gas

When consideration is restricted to gases whose fluid-frame momenta are non-relativistic, the familiar equations of fluid dynamics can be recovered in which F^a vanishes, while Q^a is non-zero. Under the assumption $p_v \ll m_0 c$:

$$\begin{aligned}\int f \frac{{p_v}^a}{m_v} d^3 p_v &\equiv 0 \\ &\simeq \int f \frac{{p_v}^a}{m_0}\left(1 - \frac{{p_v}^2}{2 m_0 c^2}\right) d^3 p_v \\ &= \frac{{F_v}^a}{m_0} - \int f \frac{{p_v}^a}{m_0} \frac{{p_v}^2}{m_0 c^2} d^3 p_v\end{aligned} \tag{8.24}$$

Thus ${F_v}^a \ll \rho_v V^a$ and the number, momentum and energy equations become, for the frame V^a:

$$\frac{\partial n_v}{\partial t} + \frac{\partial}{\partial x^a} n_v V^a = 0 \tag{8.25}$$

$$\frac{\partial}{\partial t}(\rho_v V^a) + \frac{\partial}{\partial x^b}(\rho_v V^a V^b + {P_{ab}}^v) = 0 \tag{8.26}$$

$$\frac{\partial}{\partial t}\left(T_v + \frac{1}{2}\rho_v V^2\right) + \frac{\partial}{\partial x^b}\left(\frac{1}{2}\rho_v V^2 V^a + {P_{ab}}^v V^b + T_v V^a + {Q_v}^a\right) = 0 \tag{8.27}$$

So although ${F_v}^a$ is unimportant in the momentum equation, ${F_v}^a c^2 = {Q_v}^a$ can be important in the energy equation. In the frame of the scattering centers U^a, the assumption of a non-relativistic gas doesn't eliminate G^a or F^a from the fluid equations. One merely has:

$$\begin{aligned}\rho &\simeq m_0 n \\ F^a &\simeq m_0 G^a \\ Q^a &\simeq \int f \frac{p^a}{m_0} \frac{p^2}{2 m_0} d^3 p\end{aligned} \tag{8.28}$$

8.6) Defining the Scattering Frame U^a

The choice of a 'fluid frame' is somewhat arbitrary; it is only necessary that it be specified uniquely in terms of observable quantities. The fundamental observable is the number flux seen by the inertial-frame observer, and by (8.15) we may take the conventional fluid velocity V^a as the observable. From (8.19) one sees that the difference between V^a and the frame of the scattering centers U^a is a small quantity, of order λ/L. Thus the choice of U^a as the fluid frame is only a small correction to the usual choice, V^a. Ultimately this correction mirrors the fact that the unified transport equation is correct to first order in λ/L. Using (8.10), and ignoring the magnetic field:

$$\begin{aligned} nV^a &= nU^a - \theta_1 n \frac{dU^a}{dt} - \kappa_1 \frac{\partial n}{\partial x^a} + O\left(\frac{\lambda^2}{L^2}\right) \\ &= nU^a - \theta_1 n \frac{dV^a}{dt} - \kappa_1 \frac{\partial n}{\partial x^a} + O\left(\frac{\lambda^2}{L^2}\right) \end{aligned} \tag{8.29}$$

The integrals represented by $\theta_1 n$ and $\kappa_1 \partial n/\partial x^a$ are over the variable $p = |\widetilde{p}^a - mU^a|$. To order λ/L, the variable can be taken to be $|\widetilde{p}^a - mV^a|$ instead. Thus (8.29) constitutes an expression for U^a in terms of V^a and the known scattering law. The physical interpretation of (8.29) is that a number flux is allowed in the absence of convection, due to either gradients in the number density or the presence of an acceleration of the scattering centers. If an observer in the frame of the scattering centers were to create a density enhancement, he would observe a flux of particles moving to neutralize the enhancement. Whereas only a measurement of the distribution function is necessary to specify V^a, a knowledge of the spatial and time dependence of f is necessary to specify U^a.

When electric and magnetic fields are present, they serve to define the component of U^a perpendicular to the magnetic field through the hydromagnetic condition (5.3).

$$U^a \equiv U_{\parallel}{}^a + U_{\perp}{}^a \equiv \frac{B^a}{B^2} B^b U^b + U_{\perp}{}^a = U_{\parallel}{}^a + c\epsilon_{abc}\frac{E^b B^c}{B^2}$$

Thus, using Ampere's Law (8.11), the definition of U^a in terms of the observable V^a is merely:

$$nV^a = nU_{\parallel}{}^a + nc\epsilon_{abc}\frac{E^b B^c}{B^2} + \frac{c}{4\pi q}\epsilon_{abc}\frac{\partial B^c}{\partial x^b} \tag{8.30}$$

The reason for the simplicity in the definition of U^a with a magnetic field as opposed to without, is that the electrons have been ignored. If they were included, then the current would be the sum of the fluid-frame motions of both species, and one would have to account for the scattering of both species.

9. Cosmic-Ray Transport Equation

In (7.11) I wrote down a transport equation for particles of all energies, correct to first order in λ/L. In this section I will make contact between (7.11) and the usual diffusive-convective equation of cosmic-ray transport.

If one is interested in the transport of only the fast particles, for which $p \gg mU$, then some of the terms in the general transport equation are negligible. As it stands, the general transport equation as written in (7.11) is:

$$\frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} + \frac{1}{m}\frac{\partial S^a}{\partial x^a} = \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} + m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab}) \tag{7.11}$$

$$S^a = -\frac{p^2\tau}{3m}\frac{\partial f_0}{\partial x^a} + \frac{mp\tau}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p} + \frac{q\tau}{mc}\epsilon_{abc}S^b B^c$$

$$\Pi_{ab} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} - \frac{q\tau}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)$$

9.1) The Original Cosmic-Ray Transport Equation

The expression for S^a involves two terms in f_0; the expression for Π_{ab} just one. The transport equation itself involves two terms in S^a and one in Π_{ab}.

As discussed in §7.6, the four terms introduced into the transport equation from S^a and the one from Π_{ab} are all linear in λ/L relative to f_0/t. If one is to consider mU/p as a small parameter as well, then these five are not all the same size. So I will write down from (7.11) the transport equation for cosmic rays, under the following two restrictions on mU/p.

Here is the transport equation for cosmic rays, correct to first order in λ/L and *zeroth* order in mU/p, relative to f_0/t.

$$\frac{\partial f_0}{\partial t} + U^a\frac{\partial f_0}{\partial x^a} = \frac{\partial}{\partial x^a}\left(\kappa_{ab}\frac{\partial f_0}{\partial x^b}\right) + \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} \tag{9.1}$$

$$\kappa_{ab} \equiv \frac{p^2\tau}{3m^2}\frac{1}{1+\Omega^2\tau^2}[\delta_{ab} + \tau^2\Omega^a\Omega^b + \epsilon_{abc}\tau\Omega^c]$$

$$\Omega^a \equiv \frac{qB^a}{mc}$$

Reading (9.1) from left to right, it says that the cosmic-ray distribution changes with time due to: convection with velocity U^a, diffusion described by the anisotropic diffusion tensor κ_{ab}, and adiabatic changes in particle energy due to compression of the flow. The terms omitted from the general equation (7.11) are order m^2U^2/p^2 smaller than the diffusion term and order mU/p smaller than the convection and energy change terms. (9.1) is the conventional convective-diffusive equation of cosmic-ray transport as introduced by Parker (1965). It assumes that $mU \ll p$. As a fluid description, it assumes $\lambda/L \ll 1$.

The diffusion tensor κ_{ab} contains the effects due to gradient and curvature drifts. To see this, first decompose κ_{ab} into its symmetric and antisymmetric parts: $\kappa_{ab} \equiv {\kappa_{ab}}^{(S)} + {\kappa_{ab}}^{(A)}$. Therefore:

$$\begin{aligned}
\frac{\partial}{\partial x^a}\left(\kappa_{ab}\frac{\partial f_0}{\partial x^b}\right) &= \frac{\partial}{\partial x^a}\left({\kappa_{ab}}^{(S)}\frac{\partial f_0}{\partial x^b}\right) + {\kappa_{ab}}^{(A)}\frac{\partial^2 f_0}{\partial x^a \partial x^b} + \frac{\partial {\kappa_{ab}}^{(A)}}{\partial x^a}\frac{\partial f_0}{\partial x^b} \\
&= \frac{\partial}{\partial x^a}\left({\kappa_{ab}}^{(S)}\frac{\partial f_0}{\partial x^b}\right) + \frac{p^2\tau}{3m^2}\epsilon_{abc}\frac{\partial}{\partial x^a}\left(\frac{\tau\Omega^c}{1+\Omega^2\tau^2}\right)\frac{\partial f_0}{\partial x^b} \\
&\equiv \frac{\partial}{\partial x^a}\left({\kappa_{ab}}^{(S)}\frac{\partial f_0}{\partial x^b}\right) - {U_D}^b\frac{\partial f_0}{\partial x^b}
\end{aligned}$$

The last line defines an effective velocity ${U_D}^a$, which is due to gradient and curvature drifts in the magnetic field. The importance of this term relative to convection depends entirely on the magnetic field and the scattering. In the limit of $\Omega\tau \gg 1$, the drift velocity ${U_D}^a$ becomes independent of the scattering time τ.

$${U_D}^a \Longrightarrow \frac{p^2c}{3mq}\epsilon_{abc}\frac{\partial}{\partial x^b}(B^c/B^2)$$

This expression may also be obtained by doing a pitch-angle average over the total drift velocity for a single particle. See Rossi and Olbert (1970) for the drift velocity for a single particle. Note that the drift velocity includes not only drift transverse to the magnetic field, but drift parallel to the field as well.

Using this limiting expression for the drift velocity, it is interesting to form the energy gain for the distribution drifting in the induced electric field (Jokipii, 1987):

$$\begin{aligned} qU_D{}^a \widetilde{E}^a &= \frac{p^2 c}{3m} \epsilon_{abc} \frac{\partial}{\partial x^b} (B^c/B^2) \epsilon_{ade} B^d U^e / c \\ &= \frac{p^2}{3m} \left[\frac{\partial}{\partial x^d} \left(B^d \frac{U^e B^e}{B^2} - U^d \right) + \frac{B^d}{B^2} \frac{\partial}{\partial x^e} (B^d U^e - B^e U^d) \right] \\ &= -\frac{p^2}{3m} \left[\frac{\partial U_\perp{}^d}{\partial x^d} + \frac{1}{2B^2} \frac{\partial B^2}{\partial t} \right] \end{aligned}$$

I used the induction equation to recover the final form. This is just to show that although the electric field is transformed away, its effects are still contained in the equations.

9.2) The Extended Cosmic-Ray Transport Equation

One can also write down an extended transport equation, correct now to first order in λ/L and *first* order in mU/p, relative to f_0/t. Also, one may say it is correct to first order in τ/t relative to f_0/t.

$$\begin{aligned} \frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} + \frac{1}{m} \frac{\partial S^a}{\partial x^a} = \frac{\partial U^a}{\partial x^a} \frac{p}{3} \frac{\partial f_0}{\partial p} - m \frac{dU^a}{dt} \frac{1}{p^2} \frac{\partial}{\partial p} \left(m p \kappa_{ab} \frac{\partial f_0}{\partial x^b} \right) \\ + \frac{\partial U^a}{\partial x^b} \frac{1}{p^2} \frac{\partial}{\partial p} (p \Pi_{ab}) \end{aligned} \tag{9.2}$$

$$S^a = -m\kappa_{ab} \left[\frac{\partial f_0}{\partial x^b} - \frac{m^2}{p} \frac{dU^b}{dt} \frac{\partial f_0}{\partial p} \right]$$

$$\Pi_{ab} = \frac{p^3 \tau}{15} \frac{\partial f_0}{\partial p} \Lambda_{ab} - \tau (\epsilon_{dae} \Pi_{bd} \Omega^e + \epsilon_{dbe} \Pi_{ad} \Omega^e)$$

This is the extended transport equation written down by Earl, Jokipii and Morfill (1988) (for the case with no magnetic field) and by Williams and Jokipii (1991).

The term in Π_{ab} contains the cosmic-ray viscosity. The terms proportional to the fluid acceleration have been referred to as the inertial terms. The effects of the viscous and inertial terms have been analyzed in papers by Earl, Jokipii and Morfill, 1988; Jokipii, Kota and Morfill, 1989; Jokipii and Morfill, 1990; Williams and Jokipii, 1991; Jokipii and Williams, 1992. Most of the analyses pertain to the viscous terms, because the concept of viscosity is familiar in physics. It is probably fair to say that all the implications of the inertial terms have not yet been explored.

The cosmic-ray viscosity allows for exchange of energy between the cosmic-rays and the fluid in shearing flows. This is merely an extension of the energy exchange already described in the adiabatic term, which only addresses compressions in the flow. That the adiabatic energy-change term and the viscosity term have a common character may be seen in the conservation form of the general transport equation (7.6) in §7.2:

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial x^a}\left(f_0 U^a + \frac{S^a}{m}\right) = m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f_0}{3}\delta_{ab} + p\Pi_{ab}\right) \tag{7.6}$$

The second term on the RHS contains the adiabatic and viscosity terms. Together, they constitute the partial pressure contributed to the system by particles with magnitude of momentum p. Integrating this partial pressure over all d^3p would yield the total pressure of the cosmic rays. Note that the adiabatic energy-change term arises from the isotropic part of the pressure.

The first term on the RHS of (7.6) yields one of the inertial terms. Since I ordered the fluid equations ignoring order U/c, the acceleration terms don't enter the fluid equations (see Appendix D). The inertial terms represent a coupling between the acceleration of the fluid and the fluid-frame flux of particles. That such a coupling must exist is because the streaming S^a is referred to a non-inertial frame U^a. The net effect on S^a is an acceleration equal to the negative of the acceleration of the fluid. Physics in non-inertial frames is familiar. Particle trajectories in the frame of the surface of the earth are influenced by the accleration of this frame: the influence is expressed as the Coriolis 'force' or the centrifugal 'force'. In integrating these effects up into fluid equations, one finds powers of the ratio of U/c, which is consistent with the idea that physical effects that depend on the reference frame are a matter of special relativity, which incorporates U/c as a fundamental parameter.

10. Space-Plasma Viscosity

This section will deal with applications of the concept of space-plasma viscosity. In recent years, cosmic-ray viscosity has been discussed. Recall that the transport equation (7.11) applies to particles of all energies. *Cosmic-ray* viscosity arises in (9.2) upon restricting the particle distribution to particles with $p \gg mU$. Then the viscosity coefficient, which depends on the particle distribution, can be said to be the viscosity contributed by the energetic particles.

But the real goal is an accurate description of the dynamics of space plasmas. The core idea of this thesis is that the cosmic rays and the background thermal gas may be described within the *same* transport and fluid equations. Splitting the description, and therefore the distribution, must only be done if there is some advantage to considering these two populations separately. As far as the viscous behavior of the space plasma is concerned, there is no advantage to this splitting because the flow is affected only by the integrated viscosity.

When the concept of a cosmic-ray viscosity does become useful is in the understanding of the acceleration of particles. The energetic particles of a space plasma constitute its 'heat'. Unlike neutral-gas heat, these particles typically never 'cool'. They only get hotter. So cosmic-ray viscosity is useful as an acceleration mechanism, alongside adiabatic energy change. A shear in a space-plasma flow will dissipate some of its ordered kinetic energy of flow into random cosmic-ray energy. It is this process which cosmic-ray viscosity describes.

10.1) Viscosity of the Space Plasma: Thermal vs. Cosmic Ray

As noted in §6, the viscous part of the pressure tensor is given by:

$$4\pi \int (\Pi_{ab}/m) p^2 dp$$

The integral is over momenta of all energies, so also includes the thermal particles. Using the expression for Π_{ab} found in §7.6, one may identify a coefficient of shear viscosity η:

$$\eta \equiv -4\pi \int \frac{p^3 \tau}{15m} \frac{\partial f_0}{\partial p} p^2 \, dp$$

The minus sign is introduced so that for reasonable distributions with a negative momentum slope, the viscosity is positive.

As noted in §3.1, the micro-scale in space plasmas is the particle gyro-radius. The associated timescale is the gyrofrequency Ω. I will assume that the scattering time τ is some constant factor longer than the gyro-period $2\pi/\Omega$, say α/Ω, $\alpha \geq 2\pi$. With this assumption, the viscosity may be written:

$$\eta = -\frac{4\pi\alpha c}{15qB} \int \frac{\partial f_0}{\partial p} p^5 \, dp \tag{10.1}$$

Now I want to compare typical thermal and cosmic-ray contributions to the total viscosity. First assume f_0 is a thermal distribution with characteristic (non-relativistic) temperature T:

$$f_0(p) = n(2\pi mkT)^{-3/2} \exp^{-p^2/2mkT} \tag{10.2}$$

This function is strongly peaked at kT, so relativistic momenta make no significant contribution as long as kT is non-relativistic, and the variation of mass with momentum may be ignored. n is the number density.

Using the thermal distribution (10.2) for the integral (10.1), one finds $\eta = \alpha nkTmc/2qB = \alpha nkT/2\Omega$. I'll set $n = 1/\text{cc}$, $kT = 10^{-11}$ ($T \sim 10^5 K$), $B = 10^{-6}$ Gauss. With $m = 1.7x10^{-24}$ g, $c = 3x10^{10}$ cm/s and $q = 4.8x10^{-10}$ cgs, the thermal $\eta \sim 5\alpha x10^{-10}$ gm/cm-sec. A fluid description probably requires α at least a hundred, so as a final order of magnitude of the contribution to the viscosity from the thermal particles, one has:

$$\eta_{therm} \sim 10^{-7}\mathrm{gm/cm\cdot s}$$

This is a number far removed from the realm of everyday experience. The viscosity of air in these units is of order 10^{-4}.

To check that this is the right order of magnitude for η_{therm}, let me go at it another way. A viscosity will scale with the product of the fluid density, the mean velocity of a particle, and their mean free path: $\eta_{therm} \sim v\lambda\rho$. As above, take λ to be some α of the particle gyro-radius: $\lambda = \alpha pc/qB$; and put $pv = kT$. Again one finds $\eta_{therm} \sim \alpha 10^{-10}$ gm/cm-sec.

Now let's consider the contribution the cosmic rays make to the viscosity of the space plasma. I'll model the cosmic-ray part of the distribution as a simple power law with slope -4.6, running from $pc = 10^9$ eV to 10^{20} eV. The normalization is chosen to match the data at 10^9 eV. All this is pretty much what is observed in this momentum range (see Appendix F). The actual spectrum steepens somewhat at 10^{17} eV, so this will be a generous estimate. But it may compensate for ignoring particles above and below the range.

The data is actually in the form of $p^2 f_0$ as a function of kinetic energy. This is a clumsy unit for an order-of-magnitude integral, so I will assume the kinetic energy is simply pc (it's actually $(p^2c^2 + {m_0}^2c^4)^{1/2} - m_0c^2$ and $\simeq pc$ when $p \gg m_0c$), and let this take on values between 10^9 and 10^{20} eV.

The cosmic-ray distribution will be taken as:

$$p^2 f_0 = \left(\frac{p_0}{p}\right)^{2.6} A \qquad (10.3)$$

$$p_0 = 10^9 \text{eV/c} \qquad A = 1(\text{m}^2 \cdot \text{sec} \cdot \text{st} \cdot \text{MeV})^{-1}$$

The units of A are square meters per second per steradian per million eV. Using the distribution given by (10.3) in the integral (10.1), and integrating from 10^9 to 10^{20} eV/c yields $\eta = 2x10^{32} A(\alpha c/qB)(eV/c)^3$. As for the thermal case I will choose 10^{-6} Gauss for the magnetic field strength. This implies a cosmic-ray $\eta \simeq 10^{-7}\alpha$. With $\alpha = 200$ as for the thermal estimate, one finds that the final estimate for the contribution of the cosmic rays to the space-plasma viscosity is:

$$\eta_{cr} \sim 10^{-5} \text{gm/cm} \cdot \text{sec}$$

So it seems that the energetic particles dominate the thermal particles in accounting for the viscosity of the space plasma. However, there is a caveat. The caveat is that we assume a fluid description. This implies that the microscale of the system, the particle gyro-radius, be much smaller than the scale for the change of fluid quantities: $\lambda/L \ll 1$. As one moves to higher energies, the particle gyro-radius increases without bound; proportional to particle momentum. Therefore as the fluid description encompasses particles of higher momenta, its range of validity moves to larger lengthscales.

For example, a 10^{15} eV proton in a $5x10^{-5}$ Gauss field has a gyro-radius of about 4000 AU. A 10^{20} eV proton in the same field has a gyro-radius of 600 lightyears. For a 10^{10} eV proton, it is .05 AU. So when considering the dynamics of the interplanetary flow, with total size about 100 AU and characteristic lengthscale L about .1 AU, one can only consider the contribution to the viscosity of the interplanetary flow from particles of *less than* about 10^{10} eV. Although more energetic particles are *present* in the flow, they are not coupled to it on short enough lengthscales to affect the dynamics.

This applies not only for the viscosity but for the isotropic pressure as well, at least as far as the dynamics are concerned. Certainly the energy density one measures anywhere in a space plasma will contain contributions from particles of all energies, but the dynamics of the flow in the region of the measurement is only affected by those particles that are coupled to the flow. As far as the fluid description is concerned, coupling presumes $\lambda/L \ll 1$.

Returning to the main point, the viscosity of the interstellar medium, with flows of galactic scale, is dominated by cosmic rays.

10.2) Anisotropic Viscosity of the Space Plasma

In the general transport equation (7.11), which involves Π_{ab}, I wrote out only the equation which determines Π_{ab}; not an explicit expression for it. That is because the solution is far more complicated than the equation itself. In this section I will write down the general form of the viscous stress tensor.

The equation which determines Π_{ab} is:

$$\Pi_{ab} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} - \gamma_{da}\Pi_{bd} - \gamma_{db}\Pi_{ad} \tag{10.4}$$

$$\gamma_{ab} \equiv \epsilon_{abc}\Omega^c\tau \equiv \epsilon_{abc}\phi^c$$

Note that γ_{ab} has the following properties: $\gamma_{ab} = -\gamma_{ba}$; $\gamma_{ab}\gamma_{ab} = 2\phi^2$; $\gamma_{ab}\gamma_{ac} = \delta_{bc}\phi^2 - \phi^b\phi^c$; $\gamma_{ab}\phi^b = 0$. The solution to (10.4) is:

$$\Pi_{ab} = \Upsilon_{abcd}\frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{cd} \tag{10.5}$$

$$\begin{aligned}\Upsilon_{abcd} = &\frac{1}{2}\xi_1(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{1}{2}\xi_2(\gamma_{ad}\delta_{bc} + \gamma_{bc}\delta_{ad} + \gamma_{ac}\delta_{bc} + \gamma_{bd}\delta_{ac})\\ &+ \xi_2(\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc}) + \frac{1}{2}\xi_4(\delta_{ac}\phi^b\phi^d + \delta_{ad}\phi^b\phi^c + \delta_{bc}\phi^a\phi^d + \delta_{bd}\phi^a\phi^c)\\ &+ \frac{3}{2}\xi_3(\gamma_{ac}\phi^b\phi^d + \gamma_{ad}\phi^b\phi^c + \gamma_{bc}\phi^a\phi^d + \gamma_{bd}\phi^a\phi^c) + 6\xi_3\phi^a\phi^b\phi^c\phi^d\end{aligned} \tag{10.6}$$

$$\xi_1 \equiv \frac{1+2\phi^2}{1+4\phi^2} \quad \xi_2 \equiv \frac{1}{1+4\phi^2} \quad \xi_3 \equiv \frac{1}{(1+4\phi^2)(1+\phi^2)} \quad \xi_4 \equiv \frac{1-2\phi^2}{(1+4\phi^2)(1+\phi^2)}$$

The character of Υ_{abcd} has been discussed by Lifshitz and Pitaevskii (1981), §13. It has the symmetries $\Upsilon_{abcd} = \Upsilon_{abdc} = \Upsilon_{bacd}$ and $\Upsilon_{abcd}(\phi^a) = \Upsilon_{cdab}(-\phi^a)$. Lifshitz and Pitaevskii claim that the number of viscosity coefficients is equal to the number of independent tensors it takes to construct Υ_{abcd}. They find five coefficients of shear viscosity and two of bulk viscosity. However, they forgot the tensor $\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc}$. This tensor was ignored by them because of the identity:

$$\gamma_{ab}\gamma_{cd} = \phi^2\delta_{ac}\delta_{bd} + \phi^a\phi^d\delta_{bc} + \phi^b\phi^c\delta_{ad} - \phi^2\delta_{bc}\delta_{ad} - \phi^b\phi^d\delta_{ac} - \phi^a\phi^c\delta_{bd}$$

Evidently, this relation renders $\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc}$ dependent on the other tensors. Since $\xi_4/2 = 3\xi_3/2 - \xi_2$, it appears there are indeed five different coefficients in Υ_{abcd}: $\xi_1/2$, $\xi_2/2$, ξ_2, $3\xi_3/2$, $6\xi_3$.

The general expression (10.6) can be simplified enormously by putting the magnetic field along the $\hat{z}$ direction of a cartesian coordinate system. Of 81 possible coefficients, only 14 are nonzero:

$$\Upsilon_{xxxx} = \Upsilon_{yyyy} = \frac{1}{2}(1+\mu_3) \qquad \Upsilon_{yyxx} = \Upsilon_{xxyy} = \frac{1}{2}(1-\mu_3)$$

$$\Upsilon_{xxxy} = -\Upsilon_{xyxx} = \Upsilon_{xyyy} = -\Upsilon_{yyxy} = \mu_4$$

$$\Upsilon_{xyxy} = \frac{1}{2}\mu_3 \qquad \Upsilon_{xzxz} = \Upsilon_{yzyz} = \frac{1}{2}\mu_1$$

$$\Upsilon_{xzyz} = -\Upsilon_{yzxz} = \frac{1}{2}\mu_2 \qquad \Upsilon_{zzzz} = 1 \tag{10.7}$$

$$\mu_1 \equiv \frac{1}{1+\phi^2} \qquad \mu_2 \equiv \phi\mu_1 \qquad \mu_3 \equiv \frac{1}{1+4\phi^2} \qquad \mu_4 \equiv \phi\mu_3$$

The relations (10.7) are the same as those found by Kaufman (1960) for the viscosity of a plasma in a magnetic field, and by Webb (1989) in his derivation of an extended, relativistic transport equation. Note that there are evidently five independent coefficients of viscosity: μ_1, μ_2, μ_3, μ_4, and 1.

With Υ_{abcd} defined in (10.6), the pressure tensor P_{ab}, as introduced in §6 and §8, is defined.

$$P_{ab} = P\delta_{ab} - \eta_{abcd}\Lambda_{cd} \tag{8.9}$$

$$P \equiv 4\pi \int \frac{p^2 f_0}{3m} p^2 \, dp \qquad \eta_{abcd} \equiv -4\pi \int \frac{p^3 \tau}{15m} \frac{\partial f_0}{\partial p} \Upsilon_{abcd} p^2 \, dp$$

The viscosity of the space plasma is anisotropic due to the preferred direction in the fluid introduced by the magnetic field. The effect of the magnetic field is to reduce the magnitude of the isotropic viscosity by at least a factor of $1/\Omega\tau$ (except for the field aligned component of momentum transfer, which is relevant only to compressions along the field). The effect known as shear viscosity in a fluid is due to momentum transfer across gradients in a flow. The presence of a magnetic field will act to reduce all components of momentum transfer, since it exerts a force on particles that moves them away from their instantaneous direction of motion. So each coefficient of viscosity is reduced relative to the isotropic case. However, the magnetic field also leads to momentum transfer that would not otherwise be present in an isotropic situation. All these effects are included in the viscosity tensor Υ_{abcd} whose components are written down in equation (10.6). Whether the overall viscous effect is reduced relative to isotropy depends on the boundary conditions of the situation under consideration.

The character of viscosity in this situation is attributable to the nature of the Lorentz force: it will mix divergence and shear in fluid flows. For the system under consideration, there are five independent coefficients of shear viscosity. There are no bulk viscosity coefficients here: all five are shear viscosity coefficients. The distinction between the two is that shear viscosity coefficients multiply the traceless velocity gradient tensor and that bulk viscosity coefficients multiply a pure divergence of fluid velocity.

Although bulk viscosity coefficients generally exist (Lifshitz and Pitaevskii, 1981), they are absent for systems of monatomic particles. This is because bulk viscosity is associated with a transfer of kinetic energy of the flow into internal energy associated with the internal degrees of freedom of a single particle. Monatomic particles, without internal degrees of freedom, cannot participate in such a process. Shear viscosity is associated with the transfer of large-scale kinetic energy of the flow into random translational kinetic energy of the particles; a process which monatomic particles *can* participate in.

10.3) Anisotropic Viscosity and Particle Orbits

Notice from (10.5) that some of the viscosity coefficients become independent of the scattering time τ in the limit $\Omega\tau \gg 1$, and depend only on the gyrofrequency Ω, suggesting that they are related to the particle motion in the magnetic field. In this section I will use a simple case to illustrate how these viscous effects are related to the drift motion of the particles in the combined electric and magnetic fields of the flow.

Consider the following system: a uniform magnetic field along the $\hat{z}$ direction of a cartesian coordinate system; a fluid flow initially in the $\hat{y}$ direction and depending only on the $\hat{x}$ coordinate; and uniform scalar pressure. From the momentum equation (8.12) and from (10.7), the time dependence of U^x is:

$$\rho\frac{\partial U^x}{\partial t} = \eta_{xxxy}\frac{\partial^2 U^y}{\partial x^2} = -4\pi\int\left(\frac{p^3\tau}{15m}\frac{\partial f_0}{\partial p}\frac{\Omega\tau}{1+\Omega^2\tau^2}p^2\,dp\right)\frac{\partial^2 U^y}{\partial x^2} \tag{10.8}$$

The anisotropic viscosity would yield a nonzero time derivative of the $\hat{x}$ component of the fluid velocity, even if the $\hat{x}$ component were initially zero. Moreover, the effect changes sign with the particle charge and magnetic field. Also noteworthy is the fact that the force in the $\hat{x}$ direction is zero for an initial velocity configuration with only a linear spatial dependence. All these things follow from a consideration of particle orbits.

Consider a parcel of fluid at the origin of a coordinate system in which the initial velocity field has the simple form

$$U^a = (-c/B_0)(\alpha x + \beta x^2)\hat{y} \tag{10.9}$$

and in which the magnetic field is uniform so that $B^a = B_0\hat{z}$. The associated electric field is given by the hydromagnetic condition (5.3): $E^a = (\alpha x + \beta x^2)\hat{x}$. The trajectories of charged particles in these electric and magnetic fields may be computed analytically if $\beta = 0$, and turn out to be closed (circles or ellipses). The trajectories for finite β must be done numerically.

Williams & Jokipii (1991) computed orbits for a particle in the $\hat{x}$–$\hat{y}$ plane for various combinations of α and β. The orbits for $\beta = 0$ are closed. For nonzero β, the trajectory drifts in the negative $\hat{y}$ direction. This is merely due to the 'E-cross-B' drift, integrated over an entire orbit. This drift will result in an electric current in the negative $\hat{y}$ direction and cause a body force in the negative $\hat{x}$ direction. But this is just the result (10.8) from the stress tensor calculation. For reasonable distributions, $\partial_p f_0 < 0$. Since U^y is initially negative, (10.8) describes a force on the fluid in the negative $\hat{x}$ direction.

If the electric field has a linear spatial dependence in a cartesian coordinate system centered at the instantaneous gyro-center of the particle, the particle does not drift relative to the background flow, and its orbit is closed. If the spatial dependence of the electric field is quadratic, the particle orbits are not closed but are drifting ellipses. In fact, odd powers of the spatial dependence do not produce drift, while the even powers do. This is because the gyrating particle must see a net electric field over the course of a single orbit to experience a drift. If the electric field changes sign in an orbit, there is no net drift. The hydromagnetic condition thus allows a coupling between fluid velocity and electromagnetic field that leads to a description of a particle-orbit effect in terms of fluid viscosity.

When the local electric field depends on fluid velocity relative to the particle (hydromagnetic condition), it is the spatial dependence of the fluid velocity relative to the particle that controls the drift. Viscosity enters the fluid equations with second order derivatives of the fluid velocity. This is true for anisotropic viscosity as well as for the usual isotropic viscosity appearing in the Navier–Stokes equation. There will be no viscous effect for fluid velocities with linear spatial dependence. For isotropic fluids this is because the momentum transport across shear planes is constant and although momentum is being transported across shear planes, there is no accumulation of momentum at any point. Rather, a fixed flux of momentum is cascading down the shear. Of course, linear shear will have a viscous effect near the boundaries. Analogously, for the purely anisotropic part of the momentum transport, which manifests in the presence of a magnetic field, this is because there is no drift across an electric field which varies linearly in space.

The anisotropic viscosity can then be understood as an effect of the finite gyro-radii of the particles in the fluid. When the hydromagnetic condition applies, particles in the fluid gyrating around magnetic field lines will find themselves in a spatially varying electric field if a shear is present. Since there is no rest frame for a shear flow, particles at different positions in the fluid will all see different electric fields which are referenced to the individual particle rest frames. The length scale of the electric field seen by any one particle over the course of its gyro-orbit will be determined by the length scale of the fluid shear that it sees over the course of its orbit. By definition of a shear, if any point in a shear is picked as a rest frame, then the magnitude of fluid velocity will increase away from the point that is the rest frame. By the hydromagnetic condition, so too will the electric field seen by the particle. Particles making larger excursions away from some rest-frame point will see larger relative velocities and larger electric fields. Since the drift is governed by the magnitude of the electric field at the particle orbit, particles with a larger gyro-radius will drift more rapidly. For the astrophysical plasmas under consideration in this thesis, which are mainly protons and electrons, the protons with their larger gyro-radii will drift relative to the electrons (which are effectively fixed to the fluid). This is the source of the shear–induced currents which couple the plasma to the average magnetic field and lead to body forces on the plasma.

10.4) An Effect of Anisotropic Viscosity in the Space Plasma: Rotation of Linearly-Polarized Alfvén Waves

To illustrate an effect of the anisotropic viscosity, consider the propagation of linearly polarized Alfvén waves in the limit $\Omega\tau$ is large, corresponding to the scattering time being much greater than the gyro-period. Take τ to be independent of momentum so that the terms in $\Omega\tau$ can be pulled outside the integrals that define η_{abcd}. Assume $B^a = B_0\hat{z} + b^a$, and the fluid is at rest. Then linearize the momentum equation (8.12) and the induction equation (8.14) in the small quantities $U^a(z,t)$ and $b^a(z,t)$, considering variations only along $\hat{z}$. (8.12) implies the following equations for the transverse velocity components:

$$\begin{aligned}
\rho\frac{\partial U^x}{\partial t} &= \frac{B_0}{4\pi}\frac{\partial b^x}{\partial z} + 2\eta_{xzxz}\frac{\partial^2 U^x}{\partial z^2} + 2\eta_{xzyz}\frac{\partial^2 U^y}{\partial z^2} \\
\rho\frac{\partial U^y}{\partial t} &= \frac{B_0}{4\pi}\frac{\partial b^y}{\partial z} + 2\eta_{yzxz}\frac{\partial^2 U^x}{\partial z^2} + 2\eta_{yzyz}\frac{\partial^2 U^y}{\partial z^2}
\end{aligned} \tag{10.10}$$

The induction equation (8.14) may be used to eliminate the terms in b^a to yield the following equations for U^x, U^y:

$$\begin{aligned}
\frac{\partial^2 U_y}{\partial t^2} &= {c_A}^2\frac{\partial^2 U^y}{\partial z^2} + \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\left(2\nu_{xzxz}U^y - 2\nu_{xzyz}U^x\right) \\
\frac{\partial^2 U^x}{\partial t^2} &= {c_A}^2\frac{\partial^2 U^x}{\partial z^2} + \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\left(2\nu_{xzxz}U^x + 2\nu_{xzyz}U^y\right)
\end{aligned} \tag{10.11}$$

Here the Alfvén speed ${c_A}^2 \equiv B^2/4\pi\rho$ and coefficients of *kinematic* viscosity $\nu \equiv \eta/\rho$. Going from (10.10) to (10.11) I used the fact that $\eta_{yzxz} = -\eta_{xzyz}$ and $\eta_{xzxz} = \eta_{yzyz}$, so only two coefficients of viscosity enter into this calculation. Note that the anisotropic viscosity couples the $\hat{x}$ and $\hat{y}$ components of the velocity. If a wave starts with some initial polarization state such that one of these components is zero, then the viscosity will lead to a nonzero time derivative of the other component.

To solve these equations it is useful to work in terms of the right and left circularly polarized waves:

$$U_\pm \equiv U^x \pm iU^y \tag{10.12}$$

Considering only plane-wave solutions propagating along the $\hat{z}$ direction, $U^x, U^y(z,t) \propto \exp^{i(kz-\omega t)}$, (10.12) implies the following waves:

$$U_\pm(z,t) = U_\pm e^{i(kz-\omega t)} \tag{10.13}$$

The solutions (10.13) have the following interpretation. At fixed z, U_+ represents a polarization vector of constant magnitude $(U^x)^2 + (U^y)^2$ rotating clockwise when viewed looking along the $\hat{z}$ axis. Since U_+ has a positive projection of angular momentum on the $\hat{z}$ axis, it will be called the positive helicity wave. Likewise, U_- is the negative helicity wave. With these definitions, the equations (10.11) decouple:

$$\frac{\partial^2 U_\pm}{\partial t^2} = {c_A}^2 \frac{\partial^2 U_\pm}{\partial z^2} + \frac{\partial}{\partial t}\frac{\partial^2 U_\pm}{\partial z^2}(2\nu_{xzxz} \mp i2\nu_{xzyz}) \tag{10.14}$$

The first viscosity term on the right is purely dissipative, and would act to damp the wave even if there were no magnetic field. The second viscosity term is imaginary and will thus contribute to the velocity of the polarization state. Consider the plane-wave solutions (10.13). Substituting (10.13) into (10.14) and solving algebraically for ω leads to the dispersion relation:

$$\omega = -k^2(i\nu_{xzxz} \pm \nu_{xzyz}) \pm \sqrt{k^2 {c_A}^2 + {\nu_{xzyz}}^2 - {\nu_{xzxz}}^2 \pm 2i\nu_{xzxz}\nu_{xzyz}} \tag{10.15}$$

Of the three $\pm$ symbols in (10.15), the one in front of the root sign is an artifact of the solution, allowing for wave propagation parallel or antiparallel to the $\hat{z}$ axis. The other two $\pm$ symbols refer to $U_\pm$. To simplify (10.15), one may make use of the fact that $\Omega\tau \gg 1$, and that τ is independent of momentum. This implies:

$$\begin{aligned} \nu_{xzxz} &\simeq -\frac{1}{2\Omega^2\tau^2}\frac{1}{\rho}4\pi \int \frac{p^3\tau}{15m}\frac{\partial f_0}{\partial p}p^2\,dp \equiv \frac{1}{2\Omega^2\tau^2}\nu \\ \nu_{xzyz} &\simeq -\frac{1}{2\Omega\tau}\frac{1}{\rho}4\pi \int \frac{p^3\tau}{15m}\frac{\partial f_0}{\partial p}p^2\,dp \equiv \frac{1}{2\Omega\tau}\nu \end{aligned} \tag{10.16}$$

So in the limit that $kc_A \gg \nu/\Omega\tau$, an approximation to (10.15) is:

$$\omega \simeq kc_A\left(1 + \frac{\nu^2}{2k^2 {c_A}^2 \Omega^2\tau^2}\right) \mp \frac{k^2\nu}{\Omega\tau} - i\frac{k^2\nu}{\Omega^2\tau^2} \tag{10.17}$$

This is correct to second order in $\Omega\tau$. In (10.17) I kept only the piece representing propagation along $\hat{z}$. Reading (10.17) from left to right, the interpretation is as follows. The anisotropic viscosity leads to an *enhancement* of the Alfvén speed, for both U_+ and U_-, that is quadratic in $\Omega\tau$. There is also a correction that depends on the helicity state. The speed of positive-helicity waves is reduced, and the speed of negative-helicity waves is enhanced, by a factor that is only linear in $\Omega\tau$. So in the limit of large $\Omega\tau$, the effect that lends differing propagation speeds to the two helicity states dominates the effect that enhances the propagation speed for both states. The final term represents viscous dissipation of the waves, and is the same for both helicity states.

Thus the dispersion relation (10.17) predicts two effects: simple isotropic attenuation, with the short wavelength components suffering the strongest attenuation; and a difference in propagation speeds of the two waves that leads to a net rotation of the plane of polarization of a linearly-polarized wave, with the rotation becoming more pronounced at shorter wavelengths. The difference in angular frequency of the two circularly-polarized waves is the frequency with which the plane of polarization of a linearly-polarized wave rotates. Thus the plane of polarization of a wave rotates with angular frequency:

$$\Delta\omega = \frac{2\nu k^2}{\Omega\tau} \tag{10.18}$$

The rotation due to viscosity, in which the negative-helicity wave propagates faster, should be compared to the cold plasma result (e.g., Krall and Trivelpiece 1973, eqn. 4.10.20), which makes a similar prediction, but for a different range of parameters. Although it is not apparent from (10.18), this effect vanishes as the magnetic field goes to zero. The quantity governing the rotation is η_{xzyz}. This quantity goes to zero as $\Omega\tau \to 0$, and goes to $\eta/\Omega\tau$ as $\Omega\tau \to \infty$.

With these results, I want to consider the connection between the rotation of polarized Alfvén waves and particle orbits. Put the magnetic field along the $\hat{z}$

axis of a cartesian coordinate system, so that the field and velocity perturbations are in the $\hat{x}$-$\hat{y}$ plane. At each instant of time these waves possess a shear with a spatial dependence varying sinusoidally in z. Consider perturbations in the velocity field of a parcel of fluid that are initially in the $\hat{x}$ direction. The associated instantaneous hydromagnetic electric field will be in the $\hat{y}$ direction, and the particle drift will be in the $\hat{x}$ direction. As in the simpler case discussed above, finite gyro-radius leads to a relative drift of the ions relative to the fluid. This additional drift constitutes a current in the $\hat{x}$ direction that will couple to the average magnetic field to yield a body force on the parcel in the negative $\hat{y}$ direction. If the parcel were part of a wave propagating along the magnetic field, then the wave would instantaneously gain negative helicity. This would lead to a faster propagation speed for the negative helicity component.

10.5) The Viscous Particle-Acceleration Mechanism: Evolution of the Particle Momentum Distribution in Shear Flow and Average Magnetic Field

After having looked at the effect of the anisotropic viscosity on plasma dynamics, it is now time to look at the effects on the particle distribution function. It will be shown that the viscosity is a particle acceleration mechanism, just like adiabatic compression.

The magnetic field direction is chosen to be along the $\hat{z}$ axis of a local cartesian coordinate system. Assume a velocity field which has a linear spatial dependence in the x-coordinate.

$$U^a = ax\hat{y}$$

Here a is constant. Consider the case of Ω being constant, and f_0 independent of position. Now apply the transport equation (9.2) to this situation. Under these conditions, $dU^a/dt = 0$, $\partial_a U^a = 0$, $\partial_a f_0 = 0$, $S^a = 0$; the effects of viscosity are isolated. The only component of Π_{ab} that enters is $\Pi_{xy} = \Pi_{yx}$. (9.2) becomes:

$$\frac{\partial f_0}{\partial t} = a\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{xy}) = \frac{a^2}{15}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^4\tau}{1+4\Omega^2\tau^2}\frac{\partial f_0}{\partial p}\right) \tag{10.19}$$

Now consider two limiting cases. First, for no magnetic field or for $\Omega\tau \ll 1$. For this case, assume $\tau = \tau_0(p_0/p)^2$, where τ_0 and p_0 are constants. This form for τ is not likely to be realistic, but it allows (10.19) to be cast into an illustrative form:

$$\frac{\partial(pf_0)}{\partial t} = \frac{a^2{p_0}^2\tau_0}{15}\frac{\partial^2(pf_0)}{\partial p^2} \tag{10.20}$$

This is a diffusion equation for the quantity pf_0. For an initial pulse of particles at some momentum, the shear will act to diffuse the quantity pf_0 through momentum space. This constitutes acceleration of particles, because some of the particles are moved to higher values of pf_0, which means higher values of momentum. This is to be expected. In conventional neutral fluids, viscosity acts to dissipate kinetic energy of the flow into heat. That is exactly what (10.20) is describing. The generation of heat and the acceleration of particles are described by the same mathematics.

For the other limiting case, consider $\Omega\tau \gg 1$. For τ, assume $\tau = \tau_0(p/p_0)^2$. This form is realistic, since one expects the scattering time to increase with momentum. A diffusion equation can also be obtained for this case from (10.19):

$$\frac{\partial(pf_0)}{\partial t} = \frac{a^2{p_0}^2}{60\tau_0\Omega^2}\frac{\partial^2(pf_0)}{\partial p^2} \tag{10.21}$$

Again, this is a diffusion equation in pf_0, and it demonstrates that the viscous terms of the transport equation can account for acceleration of particles. Note that the acceleration rate is inversely dependent on the magnetic field.

The above cases had both velocity and shear orthogonal to the local average magnetic field. For the cases of velocity parallel to and shear orthogonal to the field, or velocity orthogonal and shear parallel, the diffusion constant that enters is μ_1. The order of magnitude is the same. Note that all these magnetic effects are independent of the sign of the field direction. The coefficients μ_2 and μ_4, which change sign with the field, do not enter in this special case.

11. Two-Fluid Models

Cosmic rays ultimately derive their energy from the kinetic energy of the flow of space plasma. The acceleration is typically at a shock in the plasma; upstream flow kinetic energy is converted partly into energetic particles. This process can be extremely efficient. Numerical models indicate that as much as 10% of the upstream kinetic energy ends up in superthermal particles. The acceleration of particles depends on the flow. Since the flow is losing energy to energetic particles, the flow depends on the accelerated particles. Therefore the acceleration process is a non-linear one.

The fluid equations of §8 include this interaction because all fluid quantities (density, pressure, etc.) are defined in terms of integrals over particles of all energies. The general transport equation of §7 descibes evolution of the particle momentum dependence. The combination of the transport equation and the fluid equations constitutes a complete, non-linear description of the system of particles, including acceleration and its effect on the flow.

The *cosmic-ray* transport equation of §9 is restricted to particles with $p \gg mU$. Since these particles are enormously fewer in number than the thermal particles, the fluid velocity U^a is determined mainly by the thermal particles. Therefore the cosmic-ray transport equation is effectively a 'test-particle' equation since the distribution of energetic particles depends on the fluid velocity, which is approximately independent of this distribution.

The test-particle cosmic-ray transport equation predicts that large amounts of energy are taken from the flow and put into superthermal particles. Therefore the test-particle transport equation begins to lose validity if the flow is not corrected for the energy lost to energetic particles. To address this problem, two-fluid models were developed.

11.1) The Two-Fluid Models

The two fluids are the cosmic rays and the thermal plasma. The thermal particles account for the mass density and fluid velocity, and contribute a pressure and internal energy. The superthermal particles also contribute a pressure and internal energy. The cosmic rays are coupled to the momentum of the thermal plasma through their pressure. The two-fluid model was first constructed by introducing a cosmic-ray pressure into the Euler equation for the thermal plasma. The cosmic-ray transport equation was integrated over all momenta to yield an energy conservation equation in the cosmic-ray pressure and energy density. From the arguments of §9, one can see that by using the cosmic-ray transport equation to get an energy equation, two-fluid models ignore terms of order mU/p and of order λ/L.

The fluid equations of typical two-fluid models, e.g. Drury and Volk, (1981) or Drury, (1983), are as follows. The magnetic field and the anisotropic part of P_{ab} are ignored.

$$\frac{\partial \rho_{th}}{\partial t} + \frac{\partial}{\partial x^a}(\rho_{th} U^a) = 0 \tag{11.1}$$

$$\rho_{th} \frac{dU^a}{dt} = -\frac{\partial}{\partial x^a}(P_{th} + P_{cr}) \tag{11.2}$$

$$\frac{\partial T_{th}}{\partial t} + \frac{\partial}{\partial x^a}(T_{th} U^a) + P_{th} \frac{\partial U^a}{\partial x^a} = 0 \tag{11.3}$$

$$\frac{\partial T_{cr}}{\partial t} + \frac{\partial}{\partial x^a}(T_{cr} U^a) + P_{cr} \frac{\partial U^a}{\partial x^a} = \frac{\partial}{\partial x^a}\left(\overline{\kappa} \frac{\partial T_{cr}}{\partial x^a}\right) \tag{11.4}$$

The subscript cr refers to the cosmic-ray fluid while th refers to the thermal fluid. All quantities are as defined in §6: ρ is mass density, P is pressure, T is internal kinetic energy.

The momentum conservation equation implied by (11.1,2) is:

$$\frac{\partial}{\partial t}(\rho_{th}U^a) + \frac{\partial}{\partial x^b}\left(\rho_{th}U^aU^b + P_{cr} + P_{th}\right) = 0 \tag{11.5}$$

The cosmic-ray energy equation (11.4) was obtained by multiplying the zeroth-order cosmic-ray transport equation (9.1) by the energy per particle, and integrating over all momenta.

$$\frac{\partial f_{cr}}{\partial t} + \frac{\partial}{\partial x^a}\left(f_{cr}U^a - \kappa\frac{\partial f_{cr}}{\partial x^a}\right) = \frac{\partial U^a}{\partial x^a}\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3 f_{cr}}{3}\right) \tag{9.1}$$

In this form of (9.1) I have explicitly labelled f_{cr} to refer to only the energetic particles. The assumptions that went into deriving (9.1) from the general transport equation (7.11) follow into (11.4) as well. In (9.1), terms of order mU/p *and* of order λ/L were ignored. If the quantity $(\lambda/L)(p/mU) \sim 1$, then all terms in (9.1) are of similar size. The term on the RHS of (11.4) is an average value of the integral of the streaming flux term of (9.1):

$$4\pi\int(mc^2 - m_0c^2)\kappa\frac{\partial f_{cr}}{\partial x^a}p^2dp \equiv \overline{\kappa}4\pi\int(mc^2 - m_0c^2)\frac{\partial f_{cr}}{\partial x^a}p^2dp = \overline{\kappa}\frac{\partial T_{cr}}{\partial x^a} \tag{11.6}$$

This expression defines $\overline{\kappa}$.

Combining (11.3,4,5), one obtains an energy conservation equation for the two-fluid system.

$$\begin{aligned}\frac{\partial}{\partial x^a}&\left(\frac{1}{2}\rho_{th}U^2U^a + P_{th}U^a + P_{cr}U^a + T_{th}U^a + T_{cr}U^a - \overline{\kappa}\frac{\partial T_{cr}}{\partial x^a}\right)\\ &= -\frac{\partial}{\partial t}(\rho_{th}U^2/2 + T_{cr} + T_{th})\end{aligned} \tag{11.7}$$

The two-fluid equations (11.1,2,3,4) constitute 6 equations in the 8 unknowns ρ_{th}, U^a, P_{cr}, P_{th}, T_{cr}, T_{th}. To reduce the number of unknowns, γ_{cr} and γ_{th} are introduced.

$$P_{th} = (\gamma_{th} - 1)T_{th} \tag{11.8}$$

$$P_{cr} = (\gamma_{cr} - 1)T_{cr} \tag{11.9}$$

This is a weak point of the two-fluid models, because γ_{th} and γ_{cr}, as well as $\overline{\kappa}$, must in general depend on the particle distribution, which depends on U^a. Typically, γ_{th} is chosen at 5/3 and γ_{cr} at 4/3. This is not generally a very precise representation of the system.

By manipulating (11.1,3,8), one can show that they imply adiabatic behavior of the thermal gas:

$$\frac{d}{dt}\left(\frac{P_{th}}{\rho_{th}{}^{\gamma_{th}}}\right) = 0 \tag{11.10}$$

So a set of equations alternative to (11.1,2,3,4) is (11.1,5,7,10). This set is suitable for looking at the structure of cosmic-ray-modified shocks because the equations are in conservation form. They imply conservation of total energy, with the thermal plasma moving adiabatically. So the two-fluid models treat the case in which the cosmic rays exchange momentum (11.2) but not energy (11.3,4) with the thermal plasma. The two-fluid models are applied to study the effect that the energetic particles have on a shock in which they are accelerated. The conservation equations (11.1,5,7) are used to find the asymptotic upstream and downstream states. The adiabaticity (11.10) of the thermal plasma is relaxed at the shock; entropy increases across a shock. One further requirement is that P_{cr} is continuous across a shock, which follows if one wants to demand continuity of f_{cr}. This move decouples the cosmic rays at the shock from the shock itself.

11.2) Two-Fluid Models from the Single-Fluid Equations

The two-fluid description follows from the single-fluid equations of §8 by splitting the distribution into two pieces, thermal plus cosmic ray: $f_0 \equiv f_{cr} + f_{th}$. The cutoff between thermal and cosmic-ray must be at some particular value of momentum p_c such that $f_0(p < p_c) = f_{th}$ and $f_0(p > p_c) = f_{cr}$. Two-fluid models never specify p_c, but it is assumed that the characteristic temperature T of the thermal population is such that $kT \ll ({p_c}^2c^2 + {m_0}^2c^4)^{1/2} - m_0c^2$. This splitting of the distribution implies the following correspondences for the fluid quantities introduced in (6.2):

$$\begin{aligned} n &\Longrightarrow n_{cr} + n_{th} \simeq n_{th} \\ \rho &\Longrightarrow \rho_{cr} + \rho_{th} \simeq \rho_{cr} + m_0 n_{th} \\ P &\Longrightarrow P_{cr} + P_{th} \\ T &\Longrightarrow T_{cr} + T_{th} \end{aligned} \tag{11.11}$$

With the decompositions of (11.11), the one-fluid equations of §8 lead to the two-fluid equations (11.1,2,3,4). I'll take them one at a time.

The conservation equation (11.1) for ρ_{th} follows from the number conservation equation (8.2) for n:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^a}(nU^a + G^a) = 0 \tag{8.2}$$

Assuming that $n_{cr} \ll n_{th}$ and $m_0 n_{th} \simeq \rho_{th}$, the cosmic-ray part of (8.2) is ignored. In terms of the typical velocity v_{th} of a thermal particle, $G_{th} \sim n v_{th} \lambda/L$. Since $v_{th} \sim U$, G_{th} is ignored relative to nU because the cosmic-ray energy equation ignores terms of order λ/L. Thus (11.1) is recovered.

The momentum conservation equation (11.5), which implies (11.2), follows from the momentum equation (8.3):

$$\frac{\partial}{\partial t}(\rho U^a + F^a) + \frac{\partial}{\partial x^b}\left(\rho U^a U^b + P_{ab} + F^a U^b + F^b U^a\right) = \frac{q}{c}\epsilon_{abc} G^b B^c \tag{8.3}$$

The magnetic field is ignored. Quantities of order λ/L are ignored, so $P_{ab} = P\delta_{ab}$ and FU is ignored. It is necessary to use the fact that $\rho_{cr}U^2 \ll P_{cr}$. Referring to the integral definitions (6.2) of these quantities, the integrand of $\rho_{cr}U^2$ is $\leq pU^2/c$.

The integrand of P_{cr} is $\leq pc$. Thus $\rho_{cr}U^2$ is order U^2/c^2 smaller than P_{cr}. Since two-fluid models are based on a cosmic-ray transport equation which ignores terms of order mU/p, (9.1), $\rho_{cr}U^2$ is ignored in the momentum equation. Thus (11.5) is recovered.

For the energy equations (11.3,4), the approach is somewhat different. The cosmic rays and the thermal plasma were considered coupled for the momentum equation (11.2). However, they are decoupled in the energy equations (11.3,4). Therefore the energy equation for the single-fluid system is to be applied to two different fluids. This brings out the clumsiness of the two-fluid approach: the fluids are coupled in the momentum equation but not in the energy equation.

The two-fluid energy equations are obtained from the single-fluid mass–energy equation that includes F^a, (8.5). Since (8.5) includes the rest-mass energy, let me first write down (8.5) with the rest-mass energy subtracted out, using (8.2) to do it. Since this equation is more proper to §8 than here, I'll call the equation (8.5b).

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x^a}(TU^a + Q^a) + \frac{\partial U^a}{\partial x^a}P = 0 \tag{8.5b}$$

Recall the definition (6.2) for Q^a:

$$Q^a \equiv 4\pi \int \frac{S^a}{m}(mc^2 - m_0c^2)p^2 dp \tag{11.12}$$

The quantity $mc^2 - m_0c^2$ is just the kinetic energy of a particle with momentum p. For the thermal plasma, the kinetic energy of a particle is $\sim p^2/2m$, and $mU/p \sim 1$. In §7.6 it is shown that $S \sim (\lambda/L)pf_0$. Therefore $Q_{th}/(P_{th}U) \sim \lambda/L$. The two-fluid models ignore terms of order λ/L because they use the cosmic-ray transport equation (9.1), so Q_{th} is ignored in the energy equation for the thermal plasma. Thus (11.3) arises from (8.5b) with $Q_{th} = 0$.

For the cosmic rays, $mU/p \ll 1$. The kinetic energy of a cosmic-ray particle is $\sim pc$. Therefore $Q_{cr}/(P_{cr}U) \sim (\lambda/L)(p/mU)$, which for some values of p is ~ 1. Thus Q^a is kept in the cosmic-ray energy equation. Identifying $m\kappa\partial_a f_0 = S^a$ and using the definition (11.6), (8.5b) translates straight into (11.4).

For all its apparent weaknesses, the two-fluid model does lead to some insight into the effect of cosmic rays on shock structure. However, the ad hoc nature of (11.6,8,9,10) make the two-fluid description a fairly restrictive one. It is much more elegant to proceed from a single-fluid approach. Then the five conservation equations plus a single equation of state are sufficient to determine ρ, U^a, P, and T for the single fluid.

11.3) Extending the Two-Fluid Models

It is natural to extend the two-fluid formalism to include the effects contained in the extended cosmic-ray transport equation. This will include the 'viscous' and 'inertial' effects. Such an extension of the transport equation implies a refinement of the equation, because the extensions are smaller than the terms of the old equation. The old transport equation (9.1) traditionally used in the two-fluid models is correct to first order in λ/L and zeroth order in mU/p (see §9). The extended transport equation (9.2) is correct to first order in λ/L and first order in mU/p. Thus the extended transport equation begins to relax the assumption that $mU \ll p$, which is the beginning of the blurring of the distinction between the two fluids.

Furthermore, if one uses a cosmic-ray transport equation corrected to order λ/L, one is forced to reckon with the fact that the transport equation is in terms of the velocity of the scattering centers U^a, whereas fluid dynamics is typically formulated in terms of a velocity V^a determined from the number flux. As discussed in

§8.3, the difference between V^a and U^a is of order λ/L. In the two-fluid model that ignores differences of λ/L, one can glibly combine the cosmic-ray energy equation with the usual fluid equations for the thermal gas. But if terms of order λ/L are kept, then non-intuitive corrections must be introduced into either the fluid equations or the transport equation to keep everything consistently in terms of the same velocity.

Because of these things, the one-fluid model of this thesis becomes much more natural to describe the system. When one attempts to build a two-fluid description based on the extended cosmic-ray transport equation, one is implicitly moving into a range of parameters in which the distinction between the two fluids is blurred. Also, naive combinations of fluid dynamics with the transport equation may be mistaken because the velocity of the scattering centers isn't exactly the velocity defined by the number flux.

Furthermore, the one-fluid model avoids the problems of the two-fluid models that come from presuming closure parameters. The results of two-fluid calculations sensitively depend on the closure parameters (Kang & Drury, 1992). Since the one-fluid model has the distribution function, problems with closure parameters are avoided.

12. Self-Consistent Calculations: Prescription and Examples

In this section I illustrate how to go about self-consistent calculations within the one-fluid framework developed in §7 and §8. In §8 I derived the fluid equations consistent with the transport equation of §7. The transport equation is valid for particles of all energies, and is correct to first order in λ/L. Here is the transport equation:

$$\frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} + \frac{1}{m}\frac{\partial S^a}{\partial x^a} = \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} + m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) + \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab}) \tag{7.11}$$

$$S^a = -m\kappa_{ab}\left[\frac{\partial f_0}{\partial x^b} - \frac{m^2}{p}\frac{\partial f_0}{\partial p}\frac{dU^b}{dt}\right] \tag{9.2}$$

$$\Pi_{ab} = \Upsilon_{abcd}\frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\Lambda_{cd} \tag{10.5}$$

This equation describes the evolution of the isotropic part of the particle distribution function in terms of the fluid velocity of the scattering centers, U^a. To complete the self-consistent description of the system, an equation for U^a is needed. That equation is the momentum equation:

$$\frac{\partial}{\partial t}(\rho U^a + F^a) + \frac{\partial}{\partial x^b}(\rho U^a U^b + P_{ab} + F^a U^b + F^b U^a + P^B{}_{ab}) = 0 \tag{8.11}$$

$$P^B{}_{ab} \equiv \frac{1}{4\pi}\left\{\frac{1}{2}B^2\delta_{ab} - B^a B^b\right\}$$

The induction equation relates the evolution of the magnetic field to U^a:

$$\frac{\partial B^a}{\partial t} = \frac{\partial}{\partial x^b}(U^a B^b - U^b B^a) \tag{8.14}$$

This completes the self-consistent description of the system. The unknowns are the particle distribution f_0, and the flow velocity of the scattering centers, U^a. Why use only the momentum equation to constrain U^a? What about the number and energy equations? The answer is that only the momentum equation is independent of the transport equation. This is because the transport equation is obtained from the zeroth moment of the kinetic equation (see §7 and Appendix E). The momentum equation is obtained from the first moment of the kinetic equation (see §8 and Appendix D). The number and energy equations are obtained from the zeroth moment of the kinetic equation. The number equation is obtained merely by integrating the transport equation over all momenta (§8 and Appendix D). The energy equation referred to the fluid frame is obtained by multiplying the transport equation by the energy per particle, and integrating over all momenta (§8 and Appendix D). The inertial-frame energy equation (8.4) that includes the flow kinetic energy is obtained from a combination of the zeroth and first moments. So the bottom line is that the transport equation, via integrals of the distribution function, contains all the information that is in the number and energy equations. The momentum equation cannot be obtained from an integral of the transport equation.

The result is a self-consistent description of the transport and dynamics of a system of particles characterized by a nearly-isotropic distribution, correct to order λ/L and for non-relativistic flows. Here are some simple example calculations to demonstrate the prescription. The emphasis here is not on the particular solutions, but on the process of finding the solutions. The main point is that one may self-consistently calculate the flow velocity and particle distribution for a given system without artificially splitting the system into populations of varying energy, or assuming equations of state.

12.1) Sound Waves

Consider small-amplitude disturbances in a medium at rest that is characterized by a known distribution function, f_α. Linearize the equations in the variations from equilibrium, and consider variations along one spatial dimension, x: $P \rightarrow P_\alpha + P(x,t)$, $\rho \rightarrow \rho_\alpha + \rho(x,t)$, $U \rightarrow U(x,t)$, $F \rightarrow F(x,t)$, $\eta \rightarrow \eta_\alpha + \eta(x,t)$, $f_0 \rightarrow f_\alpha(p) + f(x,t,p)$. We will assume a constant scattering time τ, and assume the particles are non-relativistic, so the mass m is constant and particle kinetic energies are $p^2/2m$.

For the distribution characterizing the medium at rest, we choose the so-called 'kappa' distribution. The choice is made to emphasize that we wish to allow consideration of any isotropic distribution; Scudder (1992) discusses the appropriateness of using this distribution to characterize typical astrophysical plasmas. This distribution looks like a power law at high momenta and thermal at low momenta, for any finite α. When $\alpha \rightarrow \infty$, the distribution becomes thermal at all momenta. We normalize f_α in terms of the pressure P_α obtained from f_α via (6.2). An α subscript will denote other fluid quantities obtained from f_α.

$$f_\alpha = \frac{6m}{\Gamma(5/2)} P_\alpha \frac{\Gamma(\alpha)}{\Gamma(\alpha - 5/2)} (\alpha {p_0}^2)^{-5/2} \left(\frac{p^2}{\alpha {p_0}^2} + 1 \right)^{-\alpha}$$

All our fluid quantities are obtainable as integrals of the form:

$$\int_0^\infty p^{2m} \left(\frac{p^2}{\alpha {p_0}^2} + 1 \right)^{-\alpha} dp = \frac{1}{2} (\alpha {p_0}^2)^{m+1/2} \frac{\Gamma(m+1/2)\Gamma(\alpha - m - 1/2)}{\Gamma(\alpha)}$$

$$m + 1/2 < \alpha$$

The restriction that $m + 1/2 < \alpha$ is necessary to obtain this form of the integrals. In these calculations, m is never greater than 4. The thermal distributions are included for $\alpha \rightarrow \infty$, so are unaffected by this restriction.

The two independent equations to be solved are the linearized transport equation,

$$\frac{\partial f}{\partial t} - \frac{p^2 \tau}{3m^2} \frac{\partial^2 f}{\partial x^2} = \frac{p}{3} \frac{\partial f_\alpha}{\partial p} \left[\frac{\partial U}{\partial x} - \tau \frac{\partial^2 U}{\partial x \partial t} \right]$$

and the linearized momentum equation,

$$\rho_\alpha \frac{\partial U}{\partial t} + \frac{\partial F}{\partial t} + \frac{\partial P}{\partial x} - \eta_\alpha \frac{\partial^2 U}{\partial x^2} = 0$$

We look for solutions of the form:

$$U(x,t) = U_1 \exp^{i(kx-\omega t)} \qquad f(x,t,p) = \phi(p) \exp^{i(kx-\omega t)}$$

The transport equation yields:

$$\phi = \frac{p}{3}\frac{\partial f_\alpha}{\partial p} k U_1 \frac{(i-\omega t)}{(\kappa k^2 - i\omega)} \qquad \kappa \equiv \frac{p^2 \tau}{3m^2}$$

This expression is rationalized by multiplying top and bottom by $\kappa k^2 + i\omega$. Since these equations are only valid to linear in τ, and τ is assumed to be the shortest timescale, we may write ϕ:

$$\phi = \frac{2}{3}\frac{p^2}{{p_0}^2} f_\alpha (p^2/\alpha {p_0}^2 + 1)^{-1} \frac{U_1 k}{\omega} \left[1 + i\omega\tau - i\frac{\kappa k^2}{\omega}\right] + O(\tau^2)$$

To satisfy the momentum equation, the expression for f is integrated to yield the pressure:

$$P = \frac{5}{3} P_\alpha \frac{U_1 k}{\omega} \left[1 + i\omega\tau - i\frac{7}{3}\frac{{\omega_\alpha}^2 \tau}{\omega}\right] \exp^{i(kx-\omega t)} \qquad {\omega_\alpha}^2 \equiv \frac{{p_0}^2 k^2}{2m^2} \left(\frac{\alpha}{\alpha - 7/2}\right)$$

This quantity can be substituted into the momentum equation, which will then yield a dispersion relation $\omega(k, \tau, \alpha)$.

Since τ and m are constant, it turns out that:

$$F = \int_0^\infty S p^2 dp = \int \left[-m\kappa \frac{\partial f}{\partial x} + \frac{mp\tau}{3}\frac{\partial f_\alpha}{\partial p}\frac{\partial U}{\partial t}\right] p^2 dp = -\tau \frac{\partial P}{\partial x} - \tau \rho_\alpha \frac{\partial U}{\partial t}$$

$$\eta_\alpha = -\frac{4}{3}\int_0^\infty \frac{p^3 \tau}{15m}\frac{\partial f_\alpha}{\partial p} p^2 dp = \frac{4}{3} P_\alpha \tau$$

Noting that f_α implies an equation of state:

$$P_\alpha = \frac{\rho_\alpha {p_0}^2}{2m^2}\left(\frac{\alpha}{\alpha - 5/2}\right),$$

one obtains this dispersion relation:

$$\omega^2\tau - i\omega + \frac{5}{3}\frac{{\omega_\alpha}^2}{\omega}\xi\left[i - 2\omega\tau + \frac{7}{3}\frac{{\omega_\alpha}^2}{\omega}\tau\right] + \frac{4}{3}{\omega_\alpha}^2\tau\xi = 0 \qquad \xi \equiv \frac{\alpha - 7/2}{\alpha - 5/2}$$

Since we are working to terms linear order in τ, this dispersion relation is best approached by solving first for the $\tau = 0$ solution, ω_0, and then looking for solutions of the form $\omega = \omega_0 + a\tau$, where a is a constant. Proceeding in this way, one finds the two roots:

$$\omega_\pm = \pm\left(\frac{5\xi}{3}\right)^{1/2}\omega_\alpha - i{\omega_\alpha}^2\frac{(7-\xi^2)}{6\xi}\tau$$

The imaginary term corresponds damping of the wave. The damping and the wave frequency are both k-dependent, with shorter- wavelength perturbations damping faster.

We have made the point that the transport equation contains all the information present in the number and energy equations. To reinforce this claim, we integrate our expression for f to find n, G, T, and Q, and substitute these into the linearized number and energy equations. The linearized number equation is:

$$\frac{\partial n}{\partial t} + n_\alpha\frac{\partial U}{\partial x} + \frac{\partial G}{\partial x} = 0$$

The linearized energy equation is:

$$\frac{\partial T}{\partial t} + (P_\alpha + T_\alpha)\frac{\partial U}{\partial x} + \frac{\partial Q}{\partial x} = 0$$

The integrals of f specified by (6.2) yield:

$$n = n_\alpha \frac{U_1 k}{\omega}\left[1 + i\omega\tau - i\frac{5}{3}\frac{{\omega_\alpha}^2\tau}{\omega\xi}\right]\exp^{i(kx-\omega t)}$$

$$Q = i\frac{5}{2}U_1 P_\alpha \omega\tau\left(1 - \frac{7}{3}\frac{{\omega_\alpha}^2}{\omega^2}\right)\exp^{i(kx-\omega t)}$$

Also note that $T = 3P/2$ and $G = F/m$. When these expressions are substituted into the energy and number equations, one finds that they are indeed satisfied, to linear in τ, for any k and ω. Thus the momentum equation is the one to fix the dispersion relation, and so is independent of the transport equation.

12.2) Steady-State Atmosphere

We work with the equations for the case $U^a = 0$. Consider a non-relativistic gas of one spatial dimension x, in a constant gravitational field, g. Put the base of the atmosphere at $x = 0$, with x increasing up and gravity pointing down. From (7.11), the transport equation becomes:

$$\frac{1}{m}\frac{\partial S}{\partial x} = mg\frac{1}{p^2}\frac{\partial}{\partial p}(pS)$$

$$S \equiv -\frac{p^2\tau}{3m}\frac{\partial f_0}{\partial x} + \frac{mp\tau g}{3}\frac{\partial f_0}{\partial p}$$

As can be seen from (5.5), the equation in terms of g is obtained by substituting g for dU^a/dt.

The solution to the transport equation must be consistent with the momentum equation (8.11). In the absence of the variable U^a which must be determined by the momentum equation, we merely have a constraint on the solution:

$$\frac{dP}{dx} = -\rho g$$

In this problem, the momentum equation is equivalent to a constraint on S. We may write:

$$\frac{p^2 S}{\tau} = -\frac{p^4}{3m}\frac{\partial f_0}{\partial x} + \frac{\partial}{\partial p}\left(\frac{mp^3 g}{3} f_0\right) - mgf_0 p^2$$

The momentum equation is equivalent to:

$$\int \left[\frac{p^2}{3m}\frac{\partial f_0}{\partial x} + mgf_0\right] p^2 dp = 0$$

So under the assumption that $p^3 f$ vanish at 0 and ∞, the momentum equation may be written as a constraint on S:

$$\int \frac{S}{\tau} p^2 dp = 0$$

The form of the scattering time $\tau(p)$ has yet to be specified. This piece of information is not predicted by the theory but is input a priori.

The transport equation is separable, and a general solution in terms of two free constants A and b is:

$$pS = A \exp^{bx} \exp^{bp^2/2m^2 g}$$

To satisfy the momentum constraint, A must be zero for reasonable forms of $\tau(p)$. Thus the definition of S is taken as an equation for f_0, with $S = 0$. Again, the solution is separable in terms of free constants B and a:

$$f_0 = B \exp^{ax} \exp^{ap^2/2m^2 g}$$

To set the free constants requires we specify the boundary conditions. A boundary condition $f_0(x = 0)$ is sufficient. For $f_0(x = 0) = \exp^{-p^2/2mkT}$, $a = -mg/kT$ and we recover the exponential atmosphere:

$$f_0 = \exp^{-mgx/kT} \exp^{-p^2/2mkT}$$

If the boundary condition at the base were not thermal, then the separable solutions may not be appropriate, and different solutions are sought. It is the boundary conditions that allow us to choose among solutions to the equations.

Scudder (1992) has emphasized that if the boundary condition is non-thermal, new possibilities arise. Sticking with the class of solutions for which $S = 0$, we can write down another expression for f_0:

$$f_0 = A\left[\frac{2m^2gx}{{p_0}^2} + \frac{p^2}{{p_0}^2} + 1\right]^{-\alpha}$$

This form, where A, α and p_0 are constants, is another form of the kappa distribution, and satisfies the transport equation and is consistent with momentum conservation. Of course, for this to be a real solution requires it satisfy the boundary condition:

$$f_0(x = 0) = A\left[\frac{p^2}{{p_0}^2} + 1\right]^{-\alpha}$$

The point is, a non-thermal boundary condition allows new possibilities for the equilibrium structure of a plasma held in a gravitational field. Also, the one-fluid scheme presented here is consistent with Scudder's interpretation of the temperature inversion in the corona as resulting naturally from a plasma with a non-thermal distribution being confined in a gravitational field. Unlike Scudder's approach to the non-thermal distribution, this is no 'hybrid' approach; there is a single distribution function for particles of all energies, obeying a single transport equation.

12.3) Steady-State Shear

For this case we restrict attention to a two-dimensional steady-state shear problem, in which the fluid velocity is assumed to be of the form $U^a = U(x)\hat{y}$. This implies $dU^a/dt = 0$. We further assume x to be the only spatial variable, so that

$f_0 = f_0(x,p)$. Again, the controlling equations are the transport equation and the momentum equation.

The transport equation is:

$$\frac{1}{m}\frac{\partial S^x}{\partial x} = \frac{\partial U}{\partial x}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{xy})$$

$$S^a = -\frac{p^2\tau}{3m}\frac{\partial f_0}{\partial x}\hat{x} \qquad \Pi_{xy} = \frac{p^3\tau}{15}\frac{\partial f_0}{\partial p}\frac{\partial U}{\partial x}$$

The vector momentum equation is:

$$\frac{\partial}{\partial x}(P_{ax} + F^x U^a) = 0$$

The free index a can take on the values x and y to yield the two momentum equations:

$$\frac{\partial P_{xx}}{\partial x} = 0 \qquad \frac{\partial}{\partial x}(P_{xy} + F^x U) = 0$$

$$P_{xx} = \int \frac{p^2}{3m} f p^2 dp \quad P_{xy} = \frac{\partial U}{\partial x}\int \frac{\Pi_{xy}}{m} p^2 dp \quad F^x = \int S^x p^2 dp$$

The reason there are apparently three dynamical equations for the two unknowns f and U is that our assumption for the form of U^a effectively presumes an equation, namely the xx momentum equation. The assumed form of U^a specifies a class of solutions possible in the $\hat{x}-\hat{y}$ plane, and this class of solutions requires $\partial P_{xx}/\partial x = 0$. So it is the xy component of the momentum equation that is combined with the transport equation to determine the dynamics of f and U.

As in the previous example, many solutions are possible to the transport equation. It is the boundary conditions which determine the appropriate solution. Since the transport equation is separable, we assume boundary conditions consistent with the separable solution. For $f_0(x,p) = g(x)h(p)$, the transport equation implies:

$$-\frac{1}{\Lambda(x)^2 g}\frac{d^2 g}{dx^2} = \frac{m^2}{p^2\tau h}\frac{d}{dp}\left(\frac{p^4\tau}{5}\frac{dh}{dp}\right) = k$$

Here, k is the separation constant, and $\Lambda(x) \equiv \partial U/\partial x$.

The shear condition, $\partial P_{xx}/\partial x = 0$, implies $g = const$, which then implies $k = 0$. Thus we obtain for h, in terms of constants C_1 and C_2:

$$h(p) = C_1 \int \frac{1}{\tau p^4} dp + C_2$$

For $\tau \propto p^n$, we obtain a powerlaw for f_0. Since $\partial f/\partial x = 0$, $F^x = 0$. Writing $P_{xy} = -\eta\Lambda(x)$, the momentum equation implies:

$$U(x) = Ax + B$$

We obtain a linear shear from the separable solution. In a linear shear, there is no accumulation of momentum in the fluid. A fixed flux of momentum, $-\eta\Lambda$, cascades across the shear. The momentum flux input at a boundary x_1 comes out at a boundary x_2. This solution is appropriate to power-law boundary conditions.

Note that the power-law momentum dependence of f_0 derived above is implicitly x-dependent when referred to an inertial frame, because the particle momentum is referred to a frame which depends on x. In terms of momenta $\widetilde{p}^a$ referred to an inertial frame:

$$p = \widetilde{p}\left[1 + \frac{mU(x)}{\widetilde{p}}\left(\frac{mU(x)}{\widetilde{p}} - 2\frac{\widetilde{p}^y}{\widetilde{p}}\right)\right]^{1/2}$$

So an inertial observer would see a *non-isotropic* distribution, with the complicated spatial dependence indicated above. The computation is reduced enormously by working in the frame U^a.

13. Summary and Future Work

The goal of this thesis is to unite the descriptions of cosmic rays and the space plasmas in which they propagate.

The space plasmas, the interstellar and interplanetary media, have been described in the past with the equations of magnetohydrodynamics. They were assumed to have a Maxwellian distribution, characterized by a single temperature. Sometimes separate temperatures are introduced for the electrons and the protons.

The cosmic rays have been described with a transport equation that includes convection, diffusion, adiabatic energy-change; and more recently, viscous and inertial effects. The transport equation describes the evolution of the cosmic-ray distribution function given a fluid velocity that must be specified independently. The fluid velocity is presumed to be the velocity of the space plasma. The transport equation describes acceleration of particles through a transfer of kinetic energy from the flow of the space plasma into internal degrees of freedom of the particles.

To investigate the back-reaction of cosmic-ray acceleration on the flow, the transport equation was integrated to obtain the evolution of the cosmic-ray pressure and internal energy density. The cosmic-ray pressure was added into the equations of magnetohydrodynamics describing the flow. This is the two-fluid approach. This approach lead to some insights into the back-reaction of accelerated particles on the flow of thermal particles. But a distinction always remained between the thermal and cosmic-ray particles. Due to the lack of a particle distribution in the two-fluid approach, closure parameters were introduced on which results of two-fluid calculations sensitively depend (Kang & Drury, 1992).

Numerical Monte Carlo calculations were done to study this back-reaction, and it was found that there was really no firm distinction between the thermal and cosmic-ray particles. A thermal particle could be accelerated, in a flow composed initially of thermal particles, to cosmic-ray energies. Furthermore, some of the numerical results disagree with the two-fluid results for similar parameters. For these reasons, I have worked on finding the equations governing a self-consistent description of the entire system without distinguishing between thermal particles and cosmic rays. The set of equations is non-linear, but numerical solution of differential equations is more efficient than modelling a system with Monte Carlo techniques. It is hoped that having a set of equations for the single fluid, including spectral information, will bridge the gap between the two-fluid and Monte Carlo approaches.

Given an accurate set of single-fluid equations, there remains one strength of the Monte Carlo models that the single-fluid model does not possess: fluid equations cannot describe structure on the micro-scale. But they can describe the evolution of a distribution of particles from thermal to cosmic-ray. And they can describe the effect of particle acceleration on the evolution of the space plasma, for instance the evolution of supernova remnants.

In this thesis I obtained a set of equations for the single fluid of particles of all energies. Number, momentum and energy equations were obtained for the fluid. A transport equation for the particle distribution function, for particles of all energies, was obtained. These results are summarized in §4. The transport equation is still in terms of a fluid velocity, the velocity of the scattering centers, but now this velocity is specified in terms of the particle distribution itself. It was shown in §9 how the cosmic-ray transport equation is obtained from the general transport equation when only fast particles are considered. And in §11 the two-fluid model was obtained from the single-fluid equations after splitting the fluid variables into thermal and cosmic-ray parts. In §12 a prescription for self-consistent calculations was laid out.

The next step in this work, and the work that is to be done beyond this thesis, is to perform calculations with the single-fluid equations and compare them with two-fluid results, with the Monte Carlo results, and with observation. Since the equations are non-linear, this will involve some creative mathematics or some numerical integration; probably the latter. The two-fluid and Monte Carlo models predict significant smoothing of the 1-D shock profile, and so the single-fluid model presented here may be applicable even though it requires $\lambda/L \ll 1$. This formalism also shows promise for problems where discontinuities are not a problem, such as the heating of the corona and the origin of the solar wind.

There also remains future work on the theoretical side. I treated the space plasma as consisting of 'neutralized' protons. The particle distribution function is in terms of only a single species, the protons. As a fluid, the protons were assumed to be electrically neutral. However, the electrons must be included in the dynamics. This is because there are situations in which large electric fields occur, and therefore charge separation will occur; for example, perpendicular shocks. Thus, the single-fluid equations should be modified to allow for the electrons.

One can also extend the single-fluid equations to include terms of first order in U/c, or even for the case of arbitrary fluid velocity. As relativistic effects are included, intuition will suffer. The extension to first order in U/c is instructive and useful. In Williams and Jokipii (1991), the extended cosmic-ray transport equation was derived correct to first order in U/c. Webb (1989) does the fully-relativistic derivation of the extended cosmic-ray transport equation. These corrections are important because many astrophysical flows are relativistic. The theory must allow for relativistic flow speeds before the processes operating at these sites can be made clear.

I used an approximation to the particle scattering that could conceivably be improved upon as well. The relaxation-time operator for the scattering really does not tell one very much; only that the distribution relaxes to isotropy in some time that depends on particle momenta in some way. Perhaps an understanding of the mechanism of particle scattering, presumably due to waves in the plasma, could lead to new insights or results. However, it is not clear to me how much is lost by merely assuming the relaxation operator. It is both simple and general, and would seem to be a decent approximation to any scattering mechanism given an appropriate choice of the scattering time. Given the state of understanding of the scattering process, I think a modification of the scattering term in this analysis would not be the best use of one's time.

In astrophysical situations, gravity is important to the evolution of systems. In describing the evolution of a fluid of galactic scales, gravity should be included in the fluid equations. I ignored gravity in this work, but I believe it is worth redoing the analysis to include it. It will enter all the equations as a modification to the fluid acceleration vector. In the transport equation, it will affect the streaming and the inertial terms. In the fluid equations, it will enter in the usual way. Webb (1989) derives the fully-relativistic extended cosmic-ray transport equation, and so gravity is implicitly included.

In summary, future work should include the following. Perform calculations and compare them with the Monte Carlo and two-fluid results; include electrons in the dynamics; extend the equations to include gravity and allow for relativistic flows.

Appendix A

Justification of the Kinetic Equation

In this appendix, I will attempt to justify the kinetic equation from a few different points of view. I say 'justify' instead of 'derive' because the kinetic equation is not underlain by any physical law. In my view, one can only argue for its feasibility and then look for corroboration from experiment. The archetypal kinetic equation is the Boltzmann equation. This equation has been successfully verified by observation, and this alone is justification for a kinetic equation.

I use the words 'kinetic equation' as a general term. The Boltzmann equation is a special case because of the particular form of the scattering term that it assumes. The kinetic equation used in this thesis uses a different scattering term, the relaxation term. The justifications that follow will only apply to the non-scattering elements of the kinetic equation. For a general kinetic equation:

$$\frac{Df}{Dt} = \left(\frac{\delta f}{\delta t}\right)_{scatter} \tag{A1}$$

only the LHS will be justified here. It represents the free evolution of a system of non-interacting particles. The RHS is the scattering term, and must be justified separately based on some understanding of the physics of particle interactions. Justification of a particular scattering term appears in the body of this thesis.

The different justifications for the LHS of (A1) that follow should not be mistaken for rigorous derivations based on physical law. Rather they are heuristic and will be found to relate back to each other.

A.1) Conservation of Density in Phase Space

This is the simplest approach, and the one I find most appealing. A system is to be represented by a density in phase space. Phase space is the space spanned by the all the coordinates it takes to specify the state of a single particle. For the monatomic gas of protons I consider, phase space is six-dimensional: 3 spatial coordinates, and 3 momentum coordinates. The density in phase space is written $f(t, x^a, p^a)$: a function of time, position, and momentum. Thus $d^3xd^3pf(t, x^a, p^a)$ is the number of particles, at time t, with positions in a volume element of size d^3x centered around x^a, and with momenta in a volume element of size d^3p centered around p^a. Such a definition implicitly assumes that there are enough particles in each volume element that $d^3xd^3pf(t, x^a, p^a)$ is a well-defined quantity. If the volume element were too small, then statistical fluctuations would lead to different values of $d^3xd^3pf(t, x^a, p^a)$ for identical systems.

In general, kinetic theory presupposes an averaging that is necessary for a 'fluid description': that the fluctuations in $d^3xd^3pf(t, x^a, p^a)$ are much smaller than the size of the time-averaged value of $d^3xd^3pf(t, x^a, p^a)$.

So the distribution f is the density of a fluid in a six-dimensional space. There are no sources; the number of particles in the system is fixed:

$$\int f d^3p d^3x = \text{constant} \tag{A2}$$

Therefore the fluid will obey a phase-space conservation equation, a continuity equation if you like:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x^a}\left(f\frac{dx^a}{dt}\right) + \frac{\partial}{\partial p^a}\left(f\frac{dp^a}{dt}\right) = 0 \tag{A3}$$

Now, the coordinates $p^a/m = dx^a/dt$ and x^a are independent for this fluid. The term dp^a/dt represents an external force acting on the particles, so I put it $= F^a$. Finally, one assumes the absence of any momentum-dependent forces such that $\partial F^a/\partial p^a = 0$. It turns out that the Lorentz force is such a force:

$$\frac{\partial}{\partial p^a}\epsilon_{abc}\frac{p^b}{m}B^c = \epsilon_{abc}B^c\left(\frac{1}{m}\delta_{ab} + \frac{p^a p^b}{mc^2}\right) = 0$$

See appendix B for the dependence of mass on momentum. So from (A3), one arrives at the kinetic equation.

$$\frac{\partial f}{\partial t} + \frac{p^a}{m}\frac{\partial f}{\partial x^a} + F^a\frac{\partial f}{\partial p^a} = 0 \tag{A4}$$

A.2) Liouvilles's Theorem

This theorem is found throughout the physics literature, so I won't derive it here. It is also based on arguments about conservation of particles in phase space. This time the phase space has a dimensionality equal to the total number of particles in the system multiplied by the number of coordinates it takes to specify the position of each particle. For a system of N monatomic particles, the phase space is 6N-dimensional.

One then considers an *ensemble* of such systems and defines the function $\rho(t, q_i, p_i)$ where $\rho d^{6N}qd^{6N}p$ is the number of *systems* which at time t are in the neighborhood of size $d^{6N}qd^{6N}p$ about the point (q_i, p_i); q_i are the 3N position coordinates of all the particles in the system and p_i are their 3N momentum coordinates. Since the number of systems in the ensemble is fixed, the motion of the ensemble is assumed to obey a continuity equation like (A3). Using Hamilton's equations of motion, one finds:

$$\frac{\partial \rho}{\partial t} + \frac{dq_i}{dt}\frac{\partial \rho}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial \rho}{\partial p_i} = 0 \tag{A5}$$

(A5) is Liouville's theorem.

Clearly (A5) has the form of (A4). It's just that the dimensionality is different. The connection comes by applying (A5) to an ensemble of *single-particle* systems. Then ρd^3qd^3p is interpreted as the probability of finding the particle in the volume element of size d^3pd^3q centered at (q, p). Clearly $\int \rho d^3pd^3q = 1$. If this normalization is changed to $\int \rho d^3pd^3q = N$, then ρd^3qd^3p becomes the number of particles in the volume element of size d^3pd^3q centered at (q,p). So $N\rho$ is merely the distribution function and (A5) is the kinetic equation for non-interacting particles.

A.3) Covariant Kinetic Equation

Aside from the practical predictions about the nature of gravity and motion made by the general theory of relativity, one can view relativity as a requirement that any mathematical description of a physical system must obey. Relativity is also a meta-theory into which any description of physical law must fit. The equations of physics must have the same form when written in two different frames of reference: this is covariance.

One can rightly ask if there is a covariant form for the kinetic equation. For a system of non-interacting particles there is (it is still unclear to me if scattering can be expressed covariantly).

$$\frac{df}{d\tau} = 0 \tag{A6}$$

(A6) is a covariant kinetic equation where τ is the proper time of the particles and $f = f(x^\alpha, p^a)$, a Lorentz invariant (Forman, 1970); x^α is the spacetime position of a particle, p^a is the vector component of its four-momentum (see Appendix B on relativity). Only the spatial part of the momentum four-vector is used because for particles of a fixed rest-mass m_0, the energy component is fixed by the particle momentum: ${p_0}^2 = m^2c^2 = p^2 + {m_0}^2c^2$.

One merely expands the derivative in (A6).

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} + \frac{\partial f}{\partial p^a}\frac{dp^a}{d\tau} = \frac{\partial f}{\partial t}\frac{dt}{d\tau} + \frac{\partial f}{\partial x^a}\frac{dx^a}{d\tau} + \frac{\partial f}{\partial p^a}\frac{dp^a}{d\tau}$$

Now recall that $dt/d\tau = \gamma$, the Lorentz factor; $dx^a/d\tau = \gamma dx^a/dt = p^a/m_0$; $dp^a/d\tau = \gamma dp^a/dt$. So a covariant kinetic equation for a system of non-interacting particles is:

$$\gamma\frac{\partial f}{\partial t} + \frac{p^a}{m_0}\frac{\partial f}{\partial x^a} + \gamma\frac{dp^a}{dt}\frac{\partial f}{\partial p^a} = 0 \qquad (A7)$$

Dividing through by γ yields the kinetic equation used in the thesis.

Appendix B

Results from Special Relativity

B.1) Transformation of Four-Vectors

A fundamental quantity in special relativity is the four-vector. Knowledge of the appropriate four-vector and how it transforms is sufficient to determine the transformation of any vector or scalar quantity. My approach in special relativity is simply to *assume* that flat space-time has the peculiar hyperbolic geometry that leaves the 'lengths' of four-vectors invariant upon transformation between reference frames.

A four-vector has 3 spatial components and one time component: $(x_0, x_1, x_2, x_3) = (x_0, x^a) = x^\alpha$. Small roman superscripts denote spatial components of the four-vector, and greek superscripts denote all four components. The length of a four-vector is $x^\alpha x^\alpha = {x_1}^2 + {x_2}^2 + {x_3}^2 - {x_0}^2 = x^a x^a - {x_0}^2$. This 'length' is invariant between coordinate systems. Furthermore, the contraction of any two four-vectors is invariant between coordinate systems. There is a mathematical object known as the Lorentz transformation matrix that, when contracted into any tensor, yields the transformed tensor. I refer the reader to Jackson (1975), §11.7, to see it written down. When the Lorentz matrix operates on a four-vector, the result is the following:

$$\widetilde{x}_0 = \gamma(x_0 - \beta^a x^a) \qquad (B1)$$

$$\widetilde{x}^a = x^a + \frac{(\gamma - 1)}{\beta^2}(\beta^b x^b)\beta^a - \gamma\beta^a x_0 \qquad (B2)$$

where

$$\gamma \equiv (1 - \beta^2)^{-1/2} \qquad \beta^a \equiv U^a/c$$

(B1) and (B2) describe the transformation of a four-vector between reference frames in which the tilda frame has velocity U^a relative to the no-tilda frame. The speed of light is c. The reader may check that $x^2 - {x_0}^2 = \widetilde{x}^2 - \widetilde{x}_0^2$.

Here are some typical four-vectors. The spatial coordinates and time characterizing an event: $(ct, r^a) = x^\alpha$. The energy and momentum of a particle: $(E/c, p^a) = (mc, p^a) = \gamma m_0(c, v^a)$, where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and m_0 is the rest mass of the particle. Since m_0 is a constant, $\gamma(c, v^a)$ is also a four-vector, known as the four-velocity. Defining the proper time τ of a particle as $dt/d\tau = \gamma$, the four-velocity may be written $dx^\alpha/d\tau$.

From electrodynamics, the charge density and current density constitute a four-vector: $(c\rho, J^a)$, as do the scalar and vector potentials: $(\phi, A^a) = A^\alpha$ (the units are cgs). Since ϕ and A^a represent the field, the electromagnetic field is known as a vector field: which means a four-vector field.

B.2) Transformation of the Electromagnetic Field

The electric and magnetic fields are not so fundamental as the four-vector potential. They are instead components of an antisymmetric tensor defined by: $\partial A^\alpha/\partial x^\beta - \partial A^\beta/\partial x^\alpha$. Contracting the Lorentz matrix into each index of this tensor yields the transformed tensor, and therefore the transformed electric and magnetic fields E^a and B^a. The transformations are:

$$\widetilde{E}^a = \gamma(E^a + \epsilon_{abc}\beta^b B^c) - \frac{\gamma^2}{\gamma+1}\beta^a(\beta^b E^b) \tag{B3}$$

$$\widetilde{B}^a = \gamma(B^a - \epsilon_{abc}\beta^b E^c) - \frac{\gamma^2}{\gamma+1}\beta^a(\beta^b B^b) \tag{B4}$$

The tilda frame has velocity U^a relative to the no-tilda frame.

B.3) Momentum and Mass/Energy Transformations

For transformations to a frame moving with a velocity U^a where $U \ll c$, the transformations above are particularly simple. In the body of the thesis, I transform the momentum coordinate into a frame moving with velocity U^a. I ignore terms of order U/c smaller than the largest in any equation. Under this restriction, I can use (B2) to find out how the spatial part of the energy-momentum four-vector (the momentum) transforms:

$$\widetilde{p}^a = p^a + \frac{(\gamma - 1)}{\beta^2}(\beta^b p^b)\beta^a - \gamma\beta^a p_0 \simeq p^a - mU^a \qquad (B5)$$

I put $\gamma = 1$ above, and put mc in for p_0. Now I use (B1) to find out how the energy (which is proportional to the mass) transforms:

$$\widetilde{m}c = \widetilde{p}_0 = \gamma(p_0 - \beta^a p^a) = mc - U^a p^a / c \simeq mc \qquad (B6)$$

The last step came by noting $p \leq mc$.

B.4) Mass as a Function of Momentum

I noted above that $m = m_0\gamma$. In this form, m is a function of velocity. In my work it is more convenient to have it as a function of momentum:

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0\left(1 - \frac{p^2}{m^2c^2}\right)^{-1/2} \Rightarrow m = m_0\left(1 + \frac{p^2}{{m_0}^2c^2}\right)^{1/2} \qquad (B7)$$

This has the correct limiting behavior. When $p \ll m_0 c$, $m \simeq m_0 + p^2/2m_0c^2$. One can use (B7) to find the derivative of the mass with respect to momentum.

$$\frac{dm}{dp} = \frac{p}{mc^2} \qquad (B8)$$

B.5) Transforming the Momentum-Space Volume Element

I will also need to know how the volume element in momentum space transforms, again assuming $U \ll c$. The momentum transformation is $P^a = p^a + mU^a$.

$$\begin{aligned} d^3P = dP_x dP_y dP_z &= \left(dp_x + U_x \frac{p}{mc^2} dp\right)\left(dp_y + U_y \frac{p}{mc^2} dp\right)\left(dp_z + U_z \frac{p}{mc^2} dp\right) \\ &= dp_x dp_y dp_z \left(1 + \frac{p_x U_x}{mc^2}\right)\left(1 + \frac{p_y U_y}{mc^2}\right)\left(1 + \frac{p_z U_z}{mc^2}\right) \\ &= dp_x dp_y dp_z \left(1 + \frac{p^a U^a}{mc^2}\right) + Order(U^2/c^2) \simeq dp_x dp_y dp_z \end{aligned} \tag{B9}$$

Again, since $p \leq mc$, the last approximation follows.

B.6) The Hydromagnetic Condition

The hydromagnetic condition is usually assumed for space plasmas. It is nothing more than approximating the conductivity of the plasma as infinite. Due to the high conductivity, it is a good approximation to assume that the electric field vanishes in the plasma frame. This approximation is equivalent to defining the length- and timescales on which we discuss the plasma. Fluctuations produce microscopic electric fields, which average to zero. The plasma itself, being composed of charged particles, will have an electric field on microscopic scales. Therefore the assumption of vanishing electric field implies a lower limit to the lengthscales over which calculations are valid.

So one assumes that the average electric field in the plasma frame is zero. Assume the plasma moves with velocity U^a with respect to an inertial observer, $U \ll c$, and apply (B3) to the situation:

$$0 = E^a + \epsilon_{abc} U^b B^c / c \tag{B10}$$

This expression is correct to first order in U/c. For the plasma-frame electric field to vanish, the inertial-frame electric and magnetic fields must combine in this way.

(B10) defines E^a in terms of the magnetic field B^a and the fluid velocity U^a. So the electric field observed in the inertial frame, which is induced by motion of the magnetic field embedded in the plasma, is order U/c smaller than the inertial frame magnetic field. Since E^a is order U/c smaller than B^a, it follows from (B4) that, correct to order U/c, the fluid-frame magnetic field equals the inertial-frame magnetic field.

The hydromagnetic condition will lead to an approximation of Ampere's Law:

$$\epsilon_{abc}\frac{\partial B^c}{\partial x^b} = \frac{4\pi}{c}J^a + \frac{1}{c}\frac{\partial E^a}{\partial t} \simeq \frac{4\pi}{c}J^a - \frac{\epsilon_{abc}}{c}\frac{\partial(U^b B^c)}{\partial t} \simeq \frac{4\pi}{c}J^a \qquad (B11)$$

The last approximation follows by assuming that the macroscopic timescale t is related to the macroscopic lengthscale L by $t \sim L/U$. Then the displacement current is of order U^2/c^2 smaller than the curl of B^a, and is ignored.

The hydromagnetic condition also leads to an approximation to Faraday's Law:

$$\begin{aligned} -\frac{1}{c}\frac{\partial B^a}{\partial t} &= \epsilon_{abc}\frac{\partial E^c}{\partial x^b} \simeq -\frac{1}{c}\epsilon_{abc}\epsilon_{cde}\frac{\partial}{\partial x^b}(U^d B^e) \\ \Longrightarrow \frac{\partial B^a}{\partial t} &\simeq \frac{\partial}{\partial x^b}(U^a B^b - U^b B^a) \qquad (B12) \end{aligned}$$

This equation is known as the 'induction equation'.

Thus the hydromagnetic condition completely eliminates the electric field from the dynamical equations; only the magnetic field and the fluid velocity need to be specified. Here are the equations of this section rewritten in vector notation. While not useful as a general formalism for tensor analysis, the vector notation does provide an elegant expression of electrodynamics. The hydromagnetic condition:

$$\mathbf{E} = -\mathbf{U} \times \mathbf{B}/c \qquad (B10)$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{B11}$$

The induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U} \times \mathbf{B}) \tag{B12}$$

Appendix C

Transformation of the Kinetic Equation

In these pages, I transform the momentum coordinate of the inertial-frame-referenced kinetic equation into a frame moving with non-relativistic velocity U^a. The kinetic equation is:

$$\left(\frac{\partial f}{\partial t}\right)_{x,\tilde{p}} + \frac{\tilde{p}^a}{\tilde{m}}\left(\frac{\partial f}{\partial x^a}\right)_{t,\tilde{p}} + F^a\left(\frac{\partial f}{\partial \tilde{p}^a}\right)_{x,t} = \left(\frac{\delta f}{\delta t}\right)_{scatter} \qquad (C1)$$

I have put the subscripts on the partial derivatives to remind the reader that the partials are taken while the subscripted variables are held constant. t, x^a, $\tilde{p}^a$ are independent variables and are referenced to the inertial frame. The tilda on the momentum coordinate is in anticipation of its transformation to a new coordinate.

The transformation of the momentum coordinate is:

$$\tilde{p}^a = p^a + m(p)U^a(x^b, t) \qquad (C2)$$

I assume $U \ll c$, so I will be ignoring terms of order U/c in the transformation. Let's take the three partial derivatives one at a time. The fluid velocity U^a is allowed to depend on position and time.

Start with the time derivative.

$$\left(\frac{\partial f}{\partial t}\right)_{x,\tilde{p}} = \left(\frac{\partial f}{\partial t}\right)_{x,p} + \left(\frac{\partial f}{\partial p^a}\right)_{x,t}\frac{\partial p^a}{\partial t}$$

Since $\tilde{p}^a$ and t are independent coordinates, (C2) implies:

$$\frac{\partial \tilde{p}^a}{\partial t} = 0 = \frac{\partial p^a}{\partial t} + m\frac{\partial U^a}{\partial t}$$

So therefore:

$$\left(\frac{\partial f}{\partial t}\right)_{x,\tilde{p}} = \left(\frac{\partial f}{\partial t}\right)_{x,p} - \left(\frac{\partial f}{\partial p^a}\right)_{x,t} m\frac{\partial U^a}{\partial t} \qquad (C3)$$

Now do the space derivative.

$$\left(\frac{\partial f}{\partial x^a}\right)_{t,\widetilde{p}} = \left(\frac{\partial f}{\partial x^a}\right)_{t,p} + \left(\frac{\partial f}{\partial p^b}\right)_{x,t} \frac{\partial p^b}{\partial x^a}$$

Again, since $\widetilde{p}^a$ and x^a are independent:

$$\frac{\partial \widetilde{p}^b}{\partial x^a} = 0 = \frac{\partial p^b}{\partial x^a} + m\frac{\partial U^b}{\partial x^a}$$

Therefore:

$$\left(\frac{\partial f}{\partial x^a}\right)_{t,\widetilde{p}} = \left(\frac{\partial f}{\partial x^a}\right)_{t,p} - \left(\frac{\partial f}{\partial p^b}\right)_{x,t} m\frac{\partial U^b}{\partial x^a} \tag{C4}$$

Finally, do the momentum derivative.

$$\left(\frac{\partial f}{\partial \widetilde{p}^a}\right)_{x,t} = \left(\frac{\partial f}{\partial p^b}\right)_{x,t} \frac{\partial p^b}{\partial \widetilde{p}^a}$$

(C2) implies:

$$\begin{aligned}\frac{\partial \widetilde{p}^b}{\partial p^a} &= \delta_{ab} + U^b\frac{\partial m}{\partial p^a} = \delta_{ab} + U^b\frac{\partial m}{\partial p}\frac{\partial p}{\partial p^a} \\ &= \delta_{ab} + \frac{p^a U^b}{mc^2} \simeq \delta_{ab}\end{aligned}$$

For the momentum dependence of the mass, see Appendix B. The last approximation follows from ignoring terms of order U/c. Therefore one has:

$$\left(\frac{\partial f}{\partial \widetilde{p}^a}\right)_{x,t} = \left(\frac{\partial f}{\partial p^a}\right)_{x,t} \tag{C5}$$

Finally, in Appendix B it is shown that ignoring terms of order U/c, $m = \widetilde{m}$. Combining (C2-C5), the transformation of (C1) is:

$$\frac{\partial f}{\partial t} + U^a\frac{\partial f}{\partial x^a} + \frac{p^a}{m}\frac{\partial f}{\partial x^a} + \left[F^a - m\frac{dU^a}{dt} - p^b\frac{\partial U^a}{\partial x^b}\right]\frac{\partial f}{\partial p^a} = \left(\frac{\delta f}{\delta t}\right)_{scatter} \tag{C6}$$

where the convective derivative is defined:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U^b\frac{\partial}{\partial x^b}$$

In (C6), terms of order U/c have been ignored, so we are restricted to non-relativistic flows. The external force, F^a, the fluid velocity U^a, and the space and time coordinates are still referred to the inertial frame. The momentum coordinate is referred to the frame moving with velocity U^a. So this equation mixes reference frames.

Appendix D

Fluid Equations from the Kinetic Equation

The starting point is the kinetic equation:

$$\left[\frac{\partial}{\partial t} + U^a \frac{\partial}{\partial x^a} + \frac{p^a}{m}\frac{\partial}{\partial x^a} + \left\{\frac{q}{mc}\epsilon_{abc}p^b B^c - m\frac{dU^a}{dt} - p^b \frac{\partial U^a}{\partial x^b}\right\}\frac{\partial}{\partial p^a}\right] f = 0 \quad (D1)$$

The momentum coordinate p^a is referred to the frame moving with velocity $U^a(x^b, t)$ with respect to an inertial frame. The space and time coordinates x^a, t are referred to the inertial frame. The only force on the particles is the Lorentz force. The system is assumed to be composed of a single species of particles.

The procedure will generally be to integrate terms by parts and, employing the divergence theorem, to assume that all integrands vanish on the boundaries. This is equivalent to assuming that there are no particles at infinite momenta. Otherwise, about the only things needed are: $\partial p^b/\partial p^a = \delta_{ab}$ and $dm/dp = p/mc^2$. Also, recall the definitions:

$$n \equiv \int f d^3p = 4\pi \int f_0 p^2 dp$$

$$\rho \equiv \int m f d^3p = 4\pi \int m f_0 p^2 dp$$

$$P_{ab} \equiv \int f \frac{p^a p^b}{m} d^3p = 4\pi \int \left(\frac{p^2 f_0}{3m} + \frac{\Pi_{ab}}{m}\right) p^2 dp$$

$$F^a \equiv \int f p^a d^3p = 4\pi \int S^a p^2 dp$$

$$G^a \equiv \int f \frac{p^a}{m} d^3p = 4\pi \int \frac{S^a}{m} p^2 dp$$

$$T \equiv \rho c^2 - n m_0 c^2$$

$$Q^a \equiv F^a c^2 - G^a m_0 c^2$$

D.1) Number Equation

The procedure will be to integrate (D1) term by term over d^3p. The first three terms follow from definition.

$$\int \left[\frac{\partial f}{\partial t} + U^a \frac{\partial f}{\partial x^a} \right] d^3p = \frac{\partial n}{\partial t} + U^a \frac{\partial n}{\partial x^a}$$

$$\int \frac{p^a}{m} \frac{\partial f}{\partial x^a} d^3p = \frac{\partial G^a}{\partial x^a}$$

The Lorentz force term integrates to zero.

$$\frac{q}{mc} \epsilon_{abc} B^c \int p^b \frac{\partial f}{\partial p^a} d^3p = 0$$

This is because: $\delta_{ab}\epsilon_{abc} = 0$; and the contraction of a symmetric tensor with an antisymmetric tensor is zero.

To integrate the acceleration term, integrate by parts and apply the divergence theorem. It is assumed that the integrand vanishes on a boundary at ∞. Using $dm/dp = p/mc^2$ (see Appendix B), one finds:

$$-\frac{dU^a}{dt} \int m \frac{\partial f}{\partial p^a} d^3p = \frac{dU^a}{dt} \frac{G^a}{c^2}$$

For the last term, again integrate by parts and apply the divergence theorem, with the quantity evaluated at the boundary discarded:

$$-\frac{\partial U^a}{\partial x^b} \int p^b \frac{\partial f}{\partial p^a} d^3p = n \frac{\partial U^a}{\partial x^a}$$

Now just put these together, ignoring terms order U/c smaller than the largest. Under this condition, the acceleration term is ignored. There are two pieces to the acceleration term. The piece $U^b G^a/c^2(\partial U^a/\partial x^b)$ is order U^2/c^2 smaller than $\partial G^a/\partial x^a$. The piece $(\partial U^a/\partial t)(G^a/c^2)$ is order U/c smaller than $\partial n/\partial t$ since $G \leq cn$. Therefore the equation describing conservation of number of particles is:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^a}(nU^a + G^a) = 0 \tag{D2}$$

D.2) Momentum Equation

The procedure is to integrate the product of (D1) with $p^a + mU^a$ term by term over all d^3p. The first three terms follow from the definitions.

$$\int (p^a + mU^a)\left[\frac{\partial f}{\partial t} + U^a\frac{\partial f}{\partial x^a}\right] d^3p = \frac{\partial F^a}{\partial t} + U^a\frac{\partial \rho}{\partial t} + U^b\frac{\partial F^a}{\partial x^b} + U^aU^b\frac{\partial \rho}{\partial x^b}$$

$$\int (p^a + mU^a)\frac{p^b}{m}\frac{\partial f}{\partial x^b}d^3p = \frac{\partial}{\partial x^b}P_{ab} + U^a\frac{\partial F^b}{\partial x^b}$$

To integrate the Lorentz term, integrate by parts, discarding the boundary term. Four terms are obtained, of which three vanish due to the anti-symmetry properties of $\epsilon_{abc} = -\epsilon_{bac}$. The one that remains is:

$$\frac{qB^d}{mc}\epsilon_{bcd}\int (p^a + mU^a)p^c\frac{\partial f}{\partial p^a}d^3p = -\frac{q}{c}\epsilon_{abd}G^bB^d$$

To handle the acceleration term, again integrate by parts and discard the boundary term to find:

$$-\frac{dU^b}{dt}\int (p^a + mU^a)m\frac{\partial f}{\partial p^b}d^3p = \frac{dU^b}{dt}\left(\rho\delta_{ab} + \frac{P_{ab}}{c^2} + \frac{2}{c^2}U^aF^b\right)$$

And again for the final term one finds:

$$\int (p^a + mU^a)p^b\frac{\partial U^c}{\partial x^b}\frac{\partial f}{\partial p^c}d^3p = F^b\frac{\partial U^a}{\partial x^b} + F^a\frac{\partial U^b}{\partial x^b} + \rho U^a\frac{\partial U^b}{\partial x^b} + \frac{U^a}{c^2}P_{bc}\frac{\partial U^c}{\partial x^b}$$

These are all to be combined. The equation will ignore terms of order U/c, so the three terms which have factors of c^2 in the denominator are ignored. The momentum equation may be written:

$$\frac{\partial}{\partial t}(F^a + \rho U^a) + \frac{\partial}{\partial x^b}(P_{ab} + \rho U^a U^b + F^a U^b + F^b U^a) = \frac{q}{c}\epsilon_{abc}G^b B^c \qquad (D3)$$

D.3) Mass Equation

Before forming the kinetic energy equation, we will first form the mass conservation equation. Since the particles may be relativistic, this is not simply proportional to the number equation. The procedure is to multiply (D1) by m and integrate over all d^3p. As above, the first three terms follow from the definitions.

$$\int m\left(\frac{\partial}{\partial t} + U^a\frac{\partial}{\partial x^a}\right) f d^3p = \frac{\partial\rho}{\partial t} + U^a\frac{\partial\rho}{\partial x^a}$$

$$\int m\frac{p^a}{m}\frac{\partial f}{\partial x^a}d^3p = \frac{\partial F^a}{\partial x^a}$$

Due to its anti-symmetry property, the Lorentz term vanishes as it did in the number equation.

$$\frac{q}{mc}\epsilon_{abc}B^c\int mp^b\frac{\partial f}{\partial p^a}d^3p = 0$$

For the acceleration term, integrate by parts and discard the boundary term:

$$-\frac{dU^a}{dt}\int m^2\frac{\partial f}{\partial p^a}d^3p = 2\frac{F^a}{c^2}\frac{dU^a}{dt}$$

For the final term, integrate by parts and discard the boundary term.

$$-\int mp^b\frac{\partial U^a}{\partial x^b}\frac{\partial f}{\partial p^a}d^3p = \frac{P_{ab}}{c^2}\frac{\partial U^a}{\partial x^b} + \rho\frac{\partial U^a}{\partial x^a}$$

In combining these terms, note that the acceleration term is at least order U/c smaller than $\partial\rho/\partial t$ or $\partial F^a/\partial x^a$. So the mass equation, ignoring order U/c is:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x^a}(\rho U^a+F^a)+\frac{P_{ab}}{c^2}\frac{\partial U^a}{\partial x^b}=0 \qquad (D4)$$

D.4) Kinetic Energy Equation

The kinetic energy equation is formed by multiplying (D1) by $(mc^2-m_0c^2+p^aU^a)$, and integrating over all momenta. This can be formed from the number, mass and momentum equations derived above, but the derivation is started from scratch here.

As usual, the first three terms follow simply from the definitions:

$$\int(mc^2-m_0c^2+p^aU^a)\left(\frac{\partial}{\partial t}+U^a\frac{\partial}{\partial x^a}\right)fd^3p=\frac{\partial T}{\partial t}+U^a\frac{\partial T}{\partial x^a}+U^a\frac{\partial F^a}{\partial t}+U^aU^b\frac{\partial F^a}{\partial x^b}$$

$$\int(mc^2-m_0c^2+p^bU^b)\frac{p^a}{m}\frac{\partial f}{\partial x^a}d^3p=\frac{\partial Q^a}{\partial x^a}+U^b\frac{\partial P_{ab}}{\partial x^a}$$

For the integral of the Lorentz term, integrate by parts and discard the boundary term. Six terms remain, and of these five vanish due to the anti-symmetry property.

$$\frac{q}{c}\epsilon_{abc}B^c\int(mc^2-m_0c^2+p^aU^a)\frac{p^b}{m}\frac{\partial f}{\partial p^a}d^3p=-\frac{q}{c}U^a\epsilon_{abc}G^bB^c$$

The acceleration term is integrated by parts, and the boundary term discarded:

$$\begin{aligned}-\frac{dU^a}{dt}\int(mc^2-m_0c^2+p^aU^a)m\frac{\partial f}{\partial p^a}d^3p&=\frac{dU^a}{dt}\left[2F^a-m_0G^a+\rho U^a+U^b\frac{P_{ab}}{c^2}\right]\\&=\frac{dU^a}{dt}\left[F^a+\rho U^a+\frac{Q^a}{c^2}+U^b\frac{P_{ab}}{c^2}\right]\end{aligned}$$

The same for the final term yields:

$$-\frac{\partial U^a}{\partial x^b}\int (mc^2 - m_0c^2 + p^aU^a)p^b\frac{\partial f}{\partial p^a}d^3p = T\frac{\partial U^a}{\partial x^a} + \frac{\partial U^a}{\partial x^b}\left[P_{ab} + U^aF^b + U^cF^c\delta_{ab}\right]$$

Of the four terms multiplying the acceleration vector, two may be discarded: the terms in Q^a and P_{ab}. The spatial-derivative pieces are clearly order U^2/c^2 smaller than other terms present in Q^a and P_{ab}, and discarded. The time derivative pieces are discarded as well because $Q \leq cT$, and $P \leq cF$. Combining the remaining terms yields the kinetic energy equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}(T + F^aU^a) - \frac{q}{c}U^a\epsilon_{abc}G^bB^c \\
&+ \frac{\partial}{\partial x^a}(TU^a + Q^a + P_{ab}U^b + U^aU^bF^b) + F^aU^b\frac{\partial U^b}{\partial x^a} + \rho U^a\frac{dU^a}{dt} = 0
\end{aligned} \tag{D5}$$

This equation is not a perfect divergence, but it can be made so if the mass equation (D4) is used, while remembering that $U \ll c$:

$$-\frac{1}{2}U^2\frac{\partial\rho}{\partial t} = \frac{1}{2}U^2\frac{\partial}{\partial x^a}(\rho U^a + F^a)$$

The final form of the kinetic energy equation is:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(T + F^aU^a + \frac{1}{2}\rho U^2\right) - \frac{q}{c}U^a\epsilon_{abc}G^bB^c \\
&+ \frac{\partial}{\partial x^a}\left(TU^a + Q^a + P_{ab}U^b + U^aU^bF^b + \frac{1}{2}F^aU^2 + \frac{1}{2}\rho U^aU^2\right) = 0
\end{aligned} \tag{D6}$$

Appendix E

Transport Equation from the Kinetic Equation

The starting point is the kinetic equation.

$$\frac{\partial f}{\partial t}+U^a\frac{\partial f}{\partial x^a}+\frac{p^a}{m}\frac{\partial f}{\partial x^a}+\left[\frac{q}{mc}\epsilon_{abc}p^bB^c-m\frac{dU^a}{dt}-p^b\frac{\partial U^a}{\partial x^b}\right]\frac{\partial f}{\partial p^a}=\frac{f_0-f}{\tau} \quad (E1)$$

$$f(x^a,t,p^a)=f_0(x^a,t,p)+\frac{3p^a}{p^2}S^a(x^a,t,p)+\frac{15p^ap^b}{2p^4}\Pi_{ab}(x^a,p,t) \quad (E2)$$

The momentum coordinate p^a is referred to the frame moving with velocity $U^a(x^b,t)$ with respect to an inertial frame. The space and time coordinates x^a,t are referred to the inertial frame. The only force on the particles is the Lorentz force. The system is assumed to be composed of a single species of particles. The distribution is nearly isotropic in the fluid frame; expressed as a sum of spherical harmonics, only the lowest three harmonics are kept. Recall $\Pi_{ab}=\Pi_{ba}$ and $\Pi_{aa}=0$.

Some intermediate results are needed. For $d^3p=p^2dpd\Omega$:

$$\frac{1}{4\pi}\int p^ap^bd\Omega=\frac{p^2}{3}\delta_{ab}$$

$$\frac{1}{4\pi}\int p^ap^bp^cp^dd\Omega=\frac{p^4}{15}[\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}]$$

Odd moments of this average over momentum-space solid angle are zero.

Since $\partial p^b/\partial p^a=\delta_{ab}$ and $\partial p/\partial p^a=p^a/p$, then:

$$\frac{\partial f}{\partial p^a}=\frac{p^a}{p}\frac{\partial f_0}{\partial p}+\frac{3S^a}{p^2}+\frac{3p^ap^b}{p}\frac{\partial}{\partial p}\left(\frac{S^b}{p^2}\right)+\frac{15}{2}\left\{\frac{2\Pi_{ab}p^b}{p^4}+\frac{p^ap^bp^c}{p}\frac{\partial}{\partial p}\left(\frac{\Pi_{bc}}{p^4}\right)\right\}$$

Also, recall:

$$\frac{1}{4\pi}\int fd\Omega=f_0$$

$$\frac{1}{4\pi}\int fp^ad\Omega=S^a \quad (E3)$$

$$\frac{1}{4\pi}\int fp^ap^bd\Omega=f_0\frac{p^2}{3}\delta_{ab}+\Pi_{ab}\equiv T_{ab}$$

E.1) The Zeroth Moment

Since the transport equation is obtained directly by substituting terms into the zeroth moment, I want to treat this moment a little differently. I will derive the zeroth moment without assuming isotropy of the distribution, or assuming the expansion (E2). I will only use the definitions (E3) of the zeroth, first and second moments of f. We will find that our expression for *the zeroth moment involves no moments higher than the second, regardless of isotropy.* The approximations necessary to construct a transport equation will be those that express S^a and Π_{ab} in terms of f_0. In some sense, then, the transport equation is complete, and one only needs to refine the expressions for S^a and Π_{ab}. The zeroth moment of (E1) follows by integrating it over all $d\Omega$. Since τ is a function of the magnitude of momentum only, this integral of the scattering term vanishes.

$$\frac{1}{4\pi}\int \frac{Df}{Dt} d\Omega = 0$$

The procedure will be to integrate (E1) term by term over $d\Omega$ The first three terms follow from definition.

$$\frac{1}{4\pi}\int \left[\frac{\partial f}{\partial t} + U^a \frac{\partial f}{\partial x^a}\right] d\Omega = \frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a}$$

$$\frac{1}{4\pi}\int \frac{p^a}{m}\frac{\partial f}{\partial x^a} d\Omega = \frac{1}{m}\frac{\partial S^a}{\partial x^a}$$

To handle the other terms without using the expansion (E2), first write the derivative in spherical coordinates:

$$\frac{\partial f}{\partial p^a} = \frac{\partial f}{\partial p}\hat{p} + \frac{1}{p}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{p\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

The unit vectors $\hat{p}$, $\hat{\theta}$, $\hat{\phi}$ can be expressed in terms of cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$:

$$\hat{p} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Also note that:

$$p^a = p(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})$$

Then, using a little integration by parts on the angular coordinates ($d\Omega \equiv \sin\theta d\theta d\phi$), one can show that:

$$\int \frac{\partial f}{\partial p^a} d\Omega = \int \left[\frac{\partial f}{\partial p} + \frac{2f}{p}\right] \frac{p^a}{p} d\Omega = \frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) \tag{E4}$$

$$\int p^b \frac{\partial f}{\partial p^a} d\Omega = \int \left[p\frac{\partial f}{\partial p} + 3f\right] \frac{p^a p^b}{p^2} d\Omega - \int f\delta_{ab} d\Omega = \frac{1}{p^2}\frac{\partial}{\partial p}(pT_{ab}) - f_0\delta_{ab} \tag{E5}$$

To handle the Lorentz term, use (E5). Since $\delta_{ab}\epsilon_{abc} = 0$ and $\epsilon_{abc}T_{ab} = 0$ ($\epsilon_{abc}T_{ab} = -\epsilon_{bac}T_{ab} = -\epsilon_{bac}T_{ba} = -\epsilon_{abc}T_{ab}$; since it equals its negative, it must be zero), the Lorentz force term integrates to zero.

$$\frac{q}{mc}\epsilon_{abc}B^c \int p^b \frac{\partial f}{\partial p^a} d\Omega = 0$$

For the acceleration term, use (E4) to find:

$$-\frac{dU^a}{dt}\frac{1}{4\pi}\int m\frac{\partial f}{\partial p^a} d\Omega = -m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a)$$

For the last term, use (E5):

$$\begin{aligned}
-\frac{\partial U^a}{\partial x^b}\frac{1}{4\pi}\int p^b\frac{\partial f}{\partial p^a} d^3p &= -\frac{\partial U^a}{\partial x^b}\left[\frac{1}{p^2}\frac{\partial}{\partial p}(pT_{ab}) - f_0\delta_{ab}\right] \\
&= -\frac{\partial U^a}{\partial x^b}\left[\frac{p}{3}\delta_{ab}\frac{\partial f_0}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab})\right]
\end{aligned}$$

The zeroth moment is obtained by putting these terms together.

$$\frac{\partial f_0}{\partial t} + U^a \frac{\partial f_0}{\partial x^a} - \frac{\partial U^a}{\partial x^a}\frac{p}{3}\frac{\partial f_0}{\partial p} + \frac{1}{m}\frac{\partial S^a}{\partial x^a} - m\frac{dU^a}{dt}\frac{1}{p^2}\frac{\partial}{\partial p}(pS^a) - \frac{\partial U^a}{\partial x^b}\frac{1}{p^2}\frac{\partial}{\partial p}(p\Pi_{ab}) = 0 \tag{E6}$$

Note that this equation is not restricted by keeping the first three spherical harmonics of f. The highest order harmonic that enters in the evaluation of the zeroth moment is the second harmonic T_{ab}. So (E6) *is an exact equation.*

E.2) The First Moment

Unlike the zeroth moment, for the first moment I'll use the expansion (E2) for f. Clearly, an exact expression for the first moment will involve third-order moments. The first moment is the product of (E1) with p^a, integrated over all $d\Omega$:

$$\frac{1}{4\pi}\int p^a \frac{Df}{Dt} d\Omega = -\frac{S^a}{\tau}$$

To solve this equation, I assume that the magnitude of $S \ll pf_0$. This assumption is a valid start, but must be checked later for consistency. This assumption is consistent with the idea that the distribution f can be approximated by its first three spherical harmonics: the expansion of f in spherical harmonics is a convergent expansion.

Recall that in §5.4, a single characteristic timescale t and a single characteristic lengthscale L were assumed to characterize Df/Dt (the Lorentz term excluded). These were related to the magnitude of the fluid velocity $Ut = L$. Thus $\tau/t = (\lambda/L)(mU/p)$. Since p is not restricted, one can only say $\tau/t \leq \lambda/L$.

With the assumption that $S \ll pf_0$, one may write:

$$\frac{1}{4\pi}\int p^a \frac{Df_0}{Dt} d\Omega \simeq -\frac{S^a}{\tau}$$

Therefore a solution to the first moment equation correct to zeroth order in λ/L is given by:

$$\frac{1}{4\pi}\int p^a\left\{\frac{\partial f_0}{\partial t}+U^b\frac{\partial f_0}{\partial x^b}+\frac{p^b}{m}\frac{\partial f_0}{\partial p^b}-\left[m\frac{dU^c}{dt}+p^b\frac{\partial U^c}{\partial x^b}\right]\frac{\partial f_0}{\partial p^c}\right\}d\Omega$$
$$+\frac{1}{4\pi}\frac{q}{mc}\epsilon_{bcd}B^d\int p^a p^c\frac{\partial f}{\partial x^b}d\Omega=-\frac{S^a}{\tau} \qquad (E7)$$

The Lorentz term was omitted from the approximation because I do not wish to restrict its size. Also, the reason for the approximation in the first place is to turn the first moment from a differential equation in S^a into an algebraic equation in S^a; S^a already enters algebraically in the Lorentz term.

Now take the terms in (E7) one at a time. Since odd moments vanish, the first two terms and the fifth term are zero. The third term yields:

$$\frac{1}{4\pi}\int p^a\frac{p^b}{m}\frac{\partial f_0}{\partial x^b}d\Omega=\frac{p^2}{3m}\frac{\partial f_0}{\partial x^a}$$

The acceleration term yields:

$$\frac{1}{4\pi}\frac{dU^c}{dt}\int mp^a\frac{\partial f_0}{\partial p^c}d\Omega=\frac{mp}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p}$$

The symmetry properties of $\epsilon_{abc}=-\epsilon_{bac}$ are used to show that the Lorentz term yields:

$$\frac{1}{4\pi}\frac{q}{mc}\epsilon_{bcd}B^d\int p^a p^c\frac{\partial f}{\partial p^b}d\Omega=-\frac{q}{mc}\epsilon_{abd}S^bB^d$$

Thus an approximate solution to the first moment, one that ignores terms of order λ/L, is:

$$-\frac{S^a}{\tau}=\frac{p^2}{3m}\frac{\partial f_0}{\partial x^a}-\frac{mp}{3}\frac{dU^a}{dt}\frac{\partial f_0}{\partial p}-\frac{q}{mc}\epsilon_{abc}S^bB^c \qquad (E8)$$

It is clear from (E4) that $S\sim pf_0\lambda/L$, validating the original assumption that $S\ll pf_0$.

E.3) The Second Moment

The second moment is the product of (E1) with $p^a p^b$, integrated over all $d\Omega$.

$$\frac{1}{4\pi}\int p^a p^b \frac{Df}{Dt} d\Omega = -\frac{\Pi_{ab}}{\tau}$$

As in section E.2), the exact differential equation in Π_{ab} can be turned into an approximate algebraic equation for Π_{ab}. Since the Lorentz term involves no derivatives, it is not necessary to assume any restriction on the size of this term. Assuming that $\Pi_{ab} \ll p^2 f_0 + pS$, the second moment may be approximated:

$$\frac{1}{4\pi}\int p^a p^b \left\{\frac{\partial}{\partial t} + U^c \frac{\partial}{\partial x^c} + \frac{p^c}{m}\frac{\partial}{\partial x^c} - \left[m\frac{dU^c}{dt} + p^d \frac{\partial U^c}{\partial x^d}\right]\frac{\partial}{\partial p^c}\right\}\left(f_0 + \frac{3p^c}{p^2}S^c\right) d\Omega$$
$$+ \frac{1}{4\pi}\frac{q}{mc}\epsilon_{cde}B^e \int p^a p^b p^d \frac{\partial f}{\partial p^c} d\Omega = -\frac{\Pi_{ab}}{\tau} \tag{E9}$$

Now examine this integral term by term.

The first two terms yield:

$$\frac{1}{4\pi}\int p^a p^b \left\{\frac{\partial}{\partial t} + U^c\frac{\partial}{\partial x^c}\right\}\left(f_0 + \frac{3p^c}{p^2}S^c\right) d\Omega = \frac{p^2}{3}\delta_{ab}\left(\frac{\partial f_0}{\partial t} + U^c \frac{\partial f_0}{\partial x^c}\right)$$

The third term yields:

$$\frac{1}{4\pi}\int p^a p^b \frac{p^c}{m}\frac{\partial}{\partial x^c}\left(f_0 + \frac{3p^c}{p^2}S^c\right) d\Omega = \frac{p^2}{5m}\left[\frac{\partial S^a}{\partial x^b} + \frac{\partial S^b}{\partial x^a} + \frac{\partial S^c}{\partial x^c}\delta_{ab}\right]$$

The acceleration term yields:

$$\frac{1}{4\pi}\int p^a p^b m \frac{dU^c}{dt}\frac{\partial}{\partial p^c}\left(f_0 + \frac{3p^c}{p^2}S^c\right) d\Omega$$
$$= \frac{mp^3}{5}\left[\frac{dU^a}{dt}\frac{\partial}{\partial p}\left(\frac{S^b}{p^2}\right) + \frac{dU^b}{dt}\frac{\partial}{\partial p}\left(\frac{S^a}{p^2}\right) + \frac{dU^c}{dt}\frac{\partial}{\partial p}\left(\frac{S^c}{p^2}\right)\delta_{ab}\right] + m\frac{dU^c}{dt}S^c\delta_{ab}$$

The fifth term yields:

$$-\frac{1}{4\pi}\int p^a p^b p^d \frac{\partial U^c}{\partial x^d}\frac{\partial}{\partial p^c}\left(f_0 + \frac{3p^c}{p^2}S^c\right) d\Omega = -\frac{p^3}{15}\frac{\partial f_0}{\partial p}\left[\frac{\partial U^a}{\partial x^b} + \frac{\partial U^b}{\partial x^a} + \frac{\partial U^c}{\partial x^c}\delta_{ab}\right]$$

Finally, the Lorentz term yields:

$$\frac{1}{4\pi}\frac{q}{mc}\epsilon_{cde}B^e\int p^a p^b p^d \frac{\partial f}{\partial p}d\Omega = \frac{q}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)$$

In putting these terms together, they are found to contain the product of $(p^2/3)\delta_{ab}$ with the zeroth moment, (E6), which sums to zero independently. In this way, the first and second terms, and pieces of the third, fourth, and fifth terms are lost. The pieces that are lost from the third, fourth and fifth terms constitute their trace; Π_{ab} is traceless. These pieces of the third, fourth, and fifth terms that are subtracted out are: the divergence of the streaming flux, the acceleration term, and the adiabatic energy change term, respectively, of the zeroth moment (E6); all multiplied by $(p^2/3)\delta_{ab}$. With the trace subtracted out, the zeroth-order approximation to the second moment is:

$$\begin{aligned}
-\frac{\Pi_{ab}}{\tau} &= -\frac{p^3}{15}\frac{\partial f_0}{\partial p}\Lambda_{ab} + \frac{p^2}{5m}\left[\frac{\partial S^a}{\partial x^b} + \frac{\partial S^b}{\partial x^a} - \frac{2}{3}\frac{\partial S^c}{\partial x^c}\delta_{ab}\right] \\
&\quad - \frac{mp^3}{5}\left[\frac{dU^a}{dt}\frac{\partial}{\partial p}\left(\frac{S^b}{p^2}\right) + \frac{dU^b}{dt}\frac{\partial}{\partial p}\left(\frac{S^a}{p^2}\right) - \frac{2}{3}\frac{dU^c}{dt}\frac{\partial}{\partial p}\left(\frac{S^c}{p^2}\right)\delta_{ab}\right] \qquad (E10) \\
&\quad + \frac{q}{mc}(\epsilon_{dae}\Pi_{bd}B^e + \epsilon_{dbe}\Pi_{ad}B^e)
\end{aligned}$$

$$\Lambda_{ab} \equiv \frac{\partial U^a}{\partial x^b} + \frac{\partial U^b}{\partial x^a} - \frac{2}{3}\frac{\partial U^c}{\partial x^c}\delta_{ab}$$

This approximation ignores terms of order λ/L smaller than the largest, and demonstrates that $\Pi_{ab} \ll p^2 f_0 + pS$, consistent with the original assumption. In fact, $\Pi_{ab} \sim (\tau/t)p^2 f_0$.

Appendix F

The Cosmic-Ray Spectrum

In this appendix, I want to introduce the reader to the central data set of cosmic-ray physics: the cosmic-ray spectrum. The plot in figure F1 is a flux versus kinetic energy per nucleon. The plot is for the environment outside the earth's magnetosphere. The 'kinetic energy per nucleon' recognizes the fact that cosmic rays are generally nuclei, and so a single cosmic ray particle may be composed of many nucleons.

The spectrum is of the 'primary' cosmic radiation; as opposed to the 'secondary' cosmic radiation, which is produced by primaries striking the earth's atmosphere. The primary cosmic radiation is dominated by protons; hence the consideration of only protons in the thesis. The next most abundant species is helium nuclei at about 7% of the proton abundance. The electron abundance is also in the few-percent range. All other heavy nuclei constitute about 1% of the proton abundance; it is noteworthy that the heavy nuclei abundance in the primary cosmic-radiation is enhanced relative to cosmic/solar neutral abundances for heavier nuclei. This is a key to unravelling the acceleration mechanism of cosmic rays: larger-mass particles are accelerated preferentially. The photon (gamma ray) abundance is in the tenths-of-percent range.

The quantity measured by spacecraft is a flux in units of number per area per time per solid-angle per energy, and is historically denoted dj/dT. The units in the spectrum are number per square meter per second per steradian per MeV per nucleon. The quantity appearing in the thesis is the distribution function f_0. The relation is $4\pi p^2 f_0 = dj/dT$:

$$f_0 \equiv \frac{1}{4\pi}\int f d\Omega_p = \frac{1}{4\pi}\int \frac{N}{d^3pd^3r}d\Omega_p = \frac{1}{4\pi}\frac{N}{d^3rp^2dp}$$

$$\begin{aligned} 4\pi p^2 f_0 &= N/r^2 dr d\Omega_r dp \\ &= Nv/r^2 dr d\Omega_r dE \\ &= N/r^2 dt d\Omega_r dE \\ &= N/r^2 dt d\Omega_r dT \end{aligned}$$

The successive steps followed by using: $dp/dE = 1/v$ (Appendix B), $v = dr/dt$, $dE/dp = dT/dp$. The kinetic energy $T \equiv \sqrt{p^2c^2 + {m_0}^2c^4} - m_0c^2$.

As the spectrum shows, beyond 10^{10} eV the spectrum is a powerlaw in kinetic energy. At these energies, $T \simeq pc$ so $p^2 f_0 \propto p^{-2.6} \Longrightarrow f_0 \propto p^{-4.6}$. Beyond 10^{15} eV the slope steepens somewhat. The origin of this steepening is unknown. The steepening is sometimes referred to as the 'knee'.

Particles with energies in the powerlaw region below the knee are thought to be galactic in origin, being accelerated at supernova shockwaves (Drury, etal, 1989). The acceleration mechanism is diffusive shock acceleration, and roughly accounts for the powerlaw. The limiting factor in the mechanism for diffusive shock acceleration is the time a particle can spend in the acceleration region. The lifetime of supernova remnants implies that supernovas can only accelerate particles up to the knee.

The anomalous component is thought to arise from particles accelerated at the termination shock of the solar wind. The particles themselves are medium-weight interstellar neutrals that are singly ionized by solar ultraviolet radiation after they enter the heliosphere. Becoming charged, they are then picked up by the solar wind and convected out to the termination shock where they are accelerated by the mechanism of diffusive shock acceleration.

The low-energy turnup is attributed to the sun. These particles are produced in various processes in the sun, or perhaps in corotating interaction regions in propagating heliospheric shock waves. Again, the mechanism is presumed to be diffusive shock acceleration. In fact, diffusive shock acceleration remains the only *verified* mechanism for accelerating particles. It has been verified in observations of the earth's bow shock.

To account for the ultra-high energy particles beyond the knee, some workers have looked for a longer-lasting shock at which the mechanism of diffusive shock acceleration may move particles up to those enormous energies. One idea is to propose a termination shock around the galaxy, analogous to the solar termination, driven by a galactic wind. Such a structure would have the size and lifespan necessary to feed particles into the energies beyond the knee.

A major issue in determining the sources of the high-energy cosmic rays is how far from our galaxy they are accelerated. There are limits to the energy a particle can obtain set by the strength of whatever magnetic fields exist in the universe, and by the existence of the cosmic microwave background radiation.

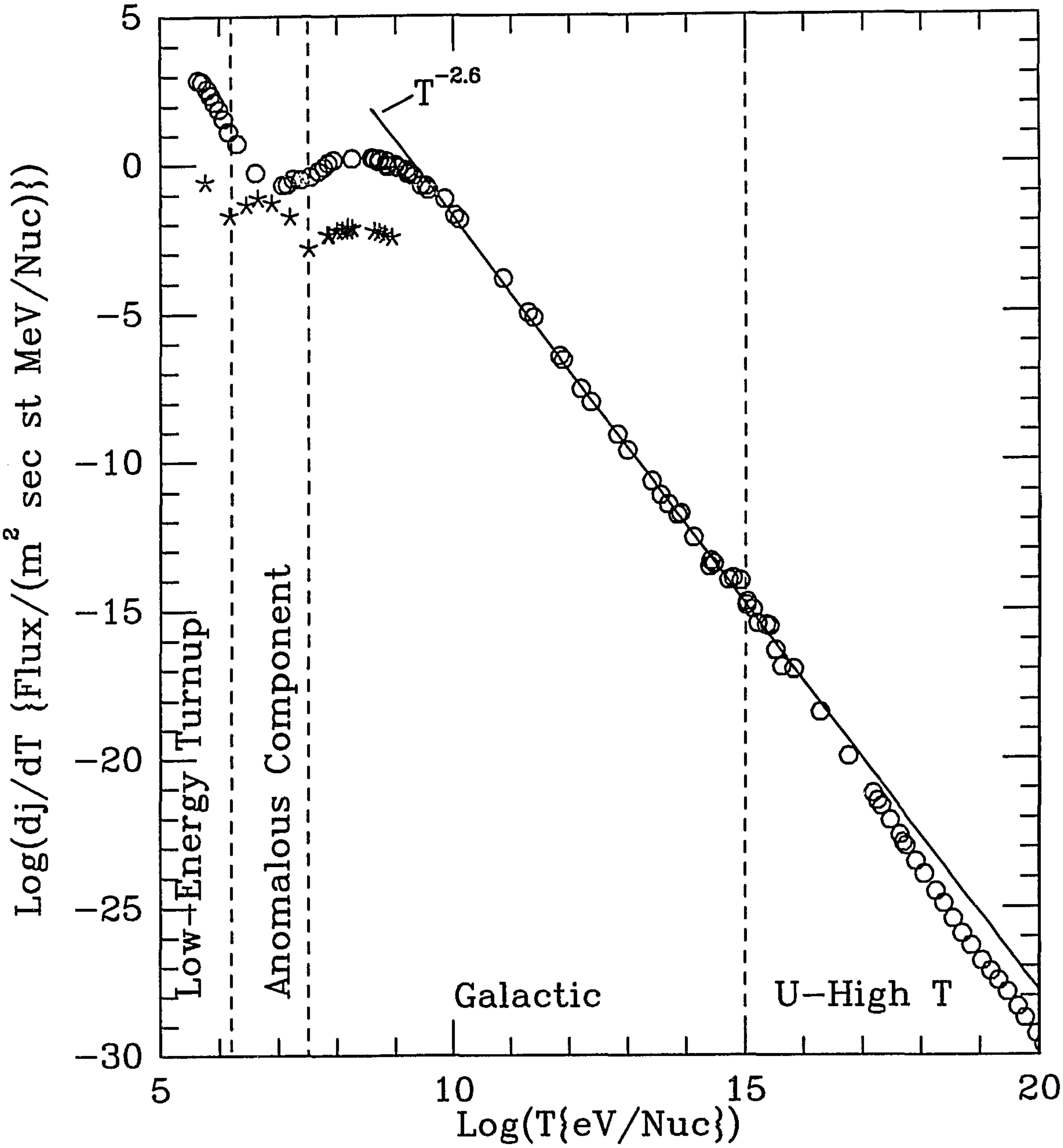
Greisen (1966) has considered the effect of the microwave background on cosmic rays. Interaction with the radiation field will cause cosmic rays to lose energy to pair production and pion production. The universe will be opaque to gamma rays above 10^{14} eV due to pair production from photon–photon interactions. Pair production will also start to depress the proton spectrum at about 10^{19} eV due to proton–photon interactions. However, this effect is small compared to energy losses due to pion production from proton-photon interactions. This effect should produce a very sharp cutoff above 10^{20} eV. Evidently, the data is not yet unequivocal that this cutoff exists. The microwave background will induce photodisintegration of heavy ions above 10^{19} eV. For electrons, they are limited by inverse Compton scattering.

If the source of high-energy cosmic rays is 'local', then these effects are negligible and particle energies may conceivably be much higher than 10^{20} eV. Stecker (1968) estimates that cosmic rays of all energies may reach us from distances of 10 Megaparsecs essentially unattenuated by photomeson production.

If the high-energy cosmic rays are local, then they would only be limited by synchrotron radiation. The magnetic field strength is poorly constrained within the galaxy and essentially unknown outside the galaxy. Using a 'reasonable' value of $5\text{x}10^{-5}$ Gauss, I calculate that energies of 10^{19} eV are required for a proton to radiate 10% of its energy in 10^{17} seconds, the age of the universe. Electrons are much more susceptible to synchrotron radiation, because the fourth power of the rest mass enters the formula (Rybicki and Lightman, 1979). Only 10^{6} eV are required for an electron to radiate 10% of its energy in 10^{17} seconds.

The bottom line is that the highest-energy cosmic rays are likely protons. Their energy is unlimited if they originate locally both in time and space. The question of locality of origin may be constrained by resolving chemical composition of these particles, but such data does not exist yet.

FIGURE F1


5
0
−5
−10
−15
−20
−25
−30
5
10
15
20
Log(dj/dT {Flux/(m² sec st MeV/Nuc)})
Log(T{eV/Nuc})
$T^{-2.6}$
Low-Energy Turnup
Anomalous Component
Galactic
U-High T

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