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A Fourier transform technique for estimating bioelectric currents from magnetic field measurements

Schlitt, Heidi Anne, Ph.D.
The University of Arizona, 1992

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A FOURIER TRANSFORM TECHNIQUE FOR ESTIMATING BIOELECTRIC CURRENTS FROM MAGNETIC FIELD MEASUREMENTS

by

Heidi Anne Schlitt

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A Dissertation Submitted to the Faculty of the
COMMITTEE ON OPTICAL SCIENCES (GRADUATE)
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

1992
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Heidi Anne Schlitt entitled A FOURIER TRANSFORM TECHNIQUE FOR ESTIMATING BIOELECTRIC CURRENTS FROM MAGNETIC FIELD MEASUREMENTS and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director

Date
STATEMENT BY AUTHOR

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SIGNED: [Signature] 15 Jan 1992
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ABSTRACT

This dissertation presents a new noniterative technique for estimating current densities from magnetic field measurements. This Fourier-transform technique starts by forming a set of linear equations from the Fourier-transformed Maxwell equations. The set of equations is sampled according to the Whittaker-Shannon sampling theorem and solved by matrix methods. Two variations of the technique are extensions of a Fourier-transform method developed by Dallas. The first method assumes that the $x$- and $y$-components of the magnetic field in a forbidden region are zero, and that the $z$-component of the current density is zero. The second method assumes that $\nabla^2 B_x$ and $\nabla^2 B_y$ are zero in a forbidden region, and is not restricted to reconstructing currents with zero $z$-component. The assumptions about the current distribution and measurement geometry are included in the reconstruction technique by means of the sampling theorem. The effects on the reconstructions of the spatial sampling requirements and the form approximations to the differentiation operator were investigated for the two variations of the reconstruction method.

We tested the Fourier-transform reconstruction methods on computer-simulated magnetic fields derived from analytic expressions. The results of the computer simulations were confirmed by reconstructions from measured magnetic fields due to known current sources. The measured magnetic fields were due to both widely distributed and highly localized current sources. If the magnetic field was undersampled, then the reconstructed current was a larger size than it should have been. The form of the differentiation operator made a dramatic difference in the reconstructed current, and in some cases resulted in a current that was not physical. Both reconstruction methods were able to distinguish between large and small current densities, had excellent lateral resolution, but were unable to provide any information about the depth of the source.
Chapter 1

INTRODUCTION

The measurement of biomagnetic fields provides a noninvasive method for indirectly observing the spatial distribution of electric currents in the body. The knowledge gained from studying the body's magnetic fields will complement other methods of medical imaging. Ultrasound and computed tomography provide spatial information about the attenuation of the incident sound waves and x-rays, respectively. Another popular method, magnetic resonance imaging, provides spatial information about the proton density and magnetization relaxation times. These methods image the morphology of the organs in the body. The physiological function of the organs, as indicated by the distribution of radioactive tracers, can be studied with positron emission tomography or single-photon emission computed tomography. A knowledge of the electric currents present in the organs will aid our understanding of their function.

Researchers have been measuring the magnetic fields surrounding the human body for over 25 years. The data from these experiments have traditionally been displayed on a strip-chart recorder as spikes projecting from a baseline. This work presents two Fourier-transform algorithms that extract information about the spatial distribution of electric currents from biomagnetic fields. The first chapter introduces biomagnetism and the mathematical methods that we use in later chapters. The second chapter presents the solutions to two problems that we encounter in the development of the reconstruction algorithms: the Laplace equation with Neumann boundary conditions and the Laplace equation with Dirichlet boundary conditions. We use the solution to the Laplace equation with Neumann boundary conditions to put the measured magnetic field into a form from
which it is easier to extract the information contained in the measurements. The solution to the Laplace equation with Dirichlet boundary conditions is used to demonstrate the validity of certain assumptions we make in one of the Fourier-transform reconstruction methods. The third chapter first reviews some of the existing methods used to extract information about electric currents from the measured magnetic field and then presents our two Fourier-transform algorithms. The two Fourier-transform methods differ primarily in the assumptions made about the current density and the magnetic field. We refer to the two methods as the 'B-zero' method and the 'V²B-zero' method, where the names indicate which set of assumptions are used. The fourth chapter presents the methods we used to measure magnetic fields from known current sources and the spatial sampling requirements for biomagnetic fields. In the last chapter we present the images from two Fourier-transform algorithms that reconstruct widely distributed electric currents in a volume.

We begin this chapter with a review of the connection between electricity and magnetism including a few examples of magnetic fields and their sources. We then discuss several devices used to measure magnetic fields, and potential uses of biomagnetism. The chapter concludes with a review of the mathematical functions and methods that we use in other chapters of this dissertation.

*Electricity and Magnetism*

The interaction of electricity and magnetism was an active area of research in the first half of the 19th century. In 1820 the Danish scientist Oersted observed that a wire carrying an electric current would rotate around a magnetic pole and that a magnet tended to move around a stationary wire carrying current. By the end of that year, the French physicist Ampère showed that a coil of wire carrying a current behaved like an ordinary magnet.
Ampère described the magnetic field, $\mathbf{B}$, in terms of the total current, $I$, passing through the closed curve $C$ by the relation (known as Ampère's law):

$$\oint_{C} \mathbf{B} \cdot \mathrm{d}l = \frac{\mu_{0}}{4\pi} I. \quad (1.1)$$

(We use boldface type to indicate vectors. All quantities are in Système International (SI) units).

This interaction was also described mathematically by the French scientists Biot and Savart. For a given current density, $\mathbf{J}$, the magnetic field, $\mathbf{B}$, at some point $\mathbf{r}$ is found by (Biot–Savart law):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_{0}}{4\pi} \int_{\text{Volume}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^{3}\mathbf{r}'. \quad (1.2)$$

In 1831 the English scientist Faraday experimentally showed that one coil carrying a time-varying electric current could generate a transient current in another coil. Faraday also showed that a changing magnetic field from a moving permanent magnet could generate an electric current in a coil. The later theoretical work of the English scientist Maxwell and others unified the observations of Oersted, Faraday and Ampère in a common mathematical framework. In 1862, Maxwell put forward his electromagnetic theory of light; it elegantly
described the relationships between electricity and magnetism in four equations that have come to be known as the Maxwell equations:

\[
\begin{align*}
\nabla \cdot D &= \rho \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times H &= J + \frac{\partial D}{\partial t} \\
\nabla \cdot B &= 0.
\end{align*}
\]

(1.3)

The displacement current, \(D\), is a function of the electric field, \(E\), and the polarization, \(P\),

\[D = \varepsilon_0 E + P\]

(1.4)

where \(\varepsilon_0\) is the dielectric constant of free space. The magnetic field, \(H\), is a function of the magnetic induction \(B\), and the magnetization, \(M\),

\[H = \frac{1}{\mu_0} B - M.\]

(1.5)

where \(\mu_0\) is the permeability of free space. The magnetic induction, \(B\), is sometimes called the magnetic-flux density and is also often referred to as the magnetic field when the sources are in a region of uniform permeability.

The continuity equation,

\[\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0,\]

(1.6)

for current density and charge density, \(\rho\), is implicit in the Maxwell equations.
In biomagnetism we assume that the conditions are quasi-static: we neglect any propagation effects due to the time required for changes in the source to propagate to a field point, we assume the medium is purely resistive so any capacitance effects can be neglected, and we neglect any inductive effects (Plonsey and Heppner, 1967). Thus, to a very good approximation, the time derivatives in the Maxwell equations (equation (1.3)) and in the continuity equation (equation (1.6)) can be set to zero. Finally, we assume the permeability is constant and refer to $B$ as the magnetic field. The form of the Maxwell equations that we use in this dissertation is

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \times \mathbf{E} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \cdot \mathbf{B} &= 0 .
\end{align*}
\] (1.7a) (1.7b) (1.7c) (1.7d)

It is easy to show that if the curl of a vector field is zero then we can write the field as the gradient of a scalar potential (see, for example, Jackson, 1975). If the divergence of the vector field is also zero, then the scalar potential satisfies the Laplace equation, $\nabla^2 \phi = 0$. For example, if the current density is zero in a region of space, then the curl of the magnetic field is zero,

\[
\nabla \times \mathbf{B} = 0 .
\] (1.8)
The divergence of the magnetic field is also zero, so we can write the magnetic field as the gradient of a scalar potential, $\phi$,

$$B(r) = -\nabla \phi(r) ,$$  \hspace{1cm} (1.9)

where $\phi(r)$ satisfies the Laplace equation

$$\nabla^2 \phi = 0 .$$  \hspace{1cm} (1.10)

We shall use scalar potential theory in Chapter 2 to extrapolate the magnetic field.

**Examples of Current Sources and Their Magnetic Fields**

To illustrate the connection between electric currents and the magnetic field we present six examples: an infinitely long wire, a circular loop, a magnetic dipole, a current dipole in a homogeneous conductor of infinite extent, a current dipole in a homogeneous half-space, and a current dipole in a homogeneous sphere.

First, consider an infinitely long straight wire carrying a uniform current $I$ along the $x$-axis. Using either Ampère's law or the Biot-Savart law, the magnetic field is found to be:

$$B(x, y, z) = \frac{\mu_0 I}{2\pi} \left( -\frac{\hat{z} y}{y^2 + z^2} \right)$$  \hspace{1cm} (1.11)

The magnitude falls off inversely with distance, and the magnetic field lines are closed circles centered on the wire (Figure 1.1). The direction of $B$ is found from the right-hand-
Figure 1.1  
Magnetic field due to a long straight wire carrying current I.

Figure 1.2  
Magnetic field due to a round loop carrying current I.

Figure 1.3  
Magnetic field due to a magnetic dipole.

Figure 1.4  
Magnetic field due to a current dipole in an infinite homogeneous conductor.
rule. Note that the magnetic field does not change in the x direction.

We now bend the straight wire into a circle of diameter, d, in the x-y plane; the field lines still enclose the wire (Figure 1.2), but the overall pattern changes significantly. The expression for the magnetic field becomes complicated and is expressed in terms of the complete elliptic integrals K and E (Jackson, 1975):

\[
\begin{align*}
B_r &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \\
B_\theta &= -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\
B_\phi &= 0,
\end{align*}
\]

where

\[
A_\phi = \frac{\mu_0 I d}{\pi \sqrt{d^2 + r^2 + 2dr \sin \theta}} \left[ \frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right],
\]

and the argument of the elliptic integrals is

\[
k^2 = \frac{4dr \sin \theta}{d^2 + r^2 + 2dr \sin \theta}.
\]

Along the z-axis, the z-component of the magnetic field falls off as \(r^{-3}\).

If we shrink the diameter of the loop, or go far away from it, we get the typical magnetic-dipolar field (Figure 1.3). We call this source a magnetic dipole. If \(d\) is the diameter of the loop and \(r\) is the distance from the loop, then in the limit \(r \gg d\), the magnetic field is

\[
B(x, y, z) = \pi d^2 \frac{\mu_0}{4\pi} \left[ \hat{z} \frac{z^2}{(x^2+y^2+z^2)^{5/2}} \right].
\]

The z-component of the magnetic field along the z-axis falls off as \(r^{-3}\), just as it did for the circular loop.
The last three magnetic fields we discuss are all due to current dipoles in conducting media of various shapes. A current dipole can be thought of as a positive current source separated from a negative current source by a small distance. In magnetostatics, including the conducting medium in the calculation of the magnetic field is important because the total current must be conserved. In order to close the circuit, electricity must be able to flow in the medium surrounding the current dipole. This extra current, the current that is needed to have a divergenceless source, is often called the volume or return current.

For a current dipole of strength $p$, located at $(0,0,a)$, pointing in the $\hat{i}$ direction, and surrounded by an infinite homogeneous conductor (Figure 1.4), the magnetic field is:

$$B(x, y, z) = \frac{\mu_0 D}{4\pi} \left\{ \frac{\hat{i}(a-z) + \hat{j}y}{\left[ x^2 + y^2 + (a-z)^2 \right]^{3/2}} \right\}. \tag{1.14}$$

If the conductor is finite, then, close to the dipole, the magnetic field is determined primarily by the current dipole, and the magnitude falls off as $r^{-2}$. Far away from the conductor, the magnetic field is determined by the entire divergenceless source, and the magnitude falls off as $r^{-4}$. The two most common conductor shapes considered in biomagnetism are the sphere and the homogeneous half-space.

For a current dipole of strength $p$, located at $(0,0,a)$, pointing in the $\hat{i}$ direction so that it is tangential to the surface of the semi-infinite homogeneous conductor (Figure 1.5),
Figure 1.5
A current dipole
in a homogeneous half-space.

Figure 1.6
A current dipole
in a homogeneous sphere.
the magnetic field outside the conductor is (Cuffin and Cohen, 1977):

\[
B_x(x, y, z) = \frac{\mu_0 p_{xy}}{4\pi} \left\{ \frac{(a-z) \left[ x^2 + y^2 + 2(a-z)^2 - 2 \left[ x^2 + y^2 + (a-z)^2 \right]^{3/2} \right]}{(x^2 + y^2)^2 \left[ x^2 + y^2 + (a-z)^2 \right]^{3/2}} \right\}
\]

\[
B_y(x, y, z) = \frac{\mu_0 p_{xy}}{4\pi} \left\{ \frac{(x^2 - y^2) \left[ x^2 + y^2 + (a-z)^2 \right]^{3/2} + (a-z) \left[ (x^2 - y^2) \left[ x^2 + (a-z)^2 \right] - 2x^4 \right]}{(x^2 + y^2)^2 \left[ x^2 + y^2 + (a-z)^2 \right]^{3/2}} \right\}
\]

\[
B_z(x, y, z) = \frac{\mu_0 p}{4\pi} \left\{ \frac{y}{\left[ x^2 + y^2 + (a-z)^2 \right]^{3/2}} \right\}.
\]

Note that the z-component of the magnetic field is the same as for a current dipole in an infinite homogeneous conductor. We need only consider current dipoles tangential to the surface of the half-space because the magnetic field due to a current dipole oriented normal to the surface is zero (Cuffin and Cohen, 1977).

A single current dipole oriented parallel to the surface of the half-space is often used to model a localized current source in the heart. The heart is assumed to be near the top surface of the homogeneous conductor, and the bottom surface of the conductor is far enough away from the heart that the torso can be approximated by a homogeneous half-space.

For a current dipole of strength \( p \), located at \((0,0,a)\), pointing in the \( \hat{i} \) direction, surrounded by a homogeneous spherical conductor (Figure 1.6),
the magnetic field outside the sphere is (Grynszpan and Geselowitz, 1973 and Cuffin and Cohen, 1977)

\[
B_x(x, y, z) = \frac{\mu_0 I}{4\pi} \frac{xy}{r^6 \beta^2 / 2(x^2 + y^2)} \left\{ a(x^2 + y^2) \left[ x^2 + y^2 - z(z - a) \right] + \frac{r^2 \beta}{a} \left( z^2 + r^2 \right) \right\}
\]

\[
B_y(x, y, z) = \frac{\mu_0 I}{4\pi} \frac{y^2}{r^6 \beta^2 / 2(x^2 + y^2)} \left\{ a(x^2 + y^2) \left[ x^2 + y^2 - z(z - a) \right] + \frac{r^2 \beta}{a} \left( z^2 - \frac{x^2 r^2}{y^2} \right) \right\}
\]

\[
B_z(x, y, z) = \frac{\mu_0 I}{4\pi} \frac{y}{r^6 \beta^2 / 2(x^2 + y^2)} \left\{ a^2(x^2 + y^2) - \frac{r^2 \beta z}{a} \left( z_a - r^2 + r^2 \beta \right) \right\}, \tag{1.16}
\]

where

\[
r^2 = x^2 + y^2 + z^2
\]

and

\[
\beta = \frac{x^2 + y^2 + (z - a)^2}{x^2 + y^2 + z^2}.
\]

For any homogeneous conductor with rotational symmetry, a current dipole aligned with the axis of rotation will generate no magnetic field outside the conductor (Baule and McFee, 1963). Therefore a current dipole that is oriented along a radius of the sphere produces no magnetic field outside of the sphere (Grynszpan and Geselowitz, 1973). A single current dipole oriented tangentially to the surface of the sphere is often used to model a localized current source in the brain.

Display Techniques

Before we discuss biomagnetism, we shall introduce the techniques we use to display the magnetic fields and current densities. The display of three-dimensional information is difficult when we are limited to a two-dimensional display surface, such as a sheet of paper or a computer screen. For us, the problem is compounded by the fact that the information
we wish to display consists of a vector field in three-dimensions: at every \((x, y, z)\) point we have a magnitude and direction. Usually, the three-dimensional data are reduced to sets of two-dimensional data, i.e. a series of planes or spheres. On each two-dimensional surface we can display the magnitude of the vector field or each component of the vector field separately. Both the magnitude of the entire vector field and the components of the vector field can be displayed as surfaces, contour plots, or grey level pictures.

When two-dimensional information is displayed as a surface, the strength of the function is mapped to the height of the surface. Contour plots trace the levels of constant magnitude, and the density of lines is related to the rate of change of the strength. The function also can be displayed as grey levels, where white represents the maximum positive value and black represents the maximum negative value. The disadvantage of plotting the magnitude of the vector field is that the information about the direction is lost. If each component is plotted separately then it is possible, although it may be difficult, to visualize the direction of the field.

To illustrate the display techniques, consider the magnetic field due to a single current dipole in a homogeneous halfspace. The \(z\)-component of the magnetic field in a plane above the surface of the homogeneous half-space is displayed as a contour plot in Figure 1.7a, as a surface in Figure 1.7b, and as a grey-level picture in Figure 1.7c.

**Biomagnetism**

**Bioelectric Currents**

Bioelectric currents are created by ions (especially Potassium, \(K^+\), and Sodium, \(Na^+\), in nerve or muscle cells) moving across, inside, and outside of the cellular membranes.
Figure 1.7
Three representations of $B_z$ due to a current dipole in a homogeneous half-space.
(a) Contour plot
(b) Surface plot
(c) Grey-level picture
Much of what is known about the conduction of electricity in a nerve fiber was discovered by studying the giant axon of the squid (Cronin, 1981). An equilibrium is established in which a layer of negative ions lining the inner surface of the membrane and a layer of positive ions on the outer surface interact, generating a potential difference across the membrane. A membrane in this condition is said to be polarized, and the equilibrium is maintained by metabolic processes. The membrane of an axon is more permeable to K+ ions than to Na+ and other ions. When the axon is stimulated by a short (millisecond) current pulse, then the permeability of the membrane alters (the permeability of the membrane is not related to the permeability, \( \mu_0 \), in the Maxwell equations). The new permeability allows ions to flow across the membrane, and the concentrations of ions on the inner and outer surfaces change. Therefore, the membrane potential changes. If the amplitude of the stimulus exceeds a certain threshold, then the membrane is depolarized, the axon fires and stimulates a neighboring axon. After firing, the axon returns to its resting state, and the membrane is repolarized.

The strength of the axial depolarization current in a frog's sciatic nerve bundle is on the order of 1 \( \mu \)A (Wikswro, 1982). The strength of the current in median and ulnar nerves in the human wrist has also been estimated to be on the order of 1 \( \mu \)A (Hari, et al., 1989). This value is commonly accepted as a realistic source strength in most human organs. The magnetic field due to a single neuron firing 4 cm away from the sensor is about \( 2 \times 10^{-6} \) picoTesla (1 pT = \( 10^{-12} \) T, 1 T = \( 10^4 \) Gauss) (Romani, 1989). Neuromagnetic fields are around 0.5 to 1 pT, so the measured fields must be due to the combined effect of \( 10^4 \) or \( 10^5 \) neurons firing at the same time. If the active neurons are localized in a small area, the current dipole is a good model for the source of the biomagnetic field. If the neurons are widely distributed or if there are several localized centers of activity, a different model is required.
In comparison to neuromagnetic fields, the earth's field is about $5 \times 10^7$ pT and urban magnetic noise can be as high as $10^6$ pT (Carelli, et al., 1982). The magnetic field due to activity in the heart is about 50 times stronger than neuromagnetic fields and is correspondingly easier to measure.

**Instrumentation**

Biomagnetism has been an active field of research since the early 1960's. The first measurement of the heart's magnetic field (the magnetocardiogram, or MCG) was recorded by Baule and McFee in 1963 using a room-temperature coil detector. Cohen built one of the first magnetically shielded rooms in 1967 and used a point-contact form of a Superconducting Quantum Interference Device (SQUID) to measure the heart's (1969) and the brain's (1972) magnetic fields. Since then, the biomagnetism of other parts of the body has been measured. A few of the milestone measurements are: the lungs (Cohen, 1973), the liver (Farrel, et al., 1980), and isolated tissue (Wikswo, et al., 1980). These early measurements demonstrated that biomagnetic fields existed and could be measured with an acceptable signal-to-noise ratio. Improvements in the magnetically shielded room and the SQUID made more extensive biomagnetism studies feasible.

Faraday observed that, just as a current in a wire loop creates a magnetic field, a changing magnetic field threading a wire loop creates a current. This is the principle behind magnetometers. To measure a magnetic field, one puts a wire loop in the time-varying field and measures the current through the loop. Magnetometers are adequate for measuring the earth's magnetic field but are not intrinsically sensitive enough to detect the weak biomagnetic signals above the unwanted magnetic noise of the earth's field and the urban environment. The urban magnetic noise can be reduced significantly simply by locating the
laboratory in a rural setting, away from the noise sources. For example, 50 Hz electric lines can produce $10^6$ pT fields in an urban environment, but the field due to the power lines is reduced to 10 pT in a rural setting (Carelli, et al., 1982). Care must be taken to build the laboratory out of non-metallic materials so that the building itself doesn’t contribute to the noise level. However, in order to apply biomagnetism in a clinical setting, the equipment must be located in or near a hospital rather than in a rural setting. One way to reduce the magnetic noise is to build a magnetically shielded room. These enclosures, designed to attenuate the environmental noise (usually in the 1-100 Hz range), consist of multiple layers of high permeability μ-metal and ultra-pure aluminum. Building a magnetically shielded room is very expensive, mostly because the μ-metal used for the shielding is expensive. Due in part to the cost, the rooms are typically very small (2 to 3 m on a side), and many people find them to be claustrophobic.

Another solution to the problem of magnetic noise is to build a gradiometer (a measurement instrument that has two or more magnetometer loops). There are many different types of gradiometers and the ones commonly used in biomagnetism are first-order axial, second-order axial, and first-order planar gradiometers (Figure 1.8) (magnetometers are sometimes referred to as zeroth-order gradiometers).

A first-order axial gradiometer consists of two wire loops, called pick-up coils, mounted axially so that we measure the change in the magnetic field with $z$ ($\partial B_z/\partial z$). The distance between pick-up coils of a gradiometer, called the baseline, typically ranges from 3.5 to 10 cm. The output current is proportional to the difference between the magnetic fields threading the coils. The earth's field falls off at about 10 pT/m (Carelli, et al., 1982) and is reasonably constant over the distance between the two pick-up coils. The biomagnetic field falls off more rapidly, so when the difference between the current in the coils is taken, the earth's field cancels out and the output is proportional to just the field of interest.
Figure 1.8
Four gradiometers used for biomagnetic measurements.
A second-order axial gradiometer takes this concept one step further and uses four vertically separated pick-up coils to measure the second derivative of the magnetic field ($\partial^2 B_z / \partial z^2$). A second-order gradiometer rejects both magnetic fields that are constant over the baseline and magnetic fields that have constant first derivatives. This results in greater noise reduction but also a greater loss in the spatial resolution of the desired field.

A planar gradiometer has two pick-up coils separated horizontally instead of vertically and measures the change in magnetic field in a horizontal directions (either $\partial B_x / \partial x$ or $\partial B_y / \partial y$). It is possible to use thin-film technology to manufacture the planar coils and even to integrate the SQUID sensor and the gradiometer into a single device.

A gradiometer reduces the noise from any magnetic field approximately constant over the baseline, but we still need a very sensitive device to measure the output current. In 1970 Zimmerman built a reliable device that could detect extremely weak magnetic fields (Zimmerman, et al., 1970). That device, called a point contact Superconducting Quantum Interference Device (SQUID), consists of a superconducting ring containing a single Josephson junction.

There are several other kinds of SQUIDs used in biomagnetism, including the radio frequency or rf-SQUID and the direct current or dc-SQUID. The rf-SQUID is a superconducting ring with one weak link and is operated with rf magnetic flux bias. The dc-SQUID is a superconducting ring with two weak links and is operated with a direct current bias. Dc-SQUIDs often possess a lower noise level than rf-SQUIDs. For a good discussion of the SQUIDs used in biomagnetism, see Erné (1982) and Williamson and Kaufman (1980).

The SQUIDs used for biomagnetic measurements operate at the temperature of liquid helium (4.2 K). The gradiometer coils are also superconducting, so the intrinsic noise of the system is very low.
Measuring the biomagnetic field in just one location is not particularly useful. We need many measurements over a large area in order to have a good spatial representation of the magnetic field. If the field is stable in time or varies predictably in time, then a single detector can be moved to each measurement location until the entire field has been mapped. This is a time-consuming, tedious task and can only be used to study events that are stable from trial to trial. In addition, the total measurement time must be short to avoid tiring the patient. To overcome these problems, arrays of SQUIDs have been built that sample the biomagnetic field in several locations simultaneously. In 1984, several researchers announced that they were routinely using multi-channel systems in their laboratories. The multi-channel systems included a four-channel system designed to be used in a magnetically shielded room (Ilmoniemi, et al., 1984) and a five-channel instrument designed to be used in an unshielded environment (Williamson, et al., 1984). Other researchers have followed suit, and arrays with more than 30 elements are now available. There are as many different designs of arrays as there are designers. No one design has proven to be better than any of the rest (see for example, Erné and Romani, 1985, or Knuutila, et al., 1985). All designs use some form of gradiometer, but that is about all they have in common. The arrays have different types of gradiometers (planar or axial), with different orders of gradiometers (first or second), different baselines (from 35 to 70 mm), and different pick-up coil diameters (from 15 to 27 mm). The axes of the gradiometers are arranged so that the pick-up coils are either on a planar surface or on a spherical surface. Heart studies often use planar arrays while brain studies use spherical arrays. The noise levels are quite low, ranging from $5 \times 10^{-3} \text{pT}/\sqrt{\text{Hz}}$ to just under $10^{-1} \text{pT}/\sqrt{\text{Hz}}$ in an unshielded environment (Katila, 1989).
Applications of Magnetic Measurements

The extraction of information from measured magnetic fields has both medical and nonmedical applications because it is a noninvasive procedure. We have already alluded to two biological sources, the heart and brain, and these are the major areas of biomagnetic research. In cardiomagnetism, the most common research method has been to map the magnetic fields of patients with known cardiac abnormalities and compare them to the magnetic fields of patients with normal hearts. Some of the abnormalities that have been studied are infarctions (damage to the heart muscle caused by lack of coronary circulation), hypertrophies (enlargement of the heart when it is overworked, for example when compensating for a faulty valve or clogged arteries), and blocked branches. For a review of the many studies see, for example, Karp (1980) or Fenici (1982). In neuromagnetism, the most commonly studied brain disease is epilepsy, including generalized epilepsy (Ricci, et al., 1990), focal epilepsy (Ricci, et al., 1987), and spontaneous signals in epileptic patients (Modena, et al., 1982). The magnetic fields evoked by auditory stimuli in normal patients were first studied by Reite, et al. (1978). The auditory signals used in later studies include constant tones, clicks, white noise (Reite, et al., 1982) and frequency glides (Arlinger, et al., 1982). Somatically evoked magnetic fields in normal patients have been investigated by several groups (Mizutani and Kuriki, 1986 and Tiihonen, et al., Other origins of electrical activity in the brain have also been studied. Hari et al. (1983) measured the magnetic field due to sources in the brain prior to and during voluntary limb (leg, ankle, and finger) movements. The magnetic fields due to sleep spindles in normal patients were studied by Nakasato et al. (1990).

Although the heart and brain are the organs most commonly studied in biomagnetic research, other areas of the body are also subjects of research. Biomagnetic measurements
have been used to determine the level of magnetic particles in miners' lungs (Kalliomäki, et al., 1982 and Brauer and Stroink, 1985). Antervo et al. (1985) mapped the magnetic field maps due to voluntary eye blinking and eye movements. Hari et al. (1989) measured magnetic fields around the arm due to nerves in the wrist.

One group (Di Luzio, et al., 1989) has suggested a novel method for studying gastrointestinal activity in which patients swallow small magnets, and the magnetic field is measured as the magnet goes through the intestines. They propose using this method instead of the present method of having patients swallow solid radio-opaque materials, and then x-raying the patients.

Another novel use of biomagnetism was demonstrated by Ilmoniemi et al. (1988). In this case, a portion of an acupuncture needle was lost in a man's back. One x-ray image showed the needle, but it wasn't found in two subsequent surgeries or in 29 other x-ray images. They used a SQUID system to measure the magnetic field, and assumed the source was a magnetic dipole. The approximate location of the needle, as determined by the magnetic measurements, was used to determine where CT scans should be made to more accurately locate the needle. They were able to successfully remove the needle.

Nonmedical applications include testing electronics and geophysical sources. Roth et al. (1989) present an algorithm for estimating electric currents that could be used for nondestructive testing of printed circuit boards. Possible geophysical sources include seawater oscillating in the earth's magnetic field, and seismic stresses (Podney, 1980). However, SQUIDs are rarely used by geophysicists outside of laboratories because of the cryogenic requirements. In the laboratory, SQUIDs are used to measure the magnetic susceptibilities and remanent magnetization of rocks (Goubau, 1980).
Mathematical Groundwork

Fourier Transforms

As the title of this dissertation indicates, Fourier transforms are used extensively in our reconstruction algorithms. The notation we use is as follows. For continuous one-dimensional Fourier transforms we define \( \tilde{f}(\xi) \), the forward transform of \( f(x) \), as

\[
\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x \xi} \, dx
\]  

(1.17)

and the inverse transform of \( \tilde{f}(\xi) \) as

\[
f(x) = \int_{-\infty}^{\infty} \tilde{f}(\xi) e^{i2\pi x \xi} \, d\xi
\]  

(1.18)

where a tilde denotes the Fourier transformed function and \( i = \sqrt{-1} \). The units of \( x \) are length (i.e. cm) and the units of the spatial frequency \( \xi \) are inverse length (i.e. cm\(^{-1}\)). The Fourier transform of \( f(x) \), \( \tilde{f}(\xi) \), contains information about the distribution of spatial frequencies in \( f(x) \). We shall also use \( F \) to denote the Fourier transform operation:

\[
\tilde{f}(\xi) = F \left\{ f(x) \right\}
\]  

(1.19)

We prefer the symmetry of these particular definitions, but other definitions are just as valid. Alternative notation schemes define the frequency variable \( \xi \) so that it includes the
In this case a $1/\sqrt{2\pi}$ is put in front of both integrals or a $1/(2\pi)$ is put in front of one or the other integrals. Which definition is used doesn't matter, as long as it is used consistently.

In order for the Fourier transform of a function to exist, the function must meet certain conditions. In particular, the function must be absolutely integrable on the interval $]-\infty, \infty[$ and any discontinuity must be finite. For more detail see a text on Fourier transforms (e.g. Bracewell, 1986).

Given the two-dimensional function $f(x, y)$, we define its two-dimensional Fourier transform in Cartesian coordinates by

$$\tilde{F}(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(x\xi+y\eta)} \, dx \, dy \quad (1.20)$$

where $\xi$ and $\eta$ are the frequency variables corresponding to the x- and y- directions. The inverse transform of $\tilde{F}(\xi, \eta)$ is defined as

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}(\xi, \eta) e^{+i2\pi(x\xi+y\eta)} \, d\xi \, d\eta \quad . \quad (1.21)$$

We shall use $F_2$ to denote the two-dimensional Fourier transform operation:

$$\tilde{F}(\xi, \eta) = F_2 \{ f(x, y) \} . \quad (1.22)$$
Similarly, the three-dimensional Fourier transform in Cartesian coordinates of \( f(x, y, z) \) is defined as

\[
\hat{T}(\xi, \eta, \zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-j2\pi(x\xi + y\eta + z\zeta)} \, dx \, dy \, dz \quad (1.23)
\]

where \( \xi, \eta, \) and \( \zeta \) are the frequency variables corresponding to the \( x-, y-, \) and \( z-\)directions. The inverse transform of \( \hat{T}(\xi, \eta, \zeta) \) is defined as

\[
f(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{T}(\xi, \eta, \zeta) e^{+j2\pi(x\xi + y\eta + z\zeta)} \, d\xi \, d\eta \, d\zeta. \quad (1.24)
\]

We shall use \( F_3 \) to denote the three-dimensional Fourier transform operation:

\[
\hat{T}(\xi, \eta, \zeta) = F_3 \left\{ f(x, y, z) \right\}. \quad (1.25)
\]

**Properties of Fourier Transforms**

We now discuss some of the properties of Fourier transforms. Consider two functions, \( f(x) \) and \( g(x) \), and their Fourier transforms, \( \hat{T}(\xi) \) and \( \hat{g}(\xi) \). The linearity property implies if we Fourier transform the sum of \( f(x) \) and \( g(x) \),
the result is the sum of their Fourier transforms

\[
F \left\{ f(x) + g(x) \right\} = \int_{-\infty}^{\infty} \left[ f(x) + g(x) \right] e^{-2\pi i x \xi} \, dx
\]

\[
= \int_{-\infty}^{\infty} f(x) \, e^{-2\pi i x \xi} \, dx + \int_{-\infty}^{\infty} g(x) \, e^{-2\pi i x \xi} \, dx \quad (1.26)
\]

\[
= \tilde{f}(\xi) + \tilde{g}(\xi).
\]

Many of the functions we use in this dissertation are vectors, e.g. the magnetic field, \( B(r) \), and current density, \( J(r) \). The Fourier transform of a vector follows directly from the linearity property; each component of the vector is transformed separately. For example, the Fourier transform of the vector field \( \mathbf{u}(x) \) in one dimension is

\[
\tilde{\mathbf{u}}(\xi) = \int_{-\infty}^{\infty} \mathbf{u}(x) \, e^{-2\pi i x \xi} \, dx
\]

\[
= \int_{-\infty}^{\infty} \left[ \hat{\mathbf{u}}_x(x) + \hat{\mathbf{u}}_y(x) + \hat{\mathbf{u}}_z(x) \right] \, e^{-2\pi i x \xi} \, dx \quad (1.27)
\]

\[
= \hat{\mathbf{u}}_x(\xi) + \hat{\mathbf{u}}_y(\xi) + \hat{\mathbf{u}}_z(\xi).
\]

The extension to vectors of two or three variables is straightforward. Thus, the three-dimensional Fourier transform in Cartesian coordinates of the magnetic field is

\[
F \{ \mathbf{B}(x, y, z) \} = \hat{\mathbf{B}}_x(\xi, \eta, \zeta) + \hat{\mathbf{B}}_y(\xi, \eta, \zeta) + \hat{\mathbf{B}}_z(\xi, \eta, \zeta)
\]

\[
= \hat{\mathbf{B}}(\xi, \eta, \zeta) \quad (1.28)
\]
In contrast to the Fourier transform of the sum of functions, transforming the product of \( f(x) \) and \( g(x) \), does not simply give the product of their Fourier transforms. Rather, we get the convolution of their Fourier transforms

\[
F \left\{ f(x) g(x) \right\} = \tilde{f}(\xi) \ast \tilde{g}(\xi) = \int_{-\infty}^{\infty} \tilde{f}(\beta) \tilde{g}(\xi - \beta) \, d\beta
\]

where we have denoted the convolution operation by an asterisk \((\ast)\). In general, the convolution of two functions yields a function that is smoother than either of the original functions. The amount of smoothing depends on the functions involved.

The scaling property states that if we multiply the width of \( f(x) \) by some constant, \( a \), then the width of \( \tilde{T}(\xi) \) also changes, but in the opposite sense.

\[
F \left\{ f \left( \frac{x}{a} \right) \right\} = |a| \tilde{T}(a\xi)
\]

Thus, if we decrease the width of \( f(x) \) then the width of \( \tilde{T}(\xi) \) increases.

The shifting property states that shifting \( f(x) \) by an amount \( x_0 \), causes \( \tilde{T}(\xi) \) to be multiplied by a phase factor

\[
F \left\{ f(x \pm x_0) \right\} = e^{\pm i2\pi x_0} \tilde{T}(\xi).
\]

Another important property of Fourier transforms is that the central ordinate of the
transform, $\mathcal{F}(0)$, is the area under the function in real space

$$\mathcal{F}(0) = \int_{-\infty}^{\infty} f(x) \, dx .$$

(1.32)

This property is easy to derive; we simply substitute the value $\xi = 0$ into the definition of the Fourier transform (equation (1.17)).

An important concept for this work is band-limited and space-limited functions. If a Fourier transform $\mathcal{F}(\xi)$ is zero outside a finite frequency region then $f(x)$ is called band-limited. In Figure 1.9 we show a the Fourier transform of a band-limited function, $\mathcal{F}(\xi)$, that is symmetric about the origin: $\mathcal{F}(\xi) = 0$, for $|\xi| \geq \xi_{\text{cutoff}}$ and thus we can define a width, $W$, where $W = 2\xi_{\text{cutoff}}$. A space-limited function $f(x)$ is zero outside a finite region in real space. A function cannot be both band- and space-limited, although in practice we often assume that the functions are approximately both band- and space-limited. We can force a function to be zero beyond a certain point by multiplying it by a function that is limited. Thus, we can force a function to be band-limited by low-pass filtering the Fourier transform. There are two common ways of determining the width of function that isn’t limited. One is to measure the peak value of the original function and set an arbitrary cutoff point that is a percentage of the peak value (for example, 5% of the peak value). The other way to determine the width is to look at the signal-to-noise ratio (SNR) and set an arbitrary cutoff point based on the SNR (for example, at the point where the SNR equals one). Forcing a function to be band- or space-limited will introduce errors since some information is lost.
Figure 1.9
A band-limited function.
A Few Functions

We now present the definitions of a few functions using the same notation and definitions as Gaskill (1978). The rectangle function of width \( b \), unit height, and centered on the point \( x_0 \) (Figure 1.10) is defined to be

\[
\text{rect}\left(\frac{x - x_0}{b}\right) = \begin{cases} 
1, & \left|\frac{x - x_0}{b}\right| < \frac{1}{2} \\
\frac{1}{2}, & \left|\frac{x - x_0}{b}\right| = 0 \\
0, & \left|\frac{x - x_0}{b}\right| > \frac{1}{2}
\end{cases}
\] (1.33)

The rect function is useful for selecting just a portion of a function as done when low-pass filtering a Fourier transform.

The sinc function of unit height and centered on the origin (Figure 1.11) is defined as

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}
\]

and has the properties that \( \text{sinc}(0) = 1 \) and \( \text{sinc}(nx) = 0 \) for \( n \) a nonzero integer. The sinc function is commonly used as an interpolation function and is also the Fourier transform of the rect function.

The cylinder function is the two-dimensional, rotationally symmetric counterpart of the rect function.
Figure 1.10
The rect function.

Figure 1.11
The sinc function.

Figure 1.12
The cylinder function.

Figure 1.13
Radial profile of the sombrero function.
The cylinder function (Figure 1.12) centered on the origin, of diameter $b$ and unit height is defined as

$$\text{cyl}(r) = \begin{cases} 1, & 0 \leq r < \frac{b}{2} \\ \frac{1}{2}, & r = \frac{b}{2} \\ 0, & r > \frac{b}{2} \end{cases}$$

(1.35)

where $r = \sqrt{x^2 + y^2}$. We use the cylinder function to describe the pick-up coil of the SQUID system.

The sombrero function (Figure 1.13) is

$$\text{somb}(r) = \frac{2J_1\left(\frac{\pi r}{b}\right)}{\pi r}$$

(1.36)

where $J_1(\cdot)$ is the first-order Bessel function of the first kind. The sombrero function is the two-dimensional, rotationally symmetric counterpart of the sinc function, in the sense that it is the two-dimensional Fourier transform of the cylinder function.

The Dirac delta function is defined to have the properties

$$\delta(x - x_0) = 0 \quad \text{if} \quad x \neq x_0,$$

(1.37)

and

$$\int_{x_1}^{x_2} f(x) \delta(x - x_0) \, dx = f(x_0) \quad \text{if} \quad x_1 < x_0 < x_2,$$
where \( f(x) \) is any well-behaved function. We shall represent \( \delta(x-x_0) \) as an arrow of unit height at \( x = x_0 \) (Figure 1.14). The reader is referred to Bracewell (1986) for a rigorous treatment of the mathematical intricacies of the delta function. The delta function can be used to pick a single point out of a function

\[
f(x) \delta(x - x_0) = f(x_0) \delta(x - x_0). \tag{1.38}
\]

The delta function can also be used to shift a function (replicate it in a new location)

\[
f(x) \ast \delta(x - x_0) = f(x - x_0). \tag{1.39}
\]

The Fourier transform of the Dirac delta function is unity,

\[
F \left\{ \delta(x) \right\} = 1. \tag{1.40}
\]

If we wish to sample a function at an infinite number of points, we use a comb function (Figure 1.15). The comb is defined as

\[
\text{comb} \left( \frac{x - x_0}{b} \right) = |b| \sum_{n=-\infty}^{\infty} \delta(x - x_0 - nb) \tag{1.41}
\]

and is an array of delta functions spaced \( |b| \) units apart. The comb function has the useful property that its Fourier transform is also a comb function.
Figure 1.14
The delta function.

Figure 1.15
The comb function.
To sample a function we multiply it by a comb function,

$$f(x) \left| \frac{1}{|b|} \right| \operatorname{comb}\left(\frac{x}{b}\right) = \sum_{n=-\infty}^{\infty} f(nb)\delta(x - nb) . \quad (1.42)$$

In Fourier space the multiplication becomes a convolution

$$\tilde{f}(\xi) \circ \operatorname{comb}(b\xi) = \sum_{n=-\infty}^{\infty} \tilde{f}(\xi - nb) . \quad (1.43)$$

The convolution with the $\sim nb$ in Fourier space replicates the function $\tilde{f}(\xi)$ at regular intervals. So a sampled function in one domain gives a periodic function in the other domain. Note that the scaling property applies here, so if we have closely spaced samples in real space, the replicas in Fourier space are widely spaced.

\textit{Whittaker-Shannon Sampling Theorem}

We now consider the effects of sampling the data. In reality we don't measure a continuous, infinite number of data points. Instead, we have practical limitations such as a finite amount of time and money that restrict the number of data points we can collect. Not only that, but we can't sample at an infinitesimal point, we must take into account the finite size of the sampling aperture. So we are faced with the questions of how finely we must sample in order not to lose information and whether it is even possible to have a finite number of data points describing exactly the original continuous function.

First, consider the case of ideal sampling (Figure 1.16a-f), where we have a band-
limited function, \( f(x) \), (Figure 1.16a) and its transform, \( \tilde{f}(\xi) \), (Figure 1.16b). We sample \( f(x) \) by multiplying by a sampling function. The most obvious and common sampling function is an infinite, regularly spaced array of delta functions: the comb function. The sampled function, \( f_s(x) \), (Figure 1.16c) is then

\[
f_s(x) = f(x) \frac{1}{|x_s|} \text{comb}\left(\frac{x}{x_s}\right)
= \sum_{n=-\infty}^{\infty} f(nx_s) \delta(x - nx_s) . \tag{1.44}
\]

If the sampling interval in direct space, \( x_s \), is small enough then the replicas of the band-limited \( \tilde{f}(\xi) \) are spaced far enough apart so that they don't overlap (Figure 1.16d). If we pick any one of the replicas, we have exactly the original function \( \tilde{f}(\xi) \). The obvious way to pick a single replica is to multiply \( \tilde{f}_s(\xi) \) by a rect function that is just wide enough to cover a single replica of the Fourier transform (Figure 1.16f)

\[
\tilde{f}(\xi) = \tilde{f}_s(\xi) \text{rect}\left(\frac{\xi}{W}\right) . \tag{1.45}
\]

This means we convolve the sampled points in real space with a sinc function (Figure 1.16e)

\[
f(x) = f_s(x) \ast W \text{sinc}(Wx) . \tag{1.46}
\]
Figure 1.16
Ideal sampling at the Nyquist rate.
(a) The original band-limited function, \( f(x) \)
(b) The symmetric Fourier-transform of \( f(x) \)
(c) The sampled function, \( f_s(x) \)
(d) The periodic Fourier-transform of \( f_s(x) \)
(e) The sampled function convolved with a sinc function
(f) The low-pass filtered, periodic Fourier transform
We have recovered the original function exactly from just the sampled data as long as we have met a couple of conditions: the function must be band-limited and we must sample at a small enough interval so that the spectra don't overlap. This condition is called the Whittaker-Shannon (or Whittaker-Shannon-Kotel'nikov) sampling theorem (for a review, see Jerri, 1977), and the minimum sampling rate is called the Nyquist rate. If the width of the function in Fourier space is \( W \), then the Nyquist condition says we must sample at intervals of \( x_n = 1/W \). Note that we can sample at smaller intervals and still recover the original function exactly because oversampling results in the spectra being further apart. Undersampling results in an overlap of the spectral orders, a condition called aliasing, where high frequencies masquerade as low frequencies.

Now we complicate matters by taking only a finite number of samples. We still have a band-limited function and sample with a comb at the Nyquist rate, but we restrict the range of the comb (in direct space) to be from \(-N\) to \(N\) instead of from negative infinity to positive infinity. The sampled function is real space is now

\[
f_s(x) = f(x) \text{comb}
\left(\frac{x}{x_n}\right) \text{rect}
\left(\frac{x}{2N}\right)
\]

and in Fourier space it is

\[
\tilde{f}_s(\xi) = \tilde{f}(\xi) \ast x_n \text{comb}(x_n \xi) \ast 2N \text{ sinc}(2N\xi)
\]

Before, we had the Fourier transform of \( f(x) \) convolved with a regularly spaced array of delta functions which replicated the Fourier transform without any distortion (as long as the Nyquist sampling condition was met). Now we have the Fourier transform of \( f(x) \) convolved with a regularly spaced array of sinc functions. If \( N \) is large, then the width of
the sinc function is very narrow, so there is little distortion of \( T(\xi) \) when it is replicated. If we have few samples then the sinc functions are wide and the original function may be severely distorted.

The final complication we shall consider is the fact that a delta function is a mathematical abstraction, and we can never sample at an infinitesimal point. Including the finite size of the sampling aperture, \( A(x/x_d) \) where \( x_d \) is the size of the aperture, gives a sampled function

\[
f_s(x) = \left[ f(x) \ast A \left( \frac{x}{x_d} \right) \right] \text{comb} \left( \frac{x}{x_s} \right) \text{rect} \left( \frac{x}{2N} \right).
\]

(1.49)

We average the function \( f(x) \) over some area by convolving with \( A(x/x_d) \). We then take a finite number of samples of the smoothed function. If \( x_d \) is small, then the aperture size has little effect on the measurement, while if the aperture is large it can destroy much of the high-frequency information. If we start with a very smooth function, then the averaging over \( A(x/x_d) \) won't have much effect, and the sampling rate as determined from the original function is not much different than the sampling rate for the averaged function.

The analysis in one dimension is easily extended to two or three dimensions. In two dimensions we sample with a two-dimensional comb function, have a two-dimensional aperture, have a finite numbers of sample points in two directions and the Fourier transform must be band-limited in both directions. The sampled function, \( f_s(x,y) \), is

\[
f_s(x,y) = \left[ f(x,y) \ast A \left( \frac{x}{x_d}, \frac{y}{y_d} \right) \right] \text{comb} \left( \frac{x}{x_s}, \frac{y}{y_s} \right) \text{rect} \left( \frac{x}{2N}, \frac{y}{2M} \right).
\]

(1.50)

In general we can have different Nyquist rates and different numbers of samples for the two
directions and an asymmetric sampling aperture. In practice, there is usually a great deal of symmetry in the two directions.

**Fourier Transforms and the Differentiation Operator**

We use Fourier transforms for this problem in large part because of the ease of working with derivatives. The differentiation operator Fourier-transforms to a simple multiplication

\[
F \left\{ \frac{\partial f(x)}{\partial x} \right\} = i2\pi \xi f(\xi). \tag{1.51}
\]

And the three-dimensional Fourier transform of the gradient operator \( \nabla \) is

\[
F \{ \nabla \} = F \left\{ \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right\} = \hat{\mathbf{i}} i2\pi \xi + \hat{\mathbf{j}} i2\pi \eta + \hat{\mathbf{k}} i2\pi \zeta. \tag{1.52}
\]

Consider, for example, the problem of finding the Fourier transform of a magnetic field for a known current density. This problem arises when we wish to find the Nyquist sampling rate for measuring the magnetic field (Chapter 4). We could use the Biot-Savart law (equation (1.2)) and then Fourier transform the magnetic field. However, the integration in the Biot-Savart law is tractable for only simple current densities. We can also Fourier-transform the Maxwell equations (1.7c and d) to find the Fourier transform of the magnetic field. To use this approach, we Fourier-transform the Maxwell equation
that relates the current density to the curl of the magnetic field (equation (1.7c)),

\[
F_3 \left\{ \nabla \times \mathbf{B}(r) \right\} = F_3 \left\{ \mu_0 \mathbf{J}(r) \right\} \\
i2\pi \rho \times \mathbf{B}(\rho) = \mu_0 \mathbf{J}(\rho)
\]  

(1.53)

where \( \rho = \hat{i} \xi + \hat{j} \eta + \hat{k} \zeta \). Now we form the cross product of \( \rho \) with both sides,

\[
\rho \times \left[ i2\pi \rho \times \mathbf{B}(\rho) \right] = \rho \times \mu_0 \mathbf{J}(\rho) .
\]

(1.54)

Applying the vector identity (sometimes called the BAC-CAB rule), \( \mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \), to the left hand side of equation (1.54) yields

\[
\rho \left[ i2\pi \rho \cdot \mathbf{B}(\rho) \right] - \mathbf{B}(\rho)(i2\pi \rho \cdot \rho) = \rho \times \mu_0 \mathbf{J}(\rho) .
\]

(1.55)

The Fourier transform of the divergence equation (equation (1.7d)) is

\[
F_3 \left\{ \nabla \cdot \mathbf{B}(r) \right\} = 0 \\
i2\pi \rho \cdot \mathbf{B}(\rho) = 0
\]

(1.56)

so the first term on the left-hand side of equation (1.55) is zero. We can now solve for the Fourier transform of the magnetic field,

\[
\mathbf{B}(\rho) = \frac{-\rho \times \mu_0 \mathbf{J}(\rho)}{i2\pi (\rho \cdot \rho)} .
\]

(1.57)
This allows us to find the transform of the magnetic field by Fourier transforming the current density, instead of solving the Biot-Savart integral and then Fourier transforming the magnetic field.

**Discrete Fourier Transforms**

If the Fourier transform of a function is computed by numerical methods, then the function and its transform are evaluated only at discrete points. We won't derive the discrete Fourier transform from the continuous transform; rather we shall simply present the definitions. There are several important differences between continuous and discrete transforms. Perhaps the most important distinction is that the discrete transform is defined only at a finite number of points. The discrete Fourier transform, as implemented on a computer, assumes that all functions are sampled and periodic, with period $M$. The forward, discrete Fourier transform is defined as

$$\hat{f}(\lambda) = \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} f(\ell) e^{-i2\pi\lambda\ell/M} \quad (1.58)$$

and the inverse transform as

$$f(\ell) = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} \hat{f}(\lambda) e^{i2\pi\lambda\ell/M}. \quad (1.59)$$
Note that the exponent in the Fourier kernel has been scaled by $1/M$ and that $\ell$ and $\lambda$ are integers. The definitions of the discrete Fourier transform have not been standardized, and the limits on the sums and the normalizing factor of $1/\sqrt{M}$ are often defined slightly differently. The limits are sometimes defined to be from $-M/2$ to $(M-1)/2$ so that the indexing has negative as well as positive values. The discrete Fourier transform is periodic with period $M$, so that the values of $f(\ell)$ for $M/2 < \ell < M-1$ are in fact the same as the values for $-M/2 < \ell < 0$. Thus, the definition of the limits is not critical as long as a full period is covered. The normalization is often defined to be just a factor of $1/M$ in front of the inverse transform. As for the definitions of the continuous Fourier transform, we prefer these definitions (equations (1.58) and (1.59)), but the other definitions are equally valid.

The linearity property holds for discrete Fourier transforms, but the properties of scaling and shifting must be applied with care. Scaling a periodic discrete function (Figure 1.17) so it is larger than $M$ results in aliasing; the ends of the function overlap the next period. Shifting a function (Figure 1.18) to the right past $\ell = M-1$ is equivalent to shifting the function to the left because of the periodicity ($f(\ell) = f(\ell \pm M) = f(\ell \pm 2M)$...) of discrete Fourier transforms.
Figure 1.17
Scaling a periodic discrete function.
(a) The original function
(b) The function scaled by an amount "a" so it is larger than the period M

Figure 1.18
Shifting a periodic discrete function.
(a) The original function
(b) The function shifted by an amount "b"
The discrete Fourier transforms of specific functions are different from the continuous transforms. The discrete version of a function must be defined carefully, and it isn't always obvious what the definition should be. It is not enough to simply replace the continuous variables with the corresponding discrete ones. For example, \( F_d \left\{ \text{rect} \left( \frac{t-t_0}{a} \right) \right\} \) is not just \( |a| \text{sinc}(a\lambda) e^{-2\pi x^2 \lambda a/M} \). To see this, we calculate the discrete Fourier transform,

\[
F_d \left\{ \text{rect} \left( \frac{t-t_0}{a} \right) \right\} = \frac{1}{\sqrt{M}} \sum_{\ell=0}^{M-1} \text{rect} \left( \frac{\ell-t_0}{a} \right) e^{-i2\pi \ell \lambda /M}
\]

\[
= \frac{1}{\sqrt{M}} \sum_{\ell=\ell_0-a/2}^{\ell_0+a/2} e^{-i2\pi \ell \lambda /M}
\]

\[
= \frac{1}{\sqrt{M}} \left[ e^{-i2\pi \lambda (\ell_0+(a/2)+1)/M} - e^{-i2\pi \lambda (\ell_0-(a/2))/M} \right]
\]

\[
= \frac{1}{\sqrt{M}} e^{-i2\pi \lambda a/M} \frac{\sin(\pi \lambda (a+1)/M)}{\sin(\pi \lambda /M)} .
\]

This is a sampled, periodic sinc function, not simply \(|a| \text{sinc}(a\lambda) e^{-2\pi x^2 \lambda a/M}\) (Figure 1.19). The periodicity means that the sinc function is replicated at regular intervals and the tails from one sinc function overlap all the other sinc functions.

Finite Differences

When we form the discrete counterparts of the Maxwell equations, we must define what we mean by a 'discrete' derivative. We stress again that the discrete version of a function must be defined with care, and we cannot simply replace the continuous variable with a discrete one. In real space, the derivative of a function is approximated by finite differences.
Figure 1.19
Discrete Fourier transform of a periodic sampled rect function.
(a) A (continuous) rect function centered at the origin
(b) The Fourier transform of the rect function in (a)
(c) A periodic sampled rect function
(d) The discrete Fourier transform of the rect function in (c)
The general finite difference is defined as \( f(t+\beta) - f(t-\alpha) \) where \( \alpha \) and \( \beta \) are the displacements from the point \( f(t) \). In terms of the discrete Fourier transform, the general finite difference is

\[
f(t+\beta) - f(t-\alpha) = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} \bar{F}(\lambda) \left[ e^{i2\pi \lambda (t+\beta)/M} - e^{i2\pi \lambda (t-\alpha)/M} \right]
\]

(1.61)

If the finite difference is not symmetric \( (\beta \neq \alpha) \), then a phase factor is generated.

There are two common ways of defining a finite difference. We shall call them the symmetric and asymmetric finite differences. Some authors refer to them as the two-sided and one-sided finite differences. Central finite difference is another commonly used term for the symmetric finite difference. Asymmetric finite differences are also referred to as forward (or backward) differences. The symmetric finite difference is defined in terms of the difference between the two neighboring points \( (\beta = \alpha = 1) \),

\[
f(t+1) - f(t-1) = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} \bar{F}(\lambda) \left[ e^{i2\pi \lambda /M} - 2i \sin \left( \frac{2\pi \lambda}{M} \right) \right].
\]

(1.62)

It is symmetric about the point \( f(t) \) and therefore there is no phase factor.

The asymmetric finite difference is defined in terms of the point and one of its neighbors. The choice of which neighbor is arbitrary and determines the sign of the phase factor.
For $\beta = 1$ and $\alpha = 0$ we have,

$$f(t+1) - f(t) = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} \tilde{f}(\lambda) \, e^{i2\pi \lambda t/M} \, e^{\pi i \lambda / M} \left[ 2i \sin \left( \frac{\pi \lambda}{M} \right) \right]. \quad (1.63)$$

For $\beta = 0$ and $\alpha = 1$ we have,

$$f(t) - f(t-1) = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} \tilde{f}(\lambda) \, e^{i2\pi \lambda t/M} \, e^{-\pi i \lambda / M} \left[ 2i \sin \left( \frac{\pi \lambda}{M} \right) \right]. \quad (1.64)$$

We shall use equation (1.63) ($\beta = 1$ and $\alpha = 0$) for our asymmetric finite difference. Recall that, for the case of continuous Fourier transforms and continuous derivatives, $F \left\{ \frac{\partial f(x)}{\partial x} \right\} = i2\pi \xi \hat{f}(\xi)$ (equation (1.51)). So in the continuous case we multiply by a linear term in Fourier space, and in the discrete case we multiply by a sine. We can't just replace the continuous variable with the corresponding discrete variable.

To illustrate the effects of defining the differentiation operator in the various ways, we apply the symmetric finite differences (equation (1.62)), asymmetric finite differences (equation (1.63)) and the sampled, periodic continuous derivative ($i2\pi \lambda$) to a sampled rect function on a computer. We chose the period of the discrete Fourier transform, $M$, to be 32 and used complex arrays for the differentiation operators, the rect function and the results.

In the field of image processing, finite differences are used to enhance the edges in a scene, and that is what we expect to happen here. We expect the operators to enhance the discontinuities at the edges of the rect function and we wish to investigate the results of the different operators. Figure 1.20 shows the three operators in Fourier space; note that only the asymmetric finite difference operator has a real part, while the other two operators are
purely imaginary. The rect function was defined to be 8 points wide (Figure 1.21a). The discrete Fourier transform of the rect function was multiplied by each of the three operators. The results were then inverse Fourier-transformed and are plotted in Figure 1.21b-d. The asymmetric-finite-difference operator correctly uses a single point to indicate the location of the edges (Figure 1.21b). The symmetric-finite-difference operator correctly reproduced the location of both edges of the rect function (Figure 1.21c), but there is an ambiguity in the location of the edges. The edges appear to be two points wide, instead of the expected single point. The sampled-continuous-derivative operator result is difficult to interpret (Figure 1.21d). It is not clear where the edges are because of the ringing. The oscillations occur because the function $i2\pi\lambda$ was truncated in Fourier space and then assumed to be periodic (due to the use of the discrete Fourier transform). The finite difference operators in Fourier space are both defined to have periods that are integral multiples of the period of the discrete Fourier transform. If the periods of the sines are changed so that they no longer match the period of the discrete Fourier transform, then ringing will occur at the edges of the rect function. The truncation of $i2\pi\lambda$ can be made less abrupt by rolling it off at the high frequencies (apodization). This will reduce the ringing, and the $i2\pi\lambda$ function will start to look like a sine function. In conclusion, $i2\pi\lambda$ is a bad choice for the Fourier transform of the differentiation operator for this application because of the severe ringing. It is better to use either of the finite difference operators, and there are no clear grounds for choosing one finite difference operator over the others.
Figure 1.20
Three approximations to the differentiation operator in Fourier space.
(a) Asymmetric finite differences
(b) Symmetric finite differences
(c) Sampled, periodic continuous derivative (Note different scale for imaginary part)
Figure 1.21
Effect of applying the three approximations of the differentiation operator to a rect function.
(a) Original rect function
(b) Asymmetric finite differences
(c) Symmetric finite differences
(d) Sampled, periodic continuous derivative (Note different scale)
Chapter 2

SOLUTION TO THE LAPLACE EQUATION

In this chapter we present the solution to the Laplace equation, \( \nabla^2 \phi = 0 \), inside a bounded volume, for two different sets of boundary conditions. We encounter the Laplace equation twice in the development of our reconstruction algorithms (Chapter 3). The two problems involve different volumes and require different boundary conditions. The first problem occurs when extrapolating three components of the magnetic field from one component known everywhere on the surface (Figure 2.1). In this case, the bounded volume consists of the region of space where we can measure the magnetic field, and the boundary condition specifies the normal derivative of the magnetic scalar potential, \( \phi \), everywhere on the surface (the Neumann boundary condition). For a brief discussion of the magnetic scalar potential, see Chapter 1. The other problem involves an assumption about the behavior of the magnetic field in a volume where we cannot measure it (because the patient occupies the space, Figure 2.14), and the boundary condition specifies the function everywhere on the surface (the Dirichlet boundary condition).

Solving the Laplace equation is a classic problem, and there are many techniques for obtaining the solution. One of the most common methods for analytically solving the Laplace equation in two or three dimensions is to use separation of variables (see for example Kovack, 1984). The solution is assumed to be a product of several functions, each function depending on only one of the variables. The partial differential equation then reduces to a set of ordinary differential equations that can be readily solved. Another general method for solving the class of partial differential equations having constant coefficients and simple boundary conditions is the Fourier transform method (Richtmyer, 1967). Here, the solution
to the Laplace equation is expressed as a Fourier integral. The Fourier transform of the solution must satisfy the equation in Fourier space equivalent to the original differential equation. Both solutions to the Laplace equation presented in this chapter are based on the Fourier transform technique. The most common numerical methods for solving the Laplace equation depend upon the notion of finite-difference operators, which are analogous to differential operators (see for example Andrews, 1986). The set of finite-difference equations are then solved using almost any iteration algorithm. The rate of convergence and accuracy of the solution depends on the iteration scheme. Schlitt et al. (1989) used the classic relaxation technique known as Jacobi's method to solve the Laplace equation with Neumann boundary conditions iteratively, and found that the extrapolated field agreed quite well with both the analytic field and the Fourier-space results.

In the first section of this chapter, we present a method for extrapolating the unknown magnetic field from the measured z-component under certain readily met boundary conditions. In the second section, we discuss the appropriateness of an assumption made about the magnetic field for one of our reconstruction algorithms. Specifically, we assume that $\nabla^2 B_x = 0$ and $\nabla^2 B_y = 0$ in the region of space where we are forbidden to make measurements.

Solution to the Laplace Equation with Neumann Boundary Conditions

One of the difficulties of estimating current densities from magnetic field measurements is the lack of information about the magnetic field. In order to have a practical data collection scheme, the number of measurements must be as small as possible. In addition, some technical difficulties exist in using SQUID detectors to measure all three components of a magnetic field simultaneously at a large number of points. The
reconstruction algorithms we have developed require knowledge of all three components of
the magnetic field in the region of space where we are permitted to make measurements.
Thus, it would be useful to be able to estimate the magnetic field from a relatively small
number of known values of a single component of the field. Extrapolating the field allows
us to include in our reconstruction algorithm the a priori information about how the field is
transmitted through space. The information about the field is implicit in the Maxwell’s
equations, and the extrapolation puts that information into a form that is more accessible to
our algorithms.

In biomagnetic imaging, some researchers have suggested building complicated
gradiometers that measure both $\partial B_x/\partial x$ and $\partial B_y/\partial y$ over a plane (Hari and Ilmoniemi,
1987). They use $\nabla \times B = 0$ outside of the source region and then find $B_x$ and $B_y$ by
integration of $\partial B_x/\partial x = \partial B_x/\partial z$ and $\partial B_y/\partial y = \partial B_y/\partial z$. However, as we show in this section,
one needs only to measure $B_z$ in a plane to calculate the total field.

In this section, we address extrapolating the magnetic field when the measured data
consist of the $z$-component of the magnetic field in a single plane (Figure 2.1). The first
method presented uses a continuous Fourier-space approach, and the other two methods use
discrete Fourier-space approaches. All three methods assume that no sources of magnetic
field are present in the extrapolation region and that we know the normal derivative of the
scalar potential, which describes this magnetic field, on the boundary of the extrapolation
region.

After we discuss the extrapolation methods we present three examples. The first
example is an analytic solution in which the continuous Fourier–space method is applied to
the magnetic field due to a magnetic dipole. The other two examples are applications of
computer algorithms to two sets of data. The first set of data uses the analytic magnetic field
due to a magnetic dipole, and demonstrates that the computer programs produce the
expected results. The second set of data consists of real measurements of the z-component of the magnetic field in two planes separated by a vertical distance of 1 cm. The data from the lower plane is used to extrapolate the field to the upper plane and the extrapolated field is compared to the measured data from the upper plane. The magnetic field was generated by two current loops and the measurements were made at Philips GmbH Forschungslaboratorium in Hamburg, Germany. Details on the measurement techniques and equipment are given in Chapter 4.

**Continuous Fourier-Space Method**

The first extrapolation method we present is the continuous Fourier-space approach. The boundary conditions that must be met in order to perform the extrapolation require that the z-component of the magnetic field fall off to zero at a large distance from the source (Figure 2.1), and that the extrapolation be performed in a region free of current sources.

For our solution to the extrapolation problem to be valid, we must know the normal derivative of the scalar potential on the entire boundary of the extrapolation region. This specification is known as the Neumann boundary condition. For practical application we require that the z-component of the magnetic field, $B_z$, goes to zero on the edges of the measurement array. Thus, we assume the normal derivative of the scalar potential is zero everywhere on the boundary except on the bottom where we have nonzero measured values. We are implicitly assuming the current is confined in the x and y directions as well as in the z direction. If we allow the current to extend to infinity, then our assumption that the normal component of the scalar potential is zero on the sides and top of the extrapolation region is not valid, and this formulation is not applicable.

For biomagnetic imaging both assumptions are reasonable if the extrapolation region
Figure 2.1
Slab geometry for solution to the Laplace equation with Neumann boundary conditions. Extrapolation region above source region.
is defined to be a portion of the space outside the body (Figure 2.1). We are dealing with biological currents, which are contained within the body, and the measurements are noninvasive, so the assumption that there are no sources present in the extrapolation region is met. If we make the extrapolation region large enough so that the normal component of the scalar potential is zero on the surface except where we have measured data, then the other assumption is also met.

Since there are no sources present in the extrapolation region \( J = 0 \) we can write the magnetic field as the gradient of a scalar potential, \( \phi \),

\[
\mathbf{B}(r) = -\nabla \phi(r) ,
\]

where \( \phi(r) \) satisfies the Laplace equation

\[
\nabla^2 \phi = 0 .
\]

We can then use the differential equations relating the scalar potential and the magnetic field, Equations (2.1) and (2.2), to solve for \( \phi \) in the extrapolation region. Once we have found the scalar potential, we need only take its gradient in order to find the magnetic field.

We know physically that the field must go to zero at large distances from the source. This is equivalent to the boundary condition that the normal component of the scalar potential is zero on the top of the extrapolation region. In Figure 2.1 the extrapolation region is shown as a cube, although in general it can be any shape. We cannot extrapolate through the source region as that would violate our assumption of having no sources present in the extrapolation region. Without loss of generality, we limit ourselves to extrapolating in the positive \( z \) direction.
For our biomagnetic imaging problem we know $B_\mathbf{s}$ in only one plane 
($B_\mathbf{s}(x,y,z_m)$, where $z_m$ is the measurement plane), and we wish to calculate $B(x,y,z \geq z_m)$. Since $B_\mathbf{s}(x,y,z_m) = \frac{\partial \phi}{\partial n} |_{z_m}$ this gives us the boundary value on the bottom of the extrapolation region.

Using this information, the normal derivative of the scalar potential on the boundary of the extrapolation region is assumed to be

$$\frac{\partial}{\partial x} \phi(x,y,z) = 0 \quad \text{for } x = \pm \infty, \ -\infty \leq y \leq \infty, \text{ and } z_m \leq z \leq \infty$$

$$\frac{\partial}{\partial y} \phi(x,y,z) = 0 \quad \text{for } -\infty \leq x \leq \infty, \ y = \pm \infty, \text{ and } z_m \leq z \leq \infty$$

$$\frac{\partial}{\partial z} \phi(x,y,z) = 0 \quad \text{for } -\infty \leq x \leq \infty, \ -\infty \leq y \leq \infty, \text{ and } z = \infty$$ (2.3)

$$\frac{\partial}{\partial z} \phi(x,y,z) = -B_\mathbf{s}(x,y,z_m) \quad \text{for } -\infty \leq x \leq \infty, \ -\infty \leq y \leq \infty, \text{ and } z = z_m$$

One advantage of applying Fourier transforms to this problem is that the differentiation operator is transformed to a simple multiplication (see Chapter 1). We use this property to write the two-dimensional Fourier transform of the Laplace equation

$$\mathcal{F}_2 \left\{ \frac{\partial^2}{\partial x^2} \phi(x,y,z) + \frac{\partial^2}{\partial y^2} \phi(x,y,z) \right\} = \mathcal{F}_2 \left\{ -\frac{\partial^2}{\partial z^2} \phi(x,y,z) \right\}$$ (2.4)

or,

$$(i2\pi \xi) \hat{\phi}(\xi,\eta,z) + (i2\pi \eta) \hat{\phi}(\xi,\eta,z) = -\frac{\partial^2}{\partial z^2} \hat{\phi}(\xi,\eta,z)$$ (2.5)

where the Fourier transform has been performed in the $x$- and $y$-directions only.
We solve this second-order differential Equation (Equation (2.5)) in the standard way; we assume a solution of the form

\[ \tilde{\psi}(\xi, \eta, z) = A_1(\xi, \eta) e^{2\pi z\sqrt{\xi^2 - \eta^2}} + A_2(\xi, \eta) e^{-2\pi z\sqrt{\xi^2 + \eta^2}} \]  

(2.6)

and determine \( A_1 \) and \( A_2 \) from the boundary conditions. It is easy to verify that Equation (2.6) satisfies the second-order differential equation (Equation (2.5)). The boundary conditions for the extrapolation problem involve the first derivative with respect to \( z \), so in order to find \( A_1 \) and \( A_2 \), we must first find the derivative of \( \tilde{\psi}(\xi, \eta, z) \) with respect to \( z \),

\[ \frac{\partial}{\partial z} \tilde{\psi}(\xi, \eta, z) = A_1(\xi, \eta) 2\pi z\sqrt{\xi^2 - \eta^2} e^{2\pi z\sqrt{\xi^2 - \eta^2}} + A_2(\xi, \eta)(-2\pi z\sqrt{\xi^2 + \eta^2}) e^{-2\pi z\sqrt{\xi^2 + \eta^2}} . \]  

(2.7)

From the two-dimensional Fourier transform of the boundary conditions (Equation (2.3)) we get

\[ \frac{\partial}{\partial z} \tilde{\psi}(\xi, \eta, z) = 0 \]  

for \( z = \infty \)  

(2.8)

and

\[ \frac{\partial}{\partial z} \tilde{\psi}(\xi, \eta, z) = \tilde{B}_z(\xi, \eta, z_m) \]  

for \( z = z_m \).  

(2.9)

Evaluating Equation (2.7) at \( z = \infty \), we see that the exponential in the first term goes to infinity as \( z \) goes to infinity. In order for Equation (2.8) to be valid, the first term in Equation (2.7) must be zero, and therefore \( A_1(\xi, \eta) \) must be zero.
We find $A_2(\xi, \eta)$ from the second boundary condition, Equation (2.9),

$$A_2(\xi, \eta) = \frac{\tilde{B}_x(\xi, \eta, z_m)}{2\pi \sqrt{\xi^2 + \eta^2}} e^{2\pi i \xi_m \sqrt{\xi^2 + \eta^2}}. \tag{2.10}$$

Finally, the solution to the second-order differential equation (Equation (2.5)) is

$$\tilde{\varphi}(\xi, \eta, z) = \frac{\tilde{B}_x(\xi, \eta, z_m)}{2\pi \sqrt{\xi^2 + \eta^2}} e^{2\pi i (\xi_m - \xi) \sqrt{\xi^2 + \eta^2}}. \tag{2.11}$$

In Equation (2.11) we can identify $e^{2\pi i (\xi_m - \xi) \sqrt{\xi^2 + \eta^2}}$ as the propagation factor. The propagation factor describes how the field changes in the z-direction. As we increase the distance from the measurement plane, the propagation factor gets narrower, and the higher frequencies are reduced. In real space the field spreads out and becomes smoother.

If we inverse Fourier transform Equation (2.11), we get the scalar potential in the extrapolation region, where the negative gradient of $\varphi(x, y, z)$ is the desired biomagnetic vector field. We choose to take the derivatives in Fourier space, although we could take the derivatives in real space and arrive at the same results.

Consider the x-component first. We know in Fourier space that

$$\tilde{B}_x(\xi, \eta, z) = -i2\pi \xi \tilde{\varphi}(\xi, \eta, z) \tag{2.12}$$

because in real space

$$B_x(x, y, z) = -\frac{\partial}{\partial x} \varphi(x, y, z). \tag{2.13}$$

Since we know $\tilde{\varphi}(\xi, \eta, z)$ from Equation (2.11), we have an expression for $\tilde{B}_x(\xi, \eta, z \geq z_m)$ in terms of known quantities. To find $B_x(x, y, z \geq z_m)$ we take the two-dimensional inverse
Fourier transform

\[ B_x(x,y,z \geq z_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -i2\pi \xi \bar{B}_x(\xi,\eta, z_m) \frac{e^{2\pi i (x \eta - z \eta)} \sqrt{\xi^2 + \eta^2}}{2\pi \sqrt{\xi^2 + \eta^2}} e^{i2\pi (x + y)} \, d\xi \, d\eta. \]  

(2.14)

Similarly for \( B_y(x,y,z \geq z_m) \) and \( B_z(x,y,z \geq z_m) \):

\[ B_y(x,y,z \geq z_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -i2\pi \eta \bar{B}_y(\xi,\eta, z_m) \frac{e^{2\pi i (x \eta - z \eta)} \sqrt{\xi^2 + \eta^2}}{2\pi \sqrt{\xi^2 + \eta^2}} e^{i2\pi (x + y)} \, d\xi \, d\eta. \]  

(2.15)

\[ B_z(x,y,z \geq z_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{B}_z(\xi,\eta, z_m) \, e^{2\pi i (x \eta - z \eta)} \sqrt{\xi^2 + \eta^2} e^{i2\pi (x + y)} \, d\xi \, d\eta. \]  

(2.16)

Finally, we must be careful about what happens to the x- and y-components of the magnetic field at the origin in Fourier space; it appears that we have a fraction in the indeterminate form \( \frac{0}{0} \) when \( \xi = \eta = 0 \). We resolve this problem by applying the property that the area of a function is equal to the central ordinate of its Fourier transform (see Chapter 1), and find the area under each magnetic field component in real space. Consider
the x-component, $\bar{B}_x(0,0,z)$, first. We use the Biot-Savart law to find $B_x(x,y,z)$, and integrate over $x$ and $y$ to find the area.

\[
\bar{B}_x(0,0,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_x(x,y,z) \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\mu_0}{4\pi} \sqrt{\frac{(y-y')B_x(x',y',z) - (z-z')B_y(x',y',z)}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}} \right\} \, dx \, dy.
\]

Interchanging the order of integration yields

\[
\bar{B}_x(0,0,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\mu_0}{4\pi} \sqrt{\frac{(y-y')B_x(x',y',z) - (z-z')B_y(x',y',z)}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}} \right\} \, dx \, dy.
\]

The first term, $I_1$, is zero. To see this, we change variables to $a = (y-y')$ and note that the function of $a$ is odd over $-\infty < a < +\infty$ so the integral is zero. The second term, $I_2$, is more complicated. However, with a table of integrals (Gradshteyn and Ryzhik, 1980, eq. 3.252.7)
we find that

$$I_x = \frac{2\pi}{z-z'},$$  \hspace{1cm} (2.19)

and the central ordinate of the \(x\)-component is

$$B_x(0,0,z) = -\frac{\mu_0}{2} \int \frac{J_y(r')}{V} \, d^3r'.$$  \hspace{1cm} (2.20)

We know that the magnetic field goes to zero as \(z \rightarrow \infty\) and in particular, the \(x\)-component goes to zero. This implies that in Fourier space \(B_x(0,0,z \rightarrow \infty) = 0.\) Since the right hand side of Equation (2.20) is independent of \(z\), the left hand side must be zero for all \(z\).

For the \(y\)-component we find

$$B_y(0,0,z) = -\frac{\mu_0}{2} \int \frac{J_x(r')}{V} \, d^3r'$$  \hspace{1cm} (2.21)

and this is also zero for all \(z\) for similar reasons.
The z-component is

\[ \mathbf{B}_z(0,0,z) = \frac{\mu_0}{4\pi} \iiint_V \left\{ J_y(r') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(x-x') \, dx \, dy}{\left( (x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{3/2}} \right. \]

\[ - \left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(y-y') \, dx \, dy}{\left( (x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{3/2}} \right\} \, dr' \]

\[ = \frac{\mu_0}{4\pi} \int_V (I_3 + I_4) \, d^3r'. \]  

(2.22)

In this case, both terms are zero. For the first term, \( I_3 \), we change variables to \( \alpha = (x-x') \) and note that the function of \( \alpha \) is odd over \(-\infty < \alpha < +\infty \) so the integral is zero. For the second term, \( I_4 \), we change variables to \( \beta = (y-y') \) and note that the function of \( \beta \) is also odd over \(-\infty < \beta < +\infty \) so the integral is also zero.

Recall that for the x- and y-components of the magnetic field at the origin in Fourier space, we have a fraction in the indeterminate form \( \frac{0}{0} \) when \( \xi = \eta = 0 \). From the central-ordinate argument we know that all three components of the magnetic field are zero at the origin in Fourier space. Therefore, the numerators in the expressions for the x- and y-components go to zero faster than the denominators such that the fraction goes to zero.

We now have the extrapolated magnetic field in the half space \( z \geq z_m \). These formulas are valid only for the specific case of a current with zero divergence that is confined in the x-, y-, and z-directions. In addition, the z-component of the magnetic field must be known everywhere on a plane. If we wish to consider currents that extend to infinity in x or y, then we must measure the normal component of the scalar potential on the entire surface of the extrapolation region, not just on the plane \( z = z_m \). This formulation guarantees a unique answer in the extrapolation region (Jackson (1975)) but it provides no information about the
magnetic field outside of the extrapolation region. We are guaranteed a unique $\phi$ only when we calculate $\phi$ in the volume bounded by the surface over which the normal derivative of $\phi$ is known. In particular, we are not guaranteed a unique solution for the magnetic field if we wish to estimate an unknown current source by extrapolating the magnetic field toward the source. In the special case of a single planar current at a known distance from the measurement plane, we have placed enough restrictions on the solution to get a unique magnetic field when we extrapolate from the measurement plane toward the planar current. However, if the distance between the planar source and the measurement plane is not known, if the source has a small thickness, or if there are several sources in different planes, then the inward extrapolated field is not guaranteed to be unique.

Roth et al. (1989) have suggested a reconstruction algorithm that uses formulas that bear a resemblance to Equations (2.20)-(2.22). They assume a divergenceless planar current source with zero $z$-component confined to a thin slab centered at $z = 0$. They Fourier transform the Biot-Savart law in $x$ and $y$ only. In Fourier space, they calculate the magnetic field components to be (in our notation)

$$
\bar{B}_x(\xi,\eta, z) = \frac{\mu_0 d}{2} e^{-2\pi \xi + \eta^3} \int_y(\xi, \eta)
$$

$$
\bar{B}_y(\xi,\eta, z) = -\frac{\mu_0 d}{2} e^{-2\pi \xi + \eta^3} \int_x(\xi, \eta)
$$

$$
\bar{B}_z(\xi,\eta, z) = \frac{i\mu_0 d}{2} \frac{e^{-2\pi \xi + \eta^3}}{\sqrt{\xi^2 + \eta^2}} \left[ \eta \int_x(\xi, \eta) - \xi \int_y(\xi, \eta) \right]
$$

where $d$ is the thickness of the slab. They use these expressions to solve for the current density in terms of the magnetic field. They do not solve for the magnetic field in terms of one measured component of the magnetic field. In contrast, the extrapolation method discussed in this chapter is not used to estimate the current density; it is used only to
calculate the magnetic field from $B_4(x,y,z_m)$, and the sources of the magnetic field are not limited to planar currents a known distance from the measurement plane.

**Discrete Fourier-Space Method**

We now explore the discrete formulation of the extrapolation problem. We use discrete Fourier transforms, finite differences and sampled fields, rather than continuous Fourier transforms, derivatives and continuous fields. These necessary approximations to the continuous case are discussed in detail in Chapter 1. In particular, we saw that there are several ways of approximating the differentiation operator in Fourier space, and the different approximations produce different results.

The development of the discrete extrapolation method follows the general development of the continuous method. We begin by approximating the Laplacian operation with the general finite differences,

$$
\nabla^2 \phi \approx \phi(l+\alpha+\beta,m+n)+\phi(l-\alpha-\beta,m,n)+\phi(l,m+\alpha+\beta,n)+\phi(l,m-\alpha-\beta,n)
+\phi(l,m,n+\alpha+\beta)+\phi(l,m,n-\alpha-\beta)-6\phi(l,m,n)
\approx 0 .
$$

(2.23)

We assume that the extrapolation volume is an $M\times M \times M$ cube and that the boundary
conditions can be approximated as

\[
\phi(l+\beta,m,n) - \phi(l-\alpha,m,n) = 0 \quad \text{for } l = M-1, \ 0 \leq n \leq M-1, \ \text{and } n_m \leq n \leq M-1
\]

\[
\phi(l+\beta,m,n) - \phi(l-\alpha,m,n) = 0 \quad \text{for } l = 0, \ 0 \leq n \leq M-1, \ \text{and } n_m \leq n \leq M-1
\]

\[
\phi(l,m+\beta,n) - \phi(l,m-\alpha,n) = 0 \quad \text{for } 0 \leq l \leq M-1, \ n = M-1 \ \text{and } n_m \leq n \leq M-1
\]

\[
\phi(l,m+\beta,n) - \phi(l,m-\alpha,n) = 0 \quad \text{for } 0 \leq l \leq M-1, \ n = 0 \ \text{and } n_m \leq n \leq M-1
\]

\[
\phi(l,m,n+\beta) - \phi(l,m,n-\alpha) = 0 \quad \text{for } l = M-1, \ 0 \leq n \leq M-1, \ \text{and } n_m \leq n \leq M-1
\]

\[
\phi(l,m,n+\beta) - \phi(l,m,n-\alpha) = B_n(l,m,n_m) \quad \text{for } 0 \leq l \leq M-1, \ 0 \leq n \leq M-1, \ \text{and } n = n_m.
\] (2.24)

We write the discrete version of the Laplace equation, Equation (2.23), in terms of a
discrete, two-dimensional Fourier transform

\[
\tilde{\mathcal{L}}(\lambda,\mu,\alpha+\beta) + \tilde{\mathcal{L}}(\lambda,\mu,\alpha-\beta) - 2\tilde{\mathcal{L}}(\lambda,\mu,\alpha) =

- \tilde{\mathcal{L}}(\lambda,\mu,\alpha) \left[ \left( \frac{12\pi \gamma \lambda}{M} - e^{-\frac{12\pi \gamma \lambda}{M}} \right)^2 + \left( \frac{12\pi \gamma \mu}{M} - e^{-\frac{12\pi \gamma \mu}{M}} \right)^2 \right] \] (2.25)

where \( \gamma = (\alpha+\beta)/2 \). We solve Equation (2.25) for \( \tilde{\mathcal{L}}(\lambda,\mu,\alpha) \) in the same way we solve a
differential equation; we assume a solution and apply the boundary conditions. For this
problem, we assume a solution of the form

\[
\tilde{\mathcal{L}}(\lambda,\mu,\alpha) = A_1(\lambda,\mu) e^{-i\left(\frac{2\pi \alpha + \pi \beta}{\alpha + \beta}\right) \sin^{-1}(i\sqrt{\sin^2(\frac{2\pi \lambda \gamma}{M}) + \sin^2(\frac{2\pi \mu \gamma}{M})})}

+ A_2(\lambda,\mu) e^{i\left(\frac{2\pi \alpha + \pi \beta}{\alpha + \beta}\right) \sin^{-1}(i\sqrt{\sin^2(\frac{2\pi \lambda \gamma}{M}) + \sin^2(\frac{2\pi \mu \gamma}{M})})} \]. (2.26)

and then apply the boundary conditions (Equation (2.24)) in order to find \( A_1(\lambda,\mu) \)
and \(A_3(\lambda, \mu)\). The boundary conditions in Fourier space are

\[
\phi(l, m, n+\beta) - \phi(l, m, n-\alpha) = 0 \quad \text{at } n = M-1 \tag{2.27}
\]

and

\[
\phi(l, m, n+\beta) - \phi(l, m, n-\alpha) = B_n(l, m, n_m) \quad \text{at } n = n_m. \tag{2.28}
\]

We apply the boundary conditions to the equation

\[
\tilde{\phi}(\lambda, \mu, n+\beta) - \tilde{\phi}(\lambda, \mu, n-\alpha) = A_3(\lambda, \mu) \left[ e^{i \frac{2\pi + \alpha + \beta}{\alpha + \beta} c} - e^{i \frac{2\pi - \alpha - \beta}{\alpha + \beta} c} \right] \\
+ A_4(\lambda, \mu) \left[ e^{i \frac{2\pi + \alpha + \beta}{\alpha + \beta} c} - e^{i \frac{2\pi - \alpha - \beta}{\alpha + \beta} c} \right] \tag{2.29}
\]

where

\[
c = \sin^{-1}(i \sqrt{\sin^2(2\pi \gamma / M) + \sin^2(2\pi \mu / M)})
\]

At \(n = M-1\) the exponentials in Equation (2.29) must go to zero in order to satisfy Equation (2.27). Noticing that the arcsine of an imaginary angle can be written as a hyperbolic arcsine of a real angle, \(\sin^{-1}(iu) = i \sinh^{-1}(u)\), we rewrite the exponentials in the first term of Equation (2.29)

\[
e^{ik_1 \sin^{-1}[i \sqrt{\sin^2(2\pi \gamma / M) + \sin^2(2\pi \mu / M)}]} - e^{ik_2 \sin^{-1}[i \sqrt{\sin^2(2\pi \gamma / M) + \sin^2(2\pi \mu / M)}]} \\
= e^{k_1 \sinh^{-1}[\sqrt{\sin^2(2\pi \gamma / M) + \sin^2(2\pi \mu / M)}]} - e^{k_2 \sinh^{-1}[\sqrt{\sin^2(2\pi \gamma / M) + \sin^2(2\pi \mu / M)}]} \tag{2.30}
\]

where

\[
k_1 = \frac{2\pi + \alpha + \beta}{\alpha + \beta},
\]

and

\[
k_2 = \frac{2\pi - \alpha - \beta}{\alpha + \beta}.
\]
Now, the argument of the inverse hyperbolic sine, $\sqrt{\sin^2(2\pi \gamma \lambda/M) + \sin^2(2\pi \gamma \mu/M)}$, is always positive and real, so $\sinh^{-1}(\sqrt{\sin^2(2\pi \gamma \lambda/M) + \sin^2(2\pi \gamma \mu/M)})$ is also positive and real. At $\nu = M-1$, the exponentials in Equation (2.30) are increasing, not decreasing as required by the boundary condition (Equation (2.28)). Therefore $A_1(\lambda, \mu)$ is zero. We find $A_2(\lambda, \mu)$ by applying the boundary condition at $\nu = \nu_m$ (Equation (2.29)). After some algebra, we find

$$\mathcal{F}(\lambda, \mu, \nu) = \frac{\mathcal{F}_k(\lambda, \mu, \nu_m)}{2 \sqrt{\sin^2(2\pi \gamma \lambda/M) + \sin^2(2\pi \gamma \mu/M)}} P(\lambda, \mu), \quad (2.31)$$

where we have defined the propagation factor, $P(\lambda, \mu)$ as

$$P(\lambda, \mu) = e^{-i\left(\frac{2(\nu_m - \nu)\alpha - \beta}{\alpha + \beta}\right)\sin^{-1}(\sqrt{\sin^2(2\pi \lambda \gamma/M) + \sin^2(2\pi \mu \gamma/M)})}. \quad (2.32)$$

The propagation factor is analogous to the propagation factor defined in the continuous case. It describes how the magnetic field changes as we go away from the measurement plane.

We now have all the information we need to extrapolate the field. If we invert the Fourier transform in Equation (2.31) we get the scalar potential in the extrapolation region and the magnetic field is just the negative gradient of the scalar potential. We choose to take the finite differences in Fourier space although taking finite differences in real space can be shown to give the same results.

For symmetric finite differences, we set $\alpha = \beta = 1$ (so $\gamma = (\alpha + \beta)/2 = 1$) and the propagation factor becomes

$$P(\lambda, \mu) = e^{-i(\nu_m - \nu)\sin^{-1}(\sqrt{\sin^2(2\pi \lambda \gamma/M) + \sin^2(2\pi \mu \gamma/M)})}. \quad (2.33)$$
The extrapolation formulas for symmetric finite differences are

\[ B_k(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} \frac{(-iM)^{\lambda} \sin(2\pi \lambda / M)}{\sqrt{\sin^2(2\pi \lambda / M) + \sin^2(2\pi \mu / M)}} \tilde{B}_k(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}, \quad (2.34) \]

\[ B_y(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} \frac{(-iM)^{\mu} \sin(2\pi \mu / M)}{\sqrt{\sin^2(2\pi \lambda / M) + \sin^2(2\pi \mu / M)}} \tilde{B}_y(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}, \quad (2.35) \]

\[ B_x(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} \tilde{B}_x(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}. \quad (2.36) \]

For asymmetric finite differences, we set \( \alpha = 1 \) and \( \beta = 0 \) (so \( \gamma = 0.5 \)) to get the propagation factor

\[ P(\lambda, \mu) = e^{-i(2\omega_m - 2\omega + 1)\sin^{-1}(i\sin^2(\pi \lambda / M) + \sin^2(\pi \mu / M))}. \quad (2.37) \]

And the extrapolation formulas for asymmetric finite differences are

\[ B_k(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} \frac{-iM^{\lambda} e^{\pi\lambda}}{\sqrt{\sin^2(\pi \lambda / M) + \sin^2(\pi \mu / M)}} \tilde{B}_k(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}, \quad (2.38) \]

\[ B_y(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} \frac{-iM^{\mu} e^{\pi\mu}}{\sqrt{\sin^2(\pi \lambda / M) + \sin^2(\pi \mu / M)}} \tilde{B}_y(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}, \quad (2.39) \]

\[ B_x(\ell, \omega, \omega) = \frac{1}{M} \sum_{\lambda, \mu = 0}^{M-1} e^{i\sin^{-1}(i\sin^2(\pi \lambda / M) + \sin^2(\pi \mu / M))} \tilde{B}_x(\lambda, \mu, \omega_m) P(\lambda, \mu) e^{i2\pi(\lambda\ell + \mu\omega)/M}. \quad (2.40) \]

Comparing the discrete extrapolation formulas (both symmetric and asymmetric finite difference forms) to the continuous formulas reveals some reassuring similarities and some
Figure 2.2
Propagation factors for two distances from the measurement plane.
Right-hand column: Plane 1 pixel above measurement plane
Left-hand column: Plane 11 pixels above measurement plane
Top row: Asymmetric finite differences
Middle row: Symmetric finite differences
Bottom row: Sampled periodic version of continuous operator
important differences. In Figure 2.2, we have graphed the propagation factors at two different distances from the measurement plane for three cases: the symmetric finite difference formulas, the asymmetric finite difference formulas and a sampled, periodic version of the continuous formulas. The propagation factors are all decaying exponentials although the frequency dependence of the propagation factor varies.

There are some minor differences in factors of $2\pi$ and $M$ that are due to the differences between continuous and discrete Fourier transforms. More important differences are that the discrete forms are periodic while the continuous case is not periodic. This may introduce artifacts to the extrapolated field when the continuous formulas are implemented on a computer since Fourier transforms performed on a computer are discrete. In particular, the dimensions of the array must be large enough so that the periodicity assumed for the discrete Fourier transforms does not influence the results. For a good discussion of some of the difficulties in representing continuous functions on a computer and the periodicity of discrete Fourier transforms see Bracewell (1986).

**Analytic Example**

To illustrate the continuous Fourier-space method, we consider the magnetic field due to a magnetic dipole. A magnetic dipole can be approximated by a circular current loop of radius $a$ and an observation point $P$, a distance $r$ away from the loop (Figure 2.3). In the limit of $r \gg a$ we obtain an acceptable approximation to a magnetic dipole.

The magnetic field for a current loop at the origin, in the limit $r \gg a$ is

$$B(x, y, z) = \frac{\mu_0}{4\pi} \frac{a^2}{(x^2 + y^2 + z^2)^{5/2}} \left[ \hat{3} x z + \hat{3} y z - \hat{k}(x^2 + y^2 - 2z^2) \right],$$

(2.41)
Figure 2.3
Circular loop in the x-y plane, centered at the origin, carrying current I.
where $\pi a^2$ is the area of the loop and $I$ is the current in the loop. Following the general procedure outlined above for the Fourier-space method, we first calculate the two-dimensional Fourier transform of $B_k(x,y,z_m)$,

$$B_k(\xi, \eta, z_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{2\pi a^2} \frac{\mu_0}{4\pi} \left[ -\frac{x^2+y^2-2z_m^2}{(x^2+y^2+z_m^2)^{5/2}} \right] e^{-i2\pi(\xi x+\eta y)} \, dx \, dy.$$  

(2.42)

Letting $r^2 = x^2+y^2$ and $\rho^2 = \xi^2+\eta^2$ we see that the integral is radially symmetric and can be written as a one-dimensional Hankel Transform in $r$,

$$B_k(\rho, z_m) = -k2\pi \int_0^\infty \left[ -\frac{1}{(r^2+z_m^2)^{3/2}} - \frac{3z_m^2}{(r^2+z_m^2)^{5/2}} \right] J_0(2\pi \rho r) \, r \, dr,$$  

(2.43)

where $k = I_{2\pi a^2} \frac{\mu_0}{4\pi}$ and $J_0(\cdot)$ is a zero-order Bessel function of the first kind. Noticing that

$$\frac{\partial}{\partial z_m} \frac{z_m}{(r^2+z_m^2)^{3/2}} = \frac{1}{(r^2+z_m^2)^{3/2}} - \frac{3z_m^2}{(r^2+z_m^2)^{5/2}},$$  

(2.44)

we can write

$$B_k(\rho, z_m) = -k2\pi \frac{\partial}{\partial z_m} z_m \int_0^\infty \frac{J_0(2\pi \rho r)}{(r^2+z_m^2)^{3/2}} \, r \, dr$$  

(2.45)

where the order of integration and differentiation has been interchanged. Using a table of integrals (Gradshteyn and Ryzhik, 1980, eq. 6.554.4), we can evaluate this integral. When we
differentiate with respect to $z_m$ and write $\rho$ in terms of $\xi$ and $\eta$, we finally get the expression for the two-dimensional Fourier transform of $B_x(x, y, z_m)$.

$$\tilde{B}_x(\xi, \eta, z_m) = k(2\pi)^2 \sqrt{\xi^2 + \eta^2} e^{-2\pi x_m} \sqrt{\xi^2 + \eta^2} . \quad (2.46)$$

According to the general procedure, we now extrapolate the field using Equations (2.20)-(2.22). Consider the $x$-component first. Starting with Equation (2.20) and inserting the expression for $\tilde{B}_x(\xi, \eta, z_m)$ due to the magnetic dipole, we get

$$B_x(x, y, z \geq z_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -i2\pi \xi k(2\pi)^2 \sqrt{\xi^2 + \eta^2} e^{-2\pi x_m} \sqrt{\xi^2 + \eta^2} e^2 \pi(e_{m=0}^\pm) e^{2\pi i \eta} d\xi d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -k(2\pi)^2 i \xi e^{-2\pi \sqrt{\xi^2 + \eta^2}} e^{2\pi i \eta} (e_{x+\eta}) \ d\xi \ d\eta . \quad (2.47)$$

To evaluate this integral, we use a well known mathematical technique: take the (indefinite) integral with respect to $x$ now and differentiate with respect to $x$ later. Performing the integration generates a term, $C_1$, that does not depend on $x$. The result of integrating with respect to $x$ may be written as the sum of two integrals

$$\int_{-\infty}^{\infty} B_x(x, y, z)dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -k2\pi e^{-2\pi \sqrt{\xi^2 + \eta^2}} e^{2\pi i \eta} (e_{x+\eta}) \ d\xi d\eta + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -k2\pi e^{-2\pi \sqrt{\xi^2 + \eta^2}} C_1 \ d\xi d\eta$$

$$= I_1 + I_2 . \quad (2.48)$$

Since the first integral, $I_1$, is radially symmetric it can be written as a Hankel Transform and may be evaluated using a table of integrals (Gradshteyn and Ryzhik, 1980, eq. 6.623.2).
The result is

$$I_1 = -k(2\pi)^2 \frac{2\pi \omega}{[(2\pi)^3 + (2\pi)^2]^{3/2}}$$ (2.49)

where \( r^2 = x^2 + y^2 \). We now differentiate with respect to \( x \) to make up for the integration with respect to \( x \) earlier. As noted before, \( C_1 \) is independent of \( x \), so \( I_1 \) is independent of \( x \) and its derivative is zero. The expression for \( B_x(x,y,z_m) \) that we sought is therefore

$$B_x(x,y,z_m) = \frac{\mu_0}{4\pi} \frac{3\pi}{(x^2+y^2+z^2)^{3/2}}.$$ (2.50)

The other two components, \( B_y(x,y,z) \) and \( B_z(x,y,z) \), may be calculated by following similar steps. We insert the expression for \( B_x(\xi,\eta,z_m) \) due to the magnetic dipole into Equations (2.11) and (2.12) and integrate with respect to \( y \) (or \( z \)). The new expression can be written as a Hankel transform and evaluated using a table of integrals (Gradshteyn and Ryzhik, 1980). Upon differentiation with respect to \( y \) (or \( z \)) we obtain:

$$B_y(x,y,z_m) = \frac{\mu_0}{4\pi} \frac{3\pi y}{(x^2+y^2+z^2)^{3/2}}.$$ (2.51)

$$B_z(x,y,z_m) = \frac{\mu_0}{4\pi} \frac{-(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{3/2}}.$$ (2.52)

When we compare these expressions for the extrapolated magnetic field with the magnetic field given in Equation (2.41) we see that they are identical.
Simulated Data

We wrote several computer programs to test the performance of the three extrapolation methods. We have developed one extrapolation method using continuous mathematics and two extrapolation methods using discrete Fourier transforms and finite differences. Implementing the discrete methods on the computer was straightforward since they were formulated for the discrete arrays and Fourier transforms used by computers. On the other hand, the continuous method needed to be modified to conform with the requirements of the digital computer. Instead of approximating the differential operator with finite differences, we sampled the continuous differential operator in Fourier space. In addition to sampling the formulas at discrete points, the propagation factor was scaled to the size of the array. Thus \( P(\lambda, \nu) = e^{(-2\pi i \nu M) / \lambda^2} \) was used in the continuous method. In spite of these modifications, we shall continue to refer to the method (as implemented on the computer) as the continuous method.

The simulated data representing the magnetic field due to a magnetic dipole was calculated using Equation (2.41) and sampled at slightly greater than the Nyquist rate as determined by the methods discussed in Chapter 4. We saved the entire (sampled) analytic magnetic field and compared it to the extrapolated field. The extrapolation routines used only the \( z \)-component in one plane as input.

Qualitatively the results of all three extrapolation methods were very good; the pattern and magnitude of the extrapolated magnetic field distribution in each plane of the extrapolation region matched those of the analytic field. Two different \( x-y \) planes of the analytic and extrapolated magnetic fields are shown in Figures 2.4-2.7. The \( x \)- and \( z \)-components of the magnetic field are plotted separately as a function of the \( (x, y) \) position,
Figure 2.4
Original magnetic field, due to a magnetic dipole, used for extrapolation.
Top row: Field in plane 13 pixels above measurement plane
Bottom row: Field in plane 4 pixels above measurement plane
Left-hand column: X-component of magnetic field in x-y plane
Right-hand column: Z-component of magnetic field in x-y plane
Figure 2.5
Extrapolated magnetic field; differentiation operator approximated by asymmetric finite differences.
Top row: Field in plane 13 pixels above measurement plane
Bottom row: Field in plane 4 pixels above measurement plane
Left-hand column: X-component of magnetic field in x-y plane
Right-hand column: Z-component of magnetic field in x-y plane
Figure 2.6
Extrapolated magnetic field; differentiation operator approximated by symmetric finite differences.
Top row: Field in plane 13 pixels above measurement plane
Bottom row: Field in plane 4 pixels above measurement plane
Left-hand column: X-component of magnetic field in x-y plane
Right-hand column: Z-component of magnetic field in x-y plane
Figure 2.7
Extrapolated magnetic field; differentiation operator approximated by sampled periodic version of continuous operator.
Top row: Field in plane 13 pixels above measurement plane
Bottom row: Field in plane 4 pixels above measurement plane
Left-hand column: X-component of magnetic field in x-y plane
Right-hand column: Z-component of magnetic field in x-y plane
with the x-component at the left of each row and the z-component at the right. Note that the scales are not the same for the x- and z-components, or for the x-y planes, although for a given component and plane, the scales are the same for the analytic and extrapolated fields. For the case shown here, the y-component of the magnetic field is the same as the x-component rotated by 90 degrees. Figure 2.4 shows the components of the original magnetic field, calculated from Equation (2.41). The bottom row is the plane one unit above the measurement plane and the top row is eleven units above the measurement plane. Figure 2.7 shows the field extrapolated using the discrete method with asymmetric finite differences, Figure 2.6 shows the field extrapolated using the discrete method with symmetric finite differences, and Figure 2.7 shows the field extrapolated using the continuous method implemented on a computer. The figures of the extrapolated fields are arranged in the same order as the analytic field plots, with the x-component on the left of each row, and the z-component of the right of the row. The peak values for each component of the extrapolated fields are all very close to the peak values of the corresponding components of the original field, even for planes far away from the measurement plane. However, the values at the edges are very different.

To quantify the differences between the extrapolated and measured fields, we used two figures of merit suggested by Herman (1980): a normalized root-mean-square error, $L_2$, (Equation (2.53)), and a normalized difference, $L_1$, (Equation (2.54)). Let $A(i,j)$ be the value of the $(i,j)^{th}$ pixel of one component of the analytic field in a given plane, and $E(i,j)$ be the value of the $(i,j)^{th}$ pixel of the extrapolated field component.
The normalized difference is defined as

$$L_1 = \frac{\sum_{i,j=1}^{M} |A(i,j) - E(i,j)|}{\sum_{i,j=1}^{M} |A(i,j)|}.$$  \hfill (2.53)

The normalized root-mean-square error is defined as

$$L_2 = \left\{ \frac{\sum_{i,j=1}^{M} [A(i,j) - E(i,j)]^2}{\sum_{i,j=1}^{M} [A(i,j) - \bar{A}]^2} \right\}^{1/2}.$$  \hfill (2.54)

where $\bar{A}$ is the average value of the analytic field. For both figures of merit, registration of the two fields to be compared is critical; misregistration of just one pixel can result in inaccurate and misleading values of $L_1$ and $L_2$. A small value of $L_1$ or $L_2$ indicates that there are only small differences between the two fields. The normalized rms error will be large if there are large differences in the two fields in just a few places. The normalized difference emphasizes the importance of many small errors.

For each $x$-$y$ plane we calculated $L_1$ and $L_2$ for each component of the magnetic field. The normalized root-mean-square errors for the $x$-component range from 0.01 to 0.42 and are plotted as a function of distance from the measurement plane in Figure 2.8. The $L_2$ values for the $y$-component are identical to the $L_2$ values for the $x$-component for this
Figure 2.8
Normalized rms error of x-component of magnetic field.

Figure 2.9
Normalized rms error of z-component of magnetic field.

Figure 2.10
Normalized difference error of x-component of magnetic field.

Figure 2.11
Normalized difference error of z-component of magnetic field.
particular choice of magnetic field. The $L_2$ values for the $z$-component of the magnetic field range from $10^{-7}$ to 1.57, as plotted in Figure 2.9. The values of $L_2$ that are of the order of unity indicate that the extrapolated fields are not a good match to the original field, and may be due to the differences in the fields at the edges of the array. The three extrapolation methods have almost the same $L_2$ value when we have gone about eight pixels in the $z$-direction away from the measurement plane for the $x$-component and about six pixels for the $z$-component. This may be because there are fewer high frequencies in the magnetic field further from the measurement plane so the errors made by approximating the derivative with finite differences are small. The interesting differences occur close to the measurement plane. For the $x$-component, the asymmetric finite difference method has a much larger $L_2$ than either of the other two methods and exhibits a minimum at seven pixels in the $z$-direction from the measurement plane. The symmetric finite difference method has a slightly larger $L_2$ than the continuous method. For the $z$-component the symmetric finite difference method has a larger $L_2$ close to the measurement plane than the other two methods.

The normalized differences for the $x$- and $z$-components are plotted as a function of distance from the measurement plane in Figures 2.10 and 2.11, respectively. Again, the three extrapolation methods have almost the same $L_1$ far away from the measurement plane and differ close to the measurement plane.

**Real Data**

For a final example, we apply the extrapolation methods to real data. The previous example showed that all three extrapolation methods worked reasonably well for simulated data. We now show that they also work for real data. In this instance the real data consist
Figure 2.12
Locations of measurement planes and current sources for measured data used for extrapolation example.
Figure 2.13
Measured data and extrapolated z-component of the magnetic field.
(a) Measured data used for extrapolation to upper plane
(b) Measured data in upper plane (to be compared to extrapolated fields at right)
(c) Extrapolated z-component, asymmetric finite differences
(d) Extrapolated z-component, symmetric finite differences
(e) Extrapolated z-component, sampled periodic version of continuous operator.
of measurements of the z-component of the magnetic field in two planes separated by a vertical distance of 1 cm. The data from the lower plane were used to extrapolate the field to the upper plane, and the extrapolated field was compared to the measured data from the upper plane. The magnetic field was generated by two square (1.2 cm x 1.2 cm) planar horizontal current loops separated by 1.8 cm in the z direction and 3.3 cm in the x direction. For the lower measurement plane, the distance from the lower loop to the bottom of the cryostat was 6.9 cm (Figure 2.12). Details on the measurement equipment and techniques are in Chapter 4.

The measured data and the extrapolated z-components are plotted as functions of (x,y) in Figure 2.13. The pattern and magnitude of the extrapolated field match the measured field, and the normalized rms error is small (0.051 for the continuous derivative method, 0.061 for the symmetric finite difference method and 0.053 for the asymmetric finite difference method). These errors are comparable to the magnitude of the L₂ values one pixel above the measurement plane for the magnetic dipole example. Thus, the extrapolation methods also work for real data.

Solution to the Laplace Equation with Dirichlet Boundary Conditions

The motivation for presenting the solution to the Laplace equation with Dirichlet boundary conditions is that we shall use the result in our Fourier-transform technique for estimating the bioelectric currents. Specifically, we shall assume that the x- and y-components of the magnetic field in a specified region satisfy the Laplace equation. We can either measure the x- and y-components of the magnetic field on the boundary, or calculate them from the measured z-component. The previous section dealt with the problem of calculating all three components of the magnetic field in a region of space from the z-
component measured on a single plane. In this section, we shall present the solution using continuous Fourier transforms only, and not discuss the modifications needed for the discrete formulation. We shall not solve the problem on a computer; we simply want to show that the assumptions that $\nabla^2 B_x = 0$ and $\nabla^2 B_y = 0$ are, in some sense, reasonable.

In this section, we shall let the function $\theta$ represent either $B_x$ or $B_y$. We shall solve the Laplace equation using the general notation, since both $B_x$ and $B_y$ must satisfy the Laplace equation, and the boundary conditions for the two problems are in the same form. We wish to find the function $\theta(x,y,z)$ in some region of space, where $\theta(x,y,z)$ obeys the Laplace equation,

$$\nabla^2 \theta(x,y,z) = 0.$$ \hspace{1cm} (2.55)

The region of space we are interested in is a slab of finite extent in $z$ and infinite extent in $x$ and $y$. Let the top of the slab be at $z = z_t$ and the bottom of the slab be at $z = z_b$ (Figure 2.14). For now, we shall consider the case where the entire slab is above the origin, $0 \leq z_b \leq z \leq z_t$. At the end of this section, we shall consider the cases where the entire slab is below the origin and where the slab is partially above and partially below (but not necessarily centered about) the origin. We know the function $\theta$ on the top and bottom of the slab and assume that $\theta$ goes to zero as $x$ and $y$ go to infinity. Thus, the boundary conditions are:

$$\theta(x,y,z) = \theta_t(x,y,z_t) \hspace{1cm} \text{for } -\infty \leq x,y \leq \infty , \ z = z_t$$

$$\theta(x,y,z) = \theta_b(x,y,z_b) \hspace{1cm} \text{for } -\infty \leq x,y \leq \infty , \ z = z_b$$ \hspace{1cm} (2.56)

$$\theta(x,y,z) = 0 \hspace{1cm} \text{for } x,y = \pm \infty , \ z_b \leq z \leq z_t.$$
Figure 2.14
Slab geometry for solution to the Laplace equation with Dirichlet boundary conditions.
We solve this problem using the same Fourier-space method that we used to solve the extrapolation problem in the previous section. We Fourier-transform the partial differential equation,

$$\frac{\partial^2}{\partial x^2} \theta(x, y, z) + \frac{\partial^2}{\partial y^2} \theta(x, y, z) + \frac{\partial^2}{\partial z^2} \theta(x, y, z) = 0,$$

(2.57)

in the x and y directions only, to get,

$$-(2\pi)^2(\xi^2 + \eta^2)\mathcal{B}(\xi, \eta, z) + \frac{\partial^2}{\partial z^2} \mathcal{B}(\xi, \eta, z) = 0.$$

(2.58)

The general solution to Equation (2.58) is

$$\mathcal{B}(\xi, \eta, z) = A_1(\xi, \eta)\cosh(2\pi |z|\sqrt{\xi^2 + \eta^2}) + A_2(\xi, \eta)\sinh(2\pi |z|\sqrt{\xi^2 + \eta^2}).$$

(2.59)

The functions $A_1(\xi, \eta)$ and $A_2(\xi, \eta)$ are determined by the boundary conditions. After some algebra, we find

$$A_1(\xi, \eta) = \frac{\mathcal{B}(\xi, \eta, z_1)\sinh(2\pi |z_1|\sqrt{\xi^2 + \eta^2}) - \mathcal{B}(\xi, \eta, z_0)\sinh(2\pi |z_1|\sqrt{\xi^2 + \eta^2})}{\sinh[2\pi(|z_1| - |z_0|)\sqrt{\xi^2 + \eta^2}]}$$

(2.60)

and

$$A_2(\xi, \eta) = \frac{\mathcal{B}(\xi, \eta, z_1)\cosh(2\pi |z_1|\sqrt{\xi^2 + \eta^2}) + \mathcal{B}(\xi, \eta, z_0)\cosh(2\pi |z_1|\sqrt{\xi^2 + \eta^2})}{\sinh[2\pi(|z_1| - |z_0|)\sqrt{\xi^2 + \eta^2}]}.$$

(2.61)

Substituting the expressions for $A_1(\xi, \eta)$ and $A_2(\xi, \eta)$ (Equations (2.60) and (2.61)) into the
general solution (Equation (2.59)), yields

$$\overline{\vartheta}(\xi, \eta, z) = \frac{\overline{\vartheta}_t(\xi, \eta, z_t) \sinh[2\pi(z_t - |z|)\sqrt{\xi^2 + \eta^2}] + \overline{\vartheta}_b(\xi, \eta, z_b) \sinh[2\pi(|z| - |z_b|)\sqrt{\xi^2 + \eta^2}]}{\sinh[2\pi(|z_t| - |z_b|)\sqrt{\xi^2 + \eta^2}]} \quad (2.62)$$

The final step is to inverse-Fourier transform Equation (2.62) in the x and y directions. Recall that the function $\vartheta$ represents either the x- or y-component of the magnetic field. We have not specified what the functions $\overline{\vartheta}_t$ and $\overline{\vartheta}_b$ are, only that they are the x- (or y-) component of the magnetic field on the top and bottom of the slab. The functions can be measured or calculated from the measured z-component of the magnetic field, but we do not have, a priori, an analytic expression for the magnetic field on the surface of the slab. In order to retain the general form and avoid specifying analytic expressions for $\overline{\vartheta}_t$ and $\overline{\vartheta}_b$, we let

$$\bar{S}_1(\xi, \eta, z) = \frac{\sinh[2\pi(z_b - |z|)\sqrt{\xi^2 + \eta^2}] / \sinh[2\pi(|z_t| - |z_b|)\sqrt{\xi^2 + \eta^2}]}{\sinh[2\pi(|z_t| - |z_b|)\sqrt{\xi^2 + \eta^2}]} \quad (2.63)$$

and

$$\bar{S}_2(\xi, \eta, z) = \frac{\sinh[2\pi(|z| - |z_b|)\sqrt{\xi^2 + \eta^2}] / \sinh[2\pi(|z_t| - |z_b|)\sqrt{\xi^2 + \eta^2}]}{\sinh[2\pi(|z_t| - |z_b|)\sqrt{\xi^2 + \eta^2}]} \quad (2.64)$$

Using this notation, Equation (2.62) becomes

$$\overline{\vartheta}(\xi, \eta, z) = \overline{\vartheta}_t(\xi, \eta, z_t)\bar{S}_1(\xi, \eta, z) + \overline{\vartheta}_b(\xi, \eta, z_b)\bar{S}_2(\xi, \eta, z) \quad (2.65)$$

We saw in Chapter 1 that the Fourier transform of a product is the convolution of the Fourier transforms of the two functions being multiplied. Thus, the
The two-dimensional inverse-Fourier transform of Equation (2.65) is

\[ \theta(x, y, z) = \theta_t(x, y, z) \ast \ast \mathcal{F}_2^{-1} \{ S'_1(e, \rho, z) \} = \theta_x(x, y, z) \ast \ast \mathcal{F}_2^{-1} \{ S_2(x, y, z) \} , \]  

(2.66)

where the convolution is performed in the x and y directions. We now need to find expressions for \( S_1 \) and \( S_2 \). Consider the two-dimensional inverse-Fourier transform of \( S_1(\xi, \eta, z) \):

\[ \mathcal{F}_2^{-1} \{ S_1(\xi, \eta, z) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sinh[2\pi(z_b - |z|)\sqrt{\xi^2 + \eta^2}]}{\sinh[2\pi(z_t - |z|)\sqrt{\xi^2 + \eta^2}]} \sin(2\pi(x\xi + y\eta)) \, d\xi d\eta . \]  

(2.67)

We simplify the notation by letting

\[ \alpha = 2\pi ( |z_b| - |z| ) , \]

\[ \beta = 2\pi ( |z_t| - |z_b| ) , \]  

(2.68)

and

\[ \rho^2 = \xi^2 + \eta^2 . \]

The function \( S_1 \) in Equation (2.67) is radially symmetric and the Fourier transform can be written as a Hankel transform,

\[ S_1(x, y, z) = 2\pi \int_0^{\infty} \frac{\sinh(\alpha \rho)}{\sinh(\beta \rho)} J_0(2\pi \rho) \, \rho \, d\rho . \]  

(2.69)
We now rewrite the hyperbolic sines in terms of exponentials, where

\[ \sinh(x) = \frac{1}{2} (e^x - e^{-x}) \quad (2.70) \]

so

\[ S_1(x, y, z) = 2\pi \int_0^\infty \frac{e^{\alpha \rho} - e^{-\alpha \rho}}{e^{\beta \rho} (1 - e^{-2\beta \rho})} J_0(2\pi \rho) \, \rho \, d\rho. \quad (2.71) \]

We have written the denominator in that form because we recognize that

\[ \frac{1}{1 - e^{-2\beta \rho}} = \sum_{n=0}^{\infty} e^{-2\beta \rho n}. \quad (2.72) \]

Substituting Equation (2.72) into Equation (2.71) and simplifying the exponents yields

\[ S_1(x, y, z) = 2\pi \sum_{n=0}^{\infty} \int_0^\infty \left[ e^{(\alpha - \beta - 2\beta n) \rho} - e^{-(\alpha + \beta + 2\beta n) \rho} \right] J_0(2\pi \rho) \, \rho \, d\rho. \quad (2.73) \]

where we have interchanged the order of the summation and integration. The integral is evaluated using a table of integrals (Gradshteyn and Ryzhik, 1980, eq. 6.623.2) to yield

\[ S_1(x, y, z) = 2\pi \left[ \sum_{n=0}^{\infty} \frac{2\pi |a|}{[(2\pi)^2 + a^2]^{3/2}} - \frac{2\pi |b|}{[(2\pi)^2 + b^2]^{3/2}} \right], \quad (2.74) \]

where

\[ a = 2n ( |z_b| - |z_t| ) + |z| - |z_t| \quad (2.75) \]

and

\[ b = 2n ( |z_b| - |z_t| ) - |z| - |z_t| + 2|z_b|. \quad (2.76) \]
Similarly for \( S_2 \),

\[
S_2(x, y, z) = 2\pi \sum_{n=0}^{\infty} \frac{2\pi|c|}{[(2\pi r)^2 + c^2]^{3/2}} - \frac{2\pi|d|}{[(2\pi r)^2 + d^2]^{3/2}},
\]

(2.77)

where

\[
c = 2n \left( |z_b| - |z_t| \right) - |z| + |z_b|,
\]

(2.78)

and

\[
d = 2n \left( |z_b| - |z_t| \right) + |z| - 2|z_t| + |z_b|.
\]

(2.79)

In summary, the \( x- \) (or \( y- \)) component of the magnetic field in the slab is the sum of the \( x- \) (or \( y- \)) component of the magnetic field on the top surface convolved with \( S_1 \) and the \( x- \) (or \( y- \)) component of the magnetic field on the bottom surface convolved with \( S_2 \) (we repeat Equation (2.66) here),

\[
\theta(x, y, z) = \theta_t(x, y, z_t) \ast \ast S_1(x, y, z) + \theta_b(x, y, z_b) \ast \ast S_2(x, y, z),
\]

(2.66)

where \( S_1 \) and \( S_2 \) are defined above, and \( 0 \leq z_b \leq z \leq z_t \).

If the entire slab is below the origin, \( z_b \leq z \leq z_t \leq 0 \), we get the same result as for the entire slab above the origin. To see this note that in the expressions for \( S_1 \) and \( S_2 \), the variables \( z, z_t, \) or \( z_b \) always appear with absolute value signs.

If the slab is partially above and partially below the origin the result has the same form, but the dependence on \( z \) is slightly different. The solution to the Laplace equation
with Dirichlet boundary conditions is the same,

\[ \theta(x, y, z) = \theta(x, y, z_t) \ast \ast S_t(x, y, z) + \theta_b(x, y, z_b) \ast \ast S_b(x, y, z) , \]  

(2.80)

but we convolve the component of the magnetic field with different functions:

\[ S_t(x, y, z) = 2\pi \sum_{n=0}^{\infty} \frac{2\pi|e|}{(2\pi)^3 + s^2} - \frac{2\pi|f|}{(2\pi)^3 + f^2} , \]  

(2.81)

where

\[ e = 2n \left( |z_b| + |z| \right) - |z| + |z_t| , \]  

(2.82)

\[ f = 2n \left( |z_b| + |z| \right) + |z| + |z_t| + 2|z_b| , \]  

(2.83)

and

\[ S_b(x, y, z) = 2\pi \sum_{n=0}^{\infty} \frac{2\pi|g|}{(2\pi)^3 + g^2} - \frac{2\pi|h|}{(2\pi)^3 + h^2} , \]  

(2.84)

where

\[ g = 2n \left( |z_b| + |z| \right) + |z| + |z_b| , \]  

(2.85)

\[ h = 2n \left( |z_b| + |z| \right) - |z| + 2|z_t| + |z_b| . \]  

(2.86)

We know from the boundary conditions that at the top of the slab, \( S_t(x, y, z_t) = \delta(x)\delta(y) \) and \( S_b(x, y, z_t) = 0 \) and at the bottom of the slab, \( S_t(x, y, z_b) = 0 \) and \( S_b(x, y, z_b) = \delta(x)\delta(y) \). In other words, the magnetic field on the bottom of the slab has no effect on the magnetic field on the top of the slab. There is a gradual transition from the field on the top of the slab to the field on the bottom. Both the maximum and minimum values of \( \theta \) must occur on the boundary of the slab (Andrews, 1986). For example, if the measured magnetic field is the same on the top and bottom of the slab, then \( \theta(x, y, z) \) will be constant in \( z \).
Chapter 3

RECONSTRUCTION TECHNIQUES

The measurement of biomagnetic fields external to the body provides a noninvasive method for indirectly observing the internal distribution of electric currents. A knowledge of bioelectric currents will aid our understanding of the function of the organ. The current is related to the magnetic field through the Maxwell equations or, equivalently, the Biot-Savart law (the forward problem); we then should be able to estimate the current sources from magnetic field measurements (the inverse problem). Unfortunately, the inverse problem is ill-posed; an infinite number of current sources could cause the same measured magnetic field. If we could measure the magnetic field everywhere (including the interior of the body) then we could uniquely determine the current source, but biomagnetic imaging is noninvasive. The true source can be described in terms of equivalent sources which produce the same measured magnetic field. The equivalent sources are unique for a given set of assumptions, but the true sources, in general, cannot be uniquely calculated from the equivalent sources. A suitable equivalent source can help visualize certain features more easily than external magnetic field maps, and may have clinical value.

In this chapter, we discuss how magnetic field measurements can be used to estimate equivalent current sources. In the first section, we review the methods to locate an equivalent current dipole (ECD). Then, we review some of the alternative methods which do not limit the possible equivalent sources to a predetermined number of current dipoles. In the last section, I present my contribution to the field of biomagnetic imaging: a Fourier transform technique for estimating bioelectric currents.
Review of Estimation Methods

There are several methods for estimating the location and strength of a current source. All methods include \textit{a priori} information or assumptions about the current source in order to overcome the ill-posed nature of the inverse problem. The accuracy of an estimation method depends in part on how closely the assumptions match the real situation. An estimation method will be able to find only the types of currents that it is designed to find. Some methods are more general than others, but all estimation methods in biomagnetism merely choose one current density distribution out of the infinite number of possible distributions.

\textit{The Equivalent-Current-Dipole (ECD) Methods}

In some cases, it is reasonable to assume that the electrical activity is highly localized. For example, medical doctors are often interested in identifying areas of pathological electrical activity in the brains of epileptic patients. For a specific type of epilepsy, the pathological areas are known to be highly localized (Ricci, \textit{et al.}, 1984). In these cases, the total divergenceless current in the conducting medium is divided into two parts: an impressed, or source, current and a return, or volume, current (Geselowitz, 1970 and Grynszpan and Geselowitz, 1973). The impressed current is assumed to be highly localized while the return currents are diffuse. The impressed current is usually modelled as a single equivalent current dipole (ECD), and the primary goal is to gather information about the impressed current, while knowledge about the return currents is undesired. In addition to assuming the current is a single current dipole, a specific conductor geometry must also be assumed. In the case of focal epilepsy, the head is usually modelled as a homogeneously conducting sphere. It has been shown (Sarvas, 1987) that the component of the magnetic
field normal to the surface of a spherical conductor is due only to the impressed current. Thus, if researchers are interested in only the impressed current, they will measure the component of the magnetic field that is normal to the conductor's surface.

_The ECD and the Peak-Location Method_

When biomagnetic fields were first mapped it was noticed that, in many cases, the field pattern consisted of a single positive peak and a single negative peak of approximately equal magnitudes. The pattern of the magnetic field was suggestive of the magnetic field due to a current dipole and led researchers to try a simple estimation method: measure the extrema and assume a highly localized source that could be modelled as a single current dipole. The single current dipole is located half-way between the peaks and along a line perpendicular to the line joining the peaks (the direction is determined by the right-hand rule), the depth of the dipole is proportional to the distance between the peaks, and the strength of the dipole is proportional to the height of the peaks (Williamson and Kaufman, 1980).

Source localization based on the size and location of the magnetic field extrema is straightforward, easy to implement, and requires that the magnetic field be measured in only a few places. The magnetic field needs to be measured in a minimum of three places: at each of the extrema and at the null point between the extrema. The disadvantages of using the extrema of the magnetic field to estimate the strength and location of the source are that the method cannot accommodate more complicated current distributions (such as multiple current dipoles) or more complicated conductor geometries.
If the signal-to-noise ratio is good, then a fitting procedure can be used. In this technique (see for example, Sarvas, 1987, Cuffin, 1985, or Harrop, et al., 1987) a specific current source (strength, location, and orientation) and a specific conductor (shape and conductivity distribution) are assumed and the forward problem is solved. The calculated magnetic field is compared to the measured magnetic field. If necessary, the strength, location, and orientation of the current source are modified and the magnetic field is recalculated. The iteration process continues until a satisfactory agreement (usually defined in the least-squares sense) between the measured field and the calculated field is reached.

The least-squares iterative method is more complicated to implement and requires more measurement points than the peak-location method. At least five measurement points are needed because there are five parameters to be fit for the equivalent current dipole (strength, (x, y, z) location and one direction). Only one direction is needed because the dipole is assumed to be parallel to the surface of the conductor. It has been shown (Baule and McFee, 1963), that a homogeneous conducting body with rotational symmetry (i.e. sphere, cylinder, etc.), and a current dipole aligned with the axis of symmetry, produces no external magnetic field. Thus, the equivalent current dipole must be parallel to the surface of the conductor. The assumption that the equivalent current dipole is parallel to the surface of a realistically-shaped head has been checked by Melcher and Cohen (1988). They implanted (approximately) radial and tangential dipoles into rabbit heads, and compared the measured fields to the predicted fields. They measured $B_z$, $\partial B_z/\partial x$, and $\partial B_z/\partial y$. Their general, although not strong, conclusion was that a radially oriented current dipole produces a weaker (by a factor of 4 to 6) magnetic field than a tangential dipole.
In neuromagnetism, the equivalent current source is usually a single current dipole, and the head is modelled as a homogeneous sphere. Solving the forward problem for a single current dipole in a homogeneously conducting sphere is straightforward because an analytic expression for the magnetic field exists (Cuffin and Cohen, 1977). In cardiomagnetism, the equivalent current source is chosen to be either a current dipole or a magnetic dipole and the torso is modelled as a homogeneous half-space. Analytic expressions for both a current dipole in a homogeneous half-space and a magnetic dipole exist.

Obviously, the human head is not a homogeneous spherical conductor and the human torso is not a homogeneous half-space, so the least-squares iterative method has been improved by including more realistic, and more complicated, conductor models. An analytic expression has not been developed for the magnetic field due to a current dipole in a realistically shaped head, so the forward problem must be solved iteratively (Hämäläinen and Sarvas, 1989). Refinements to the least-squares iterative method include modelling the head as: a set of concentric spheres (each spherical shell having a different conductivity) (Hämäläinen and Sarvas, 1989 and Tuomisto, et al., 1983), eccentric spheres (each sphere having a different conductivity) (Meijs and Peters, 1987), a realistically-shaped head (where the shape of the head is extracted from computed tomography images, magnetic resonance images, actual measurements of a skull, or computer models) (Meijs, et al., 1987 and Cuffin, 1990). For cardiomagnetism, Purcell et al. (1988), developed a numerical method to calculate the magnetic field due to current dipoles in a torso. The component of the magnetic field normal to the torso surface was calculated from the electric potential. They divided the torso into three parts of different conductivities: the outer boundary, the lungs and the cardiac blood mass. They considered four torso models: a homogeneous half-space, a homogeneous torso with realistic boundary, a torso with lungs, and a torso with lungs and cardiac mass. They concluded that the inclusion of the realistic torso boundary has the greatest effect on
the magnetic field.

The validity of the ECD methods has been verified by measuring the magnetic field due to current dipoles at known locations in known conductors and comparing the estimated current dipole location to the known location. Weinberg et al. (1986) measured the magnetic field due to a current dipole in a skull filled with a conducting jelly. For the reconstructions, they assumed a single current dipole in a homogenous sphere. The radius of the sphere was chosen by measuring the skull and fitting (in a least-squares sense) a sphere to the skull. They considered two different methods for estimating the current dipole: using only the extrema of the magnetic field and using a least-squares fit to the measured data. They concluded that both methods could adequately locate the current source. In another study, (Hämäläinen, et al., 1985) the location of the equivalent current dipole obtained from the extrema differed by up to 10 mm from the location obtained using a least-squares fit. They concluded that the extrema method is inferior to the least-squares fit and can lead to large errors in the location of the equivalent current dipole. Ricci et al. (1984), verified the validity of equivalent current dipole localizations by comparing the estimated positions to the positions determined from x-ray images that showed the pathologies and to surgical findings. They assumed the equivalent source was a single current dipole in a homogeneous sphere and used a least-squares fit to the measured data. The radius of the sphere was determined by a best (in the least-squares sense) fit to measurements made of the patient's head, x-ray images, and computed-tomography images. They concluded that the equivalent-current-dipole method works very well if the source is highly localized and in an area of the brain that can be adequately modelled as a sphere. Otherwise, the equivalent-current-dipole method can incorrectly locate the source. Later studies (Hari and Ilmoniemi, 1987) have indicated that, when modelling the head as a sphere, the radius of the sphere should be the local radius of curvature of the inner surface of the skull.
Not everyone believes that using a least-squares fit is superior to using just the extrema. Mondt (1989) argues that the magnetic field due to the volume currents interferes with the magnetic field due to the impressed current, so the least-squares fit will not determine the location, orientation, and strength of impressed current but rather of some combination of the impressed current and the return current.

*Other Iterative Methods*

The electrical activity in the human body is complex, and a single current dipole is an oversimplification. For example, in order for the ECD to be a valid assumption in neuromagnetism many neurons must fire coherently and only one small portion of the brain can be active. Researchers realize that, although a single current dipole is often a good first approximation to the actual source distribution, more sophisticated models are needed. In an attempt to expand the applicability of the ECD least-squares fitting procedure, researchers have added complications such as including the effects of the return currents (Roth and Wikswo, 1986) or assuming that the current source is a multipole (Katila, 1983, Tuomisto, *et al.*, 1983, and Brauer and Stroink, 1985).

The modifications to the least-squares iterative method are straightforward if the impressed current is assumed to be a multipole instead of a single current dipole. The magnetic field is expanded by means of equivalent current multipole coefficients. The lowest-order nonzero term in the expansion is due to a current dipole and the next term is due to a quadrupole. After the single current dipole is estimated, its contribution to the magnetic field is subtracted from the measured magnetic field. The remainder is assumed to be due to higher-order moments (quadrupoles, octopoles, etc.). As many of the higher-order terms as desired can be estimated using a least-squares iterative method analogous to the
method used to find the current dipole. Although the impressed current is modelled as a more complicated source, it is still assumed to be highly localized.

If there is more than one area of electrical activity, then the current source can be modelled as multiple current dipoles (Tuomisto, et al., 1983). The existence of multiple sources is not always obvious from the magnetic field pattern (Nunez, 1986). Figure 3.1 shows a contour plot of the z-component of the magnetic field due to a single current dipole in a homogeneous half-space, where the x-y location of the dipole is indicated by an arrow. Figure 3.2 shows the z-component of the magnetic field due to two current dipoles separated in the x-direction in a homogeneous half-space. Each field pattern consists of a single negative peak and a single positive peak; if the ECD methods were applied to the pattern due to two current dipoles, they would incorrectly estimate the strength and location of the true source.

The procedures developed for a single current dipole can be modified to include multiple current dipoles only if a fixed number of dipoles is used to solve the forward problem. If the method is expected to find the number of current dipoles, as well as their strength, location and orientation, then more sophisticated procedures must be used. More general methods do not assume a specific current source (strength, location, and orientation) although a specific conductor (shape and conductivity distribution) is assumed and the currents are confined to the conductor volume.

One solution to the biomagnetic inverse problem was presented at the Fifth World Conference in Biomagnetism (Ilmoniemi, et al., 1985 and Hämäläinen, et al., 1985). This was one of the first solutions that assumed the equivalent source consisted of multiple equivalent current dipoles instead of a single current dipole. The authors determined the current density distribution using a minimum-norm estimate of the current density and lead field theory. The lead field is the link between the current density and the magnetic flux,
Figure 3.1
Contour plot of z-component of magnetic field due to single current dipole in a homogeneous half-space.

Figure 3.2
Contour plot of z-component of magnetic field due to two current dipoles in a homogeneous half-space.
where the flux is the integral of the normal component of the magnetic field over the coil area. They assumed the source current was confined to a plane a known distance from the measurement plane. The estimate of the current distribution was obtained from a weighted sum of the lead fields. The iterative technique minimizes the norm of the current density (defined as $\|J\| = \sqrt{\int J(r) \cdot J(r) \; d^3r}$). The minimum-norm estimate was applied to a measured biomagnetic field. Estimates of the current distribution indicated concentrations of currents in the same locations as obtained by the ECD method with least-squares fit.

The iterative techniques that assume the equivalent source consists of multiple dipoles, generally approximate the Biot–Savart Law with a discrete sum and the current density with a finite number of current dipoles

$$B(r_m) = \frac{\mu_0}{4\pi} \sum_{n=1}^{N} \frac{J(r_n) \times (r_m - r_n)}{|r_m - r_n|^3}$$

where $m$ and $n$ are the indices on the discrete sample points in space. The system of linear equations can be solved using any one of the well-known algorithms used in the field of image reconstruction. These algorithms use a cost function to select a single solution from the set of possible solutions. The choice of cost function determines the class of solutions.

One group (Haneishi, et al., 1989 and Sekihara, et al., 1990) has developed an iterative method that assumes the current source consists of multiple current dipoles. The cost function constrains the total current magnitude and the sum of each dipole's contribution to the measured magnetic field. The algorithm uses simulated annealing to deal with the nonlinearity of the cost function. They were able to locate the correct number of current dipoles in computer simulations.
Another approach is to constrain the solution to be consistent with the time evolution of the magnetic field. Several groups have devised ways of including information about the time-evolution of the magnetic field and allow the strengths and/or orientations of the equivalent current dipoles to change with time (Barth, et al., 1989 and Mosher, et al., 1990).

A third method (Jeffs, et al., 1987) constrains the orientation of the dipoles such that the dipoles must be parallel to the surface of the conductor and limits the reconstruction to those orientations. They use singular value decomposition to solve the system of linear equations and have gotten good results in computer simulations.

Noniterative Methods

Iterative methods tend to be slow because of the large number of calculations that must be performed. In order to decrease the computation time, researchers have developed procedures that estimate the current density by solving the inverse problem directly. When the inverse problem is solved, a specific conductor (shape and conductivity distribution) is assumed and the current is constrained to the conductor volume. A specific current distribution is not assumed; instead the reconstruction space is divided into voxels where each voxel can be thought of as a current element which has a strength and orientation associated with it. The estimated current consists of a finite, but unknown, number of current dipoles in some arbitrary three-dimensional distribution. One group (Smith, et al., 1986, Kullmann, et al., 1989 and Smith, et al., 1990) used linear estimation theory to estimate the current distribution, where the linear estimator is the generalized Wiener-Helstrom filter. The reconstruction procedure minimized the mean-square-error between the object class and the reconstruction class. The size of the reconstruction region was limited to 4x4x4 pixels due to memory constraints of the computer (a VAX8600). Experiments with
current dipoles immersed in saline solution inside a glass head phantom, and computer simulations demonstrated the viability of the linear estimation method. The disadvantage of this method is that assumptions must be made about the first and second-order statistics of the current density, and these assumptions may be hard to verify.

Two other methods assume the current is confined to a plane that is a known distance away from the measurement plane. These assumptions about the current are restrictive enough that the inverse problem has a unique current density solution. If the distance between the planar current and the measurement plane is changed, then a new current distribution is found, and the new current distribution is unique for that distance. The distance between the planar current and the measurement plane cannot be determined from just the knowledge of the magnetic field and the fact that the current is planar.

One method (Wikswo and Roth, 1988, Roth, et al., 1989, and Tan, et al., 1990) used a high-pass spatial filtering technique. The z-component of the magnetic field is measured on a plane, and the sampled field is Fourier-transformed in the x and y directions only. A high-pass spatial filter of the form $\frac{\eta}{\sqrt{\xi^2 + \eta^2}} \exp(-2\pi z\sqrt{\xi^2 + \eta^2})$ is applied to the measured magnetic field in Fourier space to yield the x-component of the current density. The z-component of the current density is assumed to be zero, so the y-component of the current density is calculated from the x-component and the fact that the total current must have zero divergence. This method works best if the distance between the planar current and the measurement plane is small and the noise level is low, because the exponential term in the filter can lead to instabilities in the reconstruction. In order to avoid amplifying the high-frequency noise, the measured magnetic field is low-pass filtered before computing the current density. They performed computer simulations in which they started with a known current density, calculated the magnetic field, added noise to the magnetic field, and then solved the inverse problem. The forward problem and the inverse problem were both solved.
using a grid of 128x128 pixels.

The second method is based on the magnetic lead field (Alvarez, 1989 and 1990). In Fourier space, the equation relating the lead field and the current is solved for the unknown current.

Our Fourier-Transform Reconstruction Methods

The Fourier-transform technique is a noniterative method that is an alternative to the other estimation methods discussed above. It has several advantages previous the other methods, including the ability to easily handle reconstruction volumes that have up to 32x32x16 pixels and widely distributed current distributions. We do not assume that the current is highly localized, or even restricted to a single plane, only that it is confined to some known or assumed volume.

The general reconstruction method consists of forming a set of linear equations from the Fourier-transformed Maxwell equations. This set of linear equations is sampled according to the Whittaker-Shannon sampling theorem and then solved by matrix methods. The sampling theorem allows the introduction of additional information concerning the current distributions and measurement geometry, which augments the information from the Maxwell's equations. The reconstruction technique, of course, uses discrete, not continuous, Fourier theory because the work is done on a computer. However, in this section we shall initially use continuous Fourier mathematics to describe the reconstruction method in order to simplify the discussion. We shall introduce the modifications needed for the discrete formulation at the end of the section.

As we saw in Chapter 1, the curl and divergence equations Fourier-transform in a straightforward manner since the differentiation operator Fourier transforms to a simple
multiplication. The three-dimensional Fourier transform of the gradient of a function \( f(x, y, z) \) is

\[
\mathcal{F}_3 \left\{ \nabla f(x, y, z) \right\} = i2\pi \rho \hat{f}(\xi, \eta, \zeta)
\]  

(3.2)

where \( \rho \) is the position vector in Fourier space, \( \rho = \hat{i}\xi + \hat{j}\eta + \hat{k}\zeta \).

The total volume is divided into three regions: the measurement region, the forbidden region, and the reconstruction region (Figure 3.3). The forbidden region includes the space occupied by the patient, where no measurements can be taken. The reconstruction region is where the current distribution is constrained (and expected) to be and is contained within the forbidden region. When we define the separate regions in this manner, the magnetic field can be decomposed into a field inside and a field outside of the forbidden volume:

\[
B(r) = U(r) + V(r)
\]  

(3.3)

where \( U \) is the known (measured and then extrapolated) field and \( V \) is the unknown field. The Fourier transform of \( B \) is the sum of the transforms of \( U \) and \( V \) because of the linearity property of Fourier transforms. Consider the curl and divergence of the magnetic field in Fourier space, with the magnetic field decomposed into the two parts. The curl equation is

\[
\mathcal{F}_3 \left\{ \nabla \times [U(r) + V(r)] \right\} = \mathcal{F}_3 \left\{ \mu_0 J(r) \right\}
\]

or,

\[
i2\pi \rho \times [\hat{U}(\rho) + \hat{V}(\rho)] = \mu_0 \hat{J}(\rho)
\]  

(3.4)
Extrapolation Region

Forbidden Region, bounded by $S(r)$

Reconstruction Region, bounded by $T(r)$

Measurement Plane

Figure 3.3
Slab geometry for Fourier-transform reconstruction algorithms.
and the divergence equation is

\[ \nabla \cdot [\mathbf{U}(r) + \mathbf{V}(r)] = 0 \]

or,

\[ i2\pi \rho \cdot \left[ \mathbf{U}(\rho) + \mathbf{V}(\rho) \right] = 0. \quad (3.5) \]

Collecting the terms involving the known field, \( \mathbf{U} \), on the left and the terms involving the unknown field, \( \mathbf{V} \), and current density, \( \mathbf{J} \), on the right yields:

\[ i2\pi \rho \times \mathbf{U}(\rho) = \mu_0 \mathbf{J}(\rho) - i2\pi \rho \times \mathbf{V}(\rho) \quad (3.6) \]

and

\[ i2\pi \rho \cdot \mathbf{U}(\rho) = -i2\pi \rho \cdot \mathbf{V}(\rho). \quad (3.7) \]

The fields, \( \mathbf{U} \) and \( \mathbf{V} \), are related by the spatial boundaries and the curl and divergence equations of the magnetic field. The spatial boundedness of the unknown field and of the current density are due to the forbidden volume and the reconstruction region, respectively. We define the spatial bounding function of the forbidden region to be \( S(r) \) and the spatial bounding function of the reconstruction region to be \( T(r) \). Thus,

\[ S = \begin{cases} 
1, & \text{inside the forbidden region} \\
0, & \text{outside the forbidden region} 
\end{cases} \quad (3.8) \]

and

\[ T = \begin{cases} 
1, & \text{inside the reconstruction region} \\
0, & \text{outside the reconstruction region} 
\end{cases} \quad (3.9) \]
Now, the Whittaker-Shannon sampling theorem is applied to $V$ and $J$ in Fourier space:

$$\mathcal{V}(\rho) = \sum_{n=-\infty}^{\infty} \mathcal{V}(n)\delta(\rho - n) \quad (3.10)$$

and

$$\mathcal{J}(\rho) = \sum_{m=-\infty}^{\infty} \mathcal{J}(m)\delta(\rho - m) \quad (3.11)$$

where $n = (n_x, n_y, n_z)$ and $m = (m_x, m_y, m_z)$, and each of the components of $n$ and $m$ ranges from $-\infty$ to $+\infty$. We have written the triple sums in this manner to simplify the notation. Note that, in general, $\mathcal{T}$ and $\delta$ will have different sampling intervals because the sampling interval is inversely proportional to the size of the bounded region.

Including the Wittaker-Shannon sampling theorem in the Fourier-transformed Maxwell's equations (Equations (3.6) and (3.7)) yields four equations (the three curl equations and the divergence equation) and six unknowns (the three components of $J$ and the three components of $V$):

$$i2\pi\rho \times \mathcal{U}(\rho) = \mu_0 \sum_{m=-\infty}^{\infty} \mathcal{J}(m)\delta(\rho - m) - i2\pi\rho \times \sum_{n=-\infty}^{\infty} \mathcal{V}(n)\delta(\rho - n) \quad (3.12)$$

and

$$i2\pi\rho \cdot \mathcal{U}(\rho) = -i2\pi\rho \cdot \sum_{n=-\infty}^{\infty} \mathcal{V}(n)\delta(\rho - n) \quad (3.13)$$

These four equations are not linearly independent; we can write one of the curl equations in terms of the other equations. We discard one of the curl equations because it doesn't add any
new information. The equations we shall keep are

\begin{equation}
\begin{split}
i2\pi [\xi \bar{U}_x (\rho) - \xi \bar{U}_y (\rho)] &= \mu_0 \sum_{m=-\infty}^{\infty} \bar{Y}_x (m) \bar{T}(\rho - m) - i2\pi \sum_{n=-\infty}^{\infty} [\xi \bar{V}_x (n) - \xi \bar{V}_y (n)] \bar{S}(\rho - n) \quad (3.14)
\end{split}
\end{equation}

\begin{equation}
\begin{split}
i2\pi [\xi \bar{U}_x (\rho) - \xi \bar{U}_y (\rho)] &= \mu_0 \sum_{m=-\infty}^{\infty} \bar{Y}_y (m) \bar{T}(\rho - m) - i2\pi \sum_{n=-\infty}^{\infty} [\xi \bar{V}_x (n) - \xi \bar{V}_y (n)] \bar{S}(\rho - n) \quad (3.15)
\end{split}
\end{equation}

\begin{equation}
\begin{split}
\xi \bar{U}_x (\rho) + \eta \bar{U}_y (\rho) + \xi \bar{U}_y (\rho) = - \sum_{n=-\infty}^{\infty} [\xi \bar{V}_x (n) + \eta \bar{V}_y (n) + \xi \bar{V}_y (n)] \bar{S}(\rho - n) . \quad (3.16)
\end{split}
\end{equation}

Thus, we are left with three equations and five unknowns, an underdetermined system showing once more that the inverse problem is ill-posed. In order to estimate the current density, we need more information about the current density, the unknown magnetic field or both. The information can be \textit{a priori} knowledge or assumptions.

The first assumption we shall make in order to solve the system of equations is that we have the special case of a slab-like geometry, where the three regions are all slabs (Figure 3.3). Thus, the forbidden region and the reconstruction region are rectangular (finite in z, infinite in x and y). In direct space,

\begin{equation}
\begin{split}
S(x, y, z) = \sigma(z) \quad (3.17)
\end{split}
\end{equation}

and

\begin{equation}
\begin{split}
T(x, y, z) = \tau(z) . \quad (3.18)
\end{split}
\end{equation}

Some other choices for the geometry might be more physically reasonable, but, the choice of
slab geometries greatly reduces the computation time.

With the assumption of slab geometries, the reconstruction problem reduces to a one-dimensional reconstruction, and the interpolation functions, \( \hat{S} \) and \( \hat{T} \), are now delta functions in the \( \xi \) and \( \eta \) directions. In Fourier space, the interpolation functions are:

\[
\hat{S}(\rho - m) = \delta(\xi - m_x)\delta(\eta - m_y)\tilde{\sigma}(\zeta - n_z) \\
\hat{T}(\rho - m) = \delta(\xi - m_x)\delta(\eta - m_y)\tilde{\gamma}(\zeta - m_z).
\]  

(3.19)  

(3.20)

The delta functions mean that we have nonzero values for \( \hat{S} \) and \( \hat{T} \) only at certain points (where \( \xi = m_x \) and \( \eta = m_y \)). Inserting the equations for the interpolation functions in Fourier space (Equations (3.19) and (3.20)) into the curl and divergence equations in Fourier space (Equations (3.14)-(3.16)) and performing the sums over \( m_x, m_y \) and \( n_x, n_y \) yields the three linearly independent reconstruction equations

\[
i2\pi \left[ \hat{\nu}_x(\rho) - \xi \hat{\nu}_y(\rho) \right] = \mu_0 \sum_{m_z = -\infty}^{\infty} \hat{J}_x(\xi, \eta, m_z)\tilde{\gamma}(\zeta - m_z) \\
- i2\pi \sum_{n_z = -\infty}^{\infty} \left[ \hat{\nu}_x(\xi, \eta, n_z) - \xi \hat{\nu}_y(\xi, \eta, n_z) \right]\tilde{\sigma}(\zeta - n_z),
\]  

(3.21)

\[
i2\pi \left[ \hat{\nu}_x(\rho) - \xi \hat{\nu}_x(\rho) \right] = \mu_0 \sum_{m_z = -\infty}^{\infty} \hat{J}_y(\xi, \eta, m_z)\tilde{\gamma}(\zeta - m_z) \\
- i2\pi \sum_{n_z = -\infty}^{\infty} \left[ \hat{\nu}_x(\xi, \eta, n_z) - \xi \hat{\nu}_x(\xi, \eta, n_z) \right]\tilde{\sigma}(\zeta - n_z),
\]  

(3.22)

and
\[ \xi \ddot{U}_x(\rho) + \eta \ddot{U}_y(\rho) + \zeta \ddot{U}_z(\rho) = - \sum_{n_x = -\infty}^{\infty} \left[ \xi \ddot{V}_x(\xi, \eta, n_x) + \eta \ddot{V}_y(\xi, \eta, n_x) + \zeta \ddot{V}_z(\xi, \eta, n_x) \right] \sigma(\zeta - n_x). \tag{3.23} \]

In order to implement the Fourier reconstruction technique on a computer, we must use a discrete formulation of Equations (3.21)-(3.23). The first modification is to evaluate the reconstruction equations at discrete points. Thus, we replace the continuous variable

\[ \rho = \hat{i} \xi + \hat{j} \eta + \hat{k} \zeta \tag{3.24} \]

with the discrete variable

\[ \rho_d = \hat{i} \lambda + \hat{j} \mu + \hat{k} \nu, \tag{3.25} \]

where the subscript \( d \) indicates that the vector components are discrete. In the discrete formulation, we must use a finite difference approximation to the partial derivative (see Chapter 1). Instead of adding even more notational complexity, we shall use the vector \( \rho_d = \hat{i} \lambda + \hat{j} \mu + \hat{k} \nu \) to denote both the position vector and the gradient operator in Fourier space. The discrete variable \( \mu \) is not related to the permeability of free space \( \mu_0 \). The total volume, assumed to be a cube (32×32×32 pixels), is divided into three regions; the measurement region, the forbidden region, and the reconstruction region. The periodicity of the discrete Fourier transforms (see also Chapter 1) means that we still have a slab geometry. The finite volume means that we can sum over \( m_x \) and \( n_x \) from 1 to 32, instead of over all
space. When these minor changes are made, the three reconstruction equations become

\[ i2\pi \left[ \mu \bar{U}_x(\rho_d) \right] - \nu \bar{U}_y(\rho_d) = \mu_0 \sum_{m_s = 1}^{32} \mathcal{J}_x(\lambda, \mu, m_s) \tilde{r}(\nu - m_s) \]

\[ -i2\pi \sum_{n_s = 1}^{32} \left[ \mu \bar{V}_x(\lambda, \mu, n_s) \right] - \nu \bar{V}_y(\lambda, \mu, n_s) \tilde{v}(\nu - n_s), \quad (3.26) \]

\[ i2\pi \left[ \mu \bar{U}_x(\rho_d) \right] - \lambda \bar{U}_s(\rho_d) = \mu_0 \sum_{m_s = 1}^{32} \mathcal{J}_y(\lambda, \mu, m_s) \tilde{r}(\nu - m_s) \]

\[ -i2\pi \sum_{n_s = 1}^{32} \left[ \nu \bar{V}_x(\lambda, \mu, n_s) \right] - \lambda \bar{V}_s(\lambda, \mu, n_s) \tilde{v}(\nu - n_s), \quad (3.27) \]

and

\[ \lambda \bar{U}_x(\rho_d) + \mu \bar{U}_y(\rho_d) + \nu \bar{U}_s(\rho_d) \]

\[ = - \sum_{n_s = 1}^{32} \left[ \lambda \bar{V}_x(\lambda, \mu, n_s) + \mu \bar{V}_y(\lambda, \mu, n_s) + \nu \bar{V}_s(\lambda, \mu, n_s) \right] \tilde{v}(\nu - n_s). \quad (3.28) \]

We simplify the notation by letting the left-hand side of the equations be represented by

\[ C'_x, \ C'_y \quad \text{and} \quad D'' \] where

\[ C'_x = i2\pi \left[ \mu \bar{U}_x(\rho_d) \right] - \nu \bar{U}_y(\rho_d), \quad (3.29) \]

\[ C'_y = i2\pi \left[ \mu \bar{U}_x(\rho_d) \right] - \lambda \bar{U}_s(\rho_d), \quad (3.30) \]

and

\[ D'' = i2\pi \left[ \lambda \bar{U}_x(\rho_d) \right] + \mu \bar{U}_y(\rho_d) + \nu \bar{U}_s(\rho_d). \quad (3.31) \]
The unknown quantities on the right-hand side of the equations are simplified by letting

\[
\begin{align*}
\tilde{y}_m^x &= \tilde{y}_x(\lambda, \mu, m_x), \\
\tilde{y}_m^y &= \tilde{y}_y(\lambda, \mu, m_x), \\
\tilde{V}_m^x &= \tilde{V}_x(\lambda, \mu, n_x), \\
\tilde{V}_m^y &= \tilde{V}_y(\lambda, \mu, n_x),
\end{align*}
\]  
(3.32)

and

\[
\tilde{V}_m^\theta = \tilde{V}_\theta(\lambda, \mu, n_x),
\]

where the dependence on \( \lambda \) and \( \mu \) has been suppressed and the subscript \( z \) on the sample points \( m_x \) and \( n_x \) has been dropped. Using this simplified notation, the reconstruction equations become

\[
\begin{align*}
C_x' &= \mu_0 \sum_{m=1}^{32} \tilde{y}_m^x \tilde{r}(\nu-m) - i2\pi \sum_{n=1}^{32} \left[ \mu \tilde{V}_n^x - \nu \tilde{V}_n^y \right] \tilde{\sigma}(\nu-n), \\
\end{align*}
\]  
(3.33)

\[
\begin{align*}
C_y' &= \mu_0 \sum_{m=1}^{32} \tilde{y}_m^y \tilde{r}(\nu-m) - i2\pi \sum_{n=1}^{32} \left[ \nu \tilde{V}_n^x - \lambda \tilde{V}_n^y \right] \tilde{\sigma}(\nu-n),
\end{align*}
\]  
(3.34)

and

\[
D'_\nu = -i2\pi \sum_{n=1}^{32} \left[ \lambda \tilde{V}_n^x + \mu \tilde{V}_n^y + \nu \tilde{V}_n^\theta \right] \tilde{\sigma}(\nu-n). 
\]  
(3.35)

We shall use Equations (3.33)-(3.35) to derive three Fourier-transform reconstruction methods. The main difference between the methods is the assumptions made about the magnetic field in the forbidden region. We shall refer to the three methods as the Dallas
method, the $B$-zero method and the $\nabla^2 B$-zero method, where Dallas refers to the researcher who originally proposed the Fourier-transform method, and $B$-zero and $\nabla^2 B$-zero refer to assumptions made about the magnetic field in the forbidden zone. The $B$-zero method and the $\nabla^2 B$-zero method are extensions of the Dallas method and are my contribution to the field of biomagnetic imaging.

Because of the slab geometry, the problem is reduced to a one-dimensional reconstruction. We choose the $\nu$ axis as the dimension in which the reconstruction will take place. Therefore, the reconstruction equations are solved for each $(\lambda, \mu)$ position.

All of the reconstructions presented in this section start with the magnetic field, either real data or analytic expressions for the magnetic field due to square, planar current loops or magnetic dipoles. We do not solve a discrete version of the forward problem. Using all three components of the magnetic field in the measurement region, the reconstruction is performed using the variations of the Fourier-transform reconstruction algorithm. None of the methods adequately localize the current density in the $z$ (depth) direction because of the assumptions we make about the behavior of the magnetic field in the forbidden region. The algorithms will be presented in chronological order.

*The Dallas Method*

The original reconstruction method was devised by Dallas (1985). This method has been presented in detail elsewhere (Dallas, 1985, Dallas, *et al.*, 1987, and Kullmann and Dallas, 1987) and shall be discussed only briefly here. The derivation of the reconstruction equations in this dissertation shall be slightly different than in the original paper to make it easier to compare with the variations presented in this dissertation. We present the original Fourier-transform technique because it is the basis for the others, but we shall not present
any results from it.

We start from Equations (3.33)-(3.35) and eliminate the variables $\mathcal{V}_x$ and $\mathcal{V}_y$ by forming a linear combination of Equations (3.33)-(3.35),

$$
\mu C_x + \lambda C_y + \nu D^\nu = \mu_0 \sum_{m=1}^{32} \left[ \frac{\tilde{J}_m^x}{2} - \frac{\tilde{J}_m^y}{2} \right] \mathcal{V}_m - i2\pi(\lambda^2 + \mu^2 + \nu^2) \sum_{n=1}^{32} \mathcal{V}_n \bar{\sigma}(\nu-n). \tag{3.36}
$$

We now assume that the current is confined to one plane at $z = z_0$; the reconstruction region in the $z$ direction is a delta function. In Fourier space, the interpolation function becomes

$$
\mathcal{T}(\nu) = e^{-i2\pi \sigma_0 \nu}. \tag{3.37}
$$

We let the location of the reconstruction plane range from the bottom of the forbidden region to the top. Thus, the current distribution in the entire forbidden region is reconstructed one plane at a time. The planar reconstruction region implies that the $z$-component of the current is zero. The planar current means that the $x$- and $y$-components of the magnetic field will be zero in the reconstruction plane.

Including the expression for the interpolation function in Fourier space (Equation (3.37)) in Equation (3.36) yields

$$
\mu C_x + \lambda C_y + \nu D^\nu = \mu_0 \sum_{m=1}^{32} \left[ \frac{\tilde{J}_m^x}{2} - \frac{\tilde{J}_m^y}{2} \right] e^{-i2\pi \sigma_0 (\nu-m)} - i2\pi(\lambda^2 + \mu^2 + \nu^2) \sum_{n=1}^{32} \mathcal{V}_n \bar{\sigma}(\nu-n). \tag{3.38}
$$

We still do not have enough equations to solve for the unknown quantities $\tilde{J}_x$, $\tilde{J}_y$, and $\mathcal{V}_x$. 
We define $A^\nu$ to be the linear combination of the reduced curl and divergence equations,

$$A^\nu = \mu C^\nu_x - \lambda C^\nu_y + \nu D^\nu .$$  \hspace{1cm} (3.39)

We also define $A^{\nu + \Delta \nu}$,

$$A^{\nu + \Delta \nu} = \mu C^{\nu + \Delta \nu}_x - \lambda C^{\nu + \Delta \nu}_y + (\nu + \Delta \nu) D^{\nu + \Delta \nu} .$$  \hspace{1cm} (3.40)

We get enough equations by sampling at the points $\nu$ and $\nu + \Delta \nu$ and forming the linear combination of $A^\nu$ and $A^{\nu + \Delta \nu}$,

$$A^{\nu + \Delta \nu} - A^\nu e^{-i2\pi x_0 \Delta \nu} = i2\pi \sum_{n=1}^{32} \left\{ (\lambda^2 + \mu^2 + \nu^2) \tilde{g}(\nu - n) e^{-i2\pi x_0 \Delta \nu} \right. \\
- \left. \left[ \lambda^2 + \mu^2 + (\nu + \Delta \nu)^2 \right] \tilde{g}(\nu + \Delta \nu - n) \right\} \tilde{V}^n .$$  \hspace{1cm} (3.41)

The term involving the current density has dropped out and we have only one sampling rate, which is determined by the size of the forbidden region.

We can make Equation (3.41) look more like a matrix inversion problem by writing the unknown quantity (the magnetic field, $\tilde{V}$) as a matrix $F$, and letting the matrices $G$ and $\mathcal{M}$ contain the known quantities (the measured magnetic field, $\tilde{U}$ and the spatial bounding function, $\tilde{g}$).
We define the matrices $G$, $F$, and $\mathcal{M}$ as

$$G^\nu = A^{\nu+\Delta\nu} - A^{\nu} e^{-i2\pi\nu_0\Delta\nu},$$

$$F^n = \bar{\nabla}^n,$$  \hspace{1cm} (3.42)

and

$$\mathcal{M}^{\nu n} = i2\pi \left[ (\lambda^2+\mu^2+\nu^2)\bar{\sigma}(\nu-n)e^{-i2\pi\nu_0\Delta\nu} - (\lambda^2+\mu^2+(\nu+\Delta\nu)^2)\bar{\sigma}(\nu+\Delta\nu-n) \right].$$

Thus, $G$ consists of two of the curl equations, each evaluated at $\nu$ and $\nu+\Delta\nu$, and the divergence equation. As mentioned in the previous section, the third curl equation is discarded because it is linearly dependent on the other two and thus adds no new information.

Using the matrix notation, the goal is to solve the set of linear equations, $G = \mathcal{M}F$, for the unknown magnetic field, $F$, using Gaussian elimination. The matrix equation is solved for each $(\lambda,\mu)$ position. The calculated field, $\bar{\nabla}^n$, is inverse-Fourier transformed and connected to the measured field in real space. This final field is then Fourier transformed, and the current density in Fourier space is calculated from the Fourier-transformed Maxwell's equation for the curl of the magnetic field. Finally, the current density is inverse-Fourier transformed and displayed as gray-levels.

The disadvantage of this reconstruction method is that the equivalent current density is restricted to a plane. In some cases, estimating the equivalent current density on a circuit board for example, assuming the current is planar is a good approximation. In other cases, such as estimating the current on the convoluted surface of the brain, it clearly is not a valid assumption.
The B-zero Method

Our next step was to generalize the algorithm so that it would accommodate reconstruction regions that were not planar. We refer to this variation of the Dallas Fourier-transform method as the B-zero method because of the assumptions we make about the magnetic field in the forbidden region. We explicitly assume that the x- and y-components of the unknown magnetic field are zero in the entire forbidden region, not just in the reconstruction region. The assumption that $V_x$ and $V_y$ are zero implies that $J_z$ is zero. We still assume a slab geometry, but we no longer assume that the reconstruction region is a plane. Instead, the reconstruction region is assumed to be a volume, and we can reconstruct either the entire forbidden region or a portion of the forbidden region. We arrange the equations in a slightly different fashion so that we're solving for the x-component of the current density rather than for the z-component of the magnetic field in the forbidden region. We use the fact that the z-component of the current density is zero and that the total current has zero divergence to calculate the y-component of the current density from the x-component.

We start with the reconstruction equations (Equations (3.33)-(3.35)) and explicitly set $\tilde{V}_x^m$ and $\tilde{V}_y^n$ to zero,

$$C_x^m = \mu_0 \sum_{m=1}^{32} \hat{J}_x^m \tilde{r}(\nu-m) - i2\pi \sum_{n=1}^{32} \mu \bar{V}_y^n \tilde{\sigma}(\nu-n),$$

(3.43)

$$C_y^m = \mu_0 \sum_{m=1}^{32} \hat{J}_y^m \tilde{r}(\nu-m) + i2\pi \sum_{n=1}^{32} \lambda \bar{V}_z^n \tilde{\sigma}(\nu-n),$$

(3.44)

and

$$D^\nu = -i2\pi \sum_{n_x=1}^{32} \nu \bar{V}_z^n (\lambda, \mu, n_x) \tilde{\sigma}(\nu-n_x).$$

(3.45)
We eliminate the variable $\tilde{V}_a$ by forming a linear combination of Equations (3.43)-(3.45),

$$
\lambda_\nu C_{\nu} + 2\mu_\nu C_{\nu} + \lambda_\nu D_{\nu} = \mu_0 \sum_{m=1}^{32} \nu \left[ \lambda_\nu \tilde{J}_m^x + 2\mu_\nu \tilde{J}_m^y \right] \tilde{T}(\nu-m). \tag{3.46}
$$

When we eliminate $\tilde{V}_a^2$, we are left with only one sampling rate, determined by the size of the reconstruction region, while in the Dallas Fourier-transform method, the sampling rate was determined by the size of the forbidden region.

We still have two unknown variables, $\tilde{J}_x$ and $\tilde{J}_y$, that we want to determine, and only one equation. Fortunately, we know that the total current must have zero divergence. In Fourier space, the equation for the divergence of the current density is

$$
\xi \tilde{J}_x(\rho) + \eta \tilde{J}_y(\rho) + \zeta \tilde{J}_z(\rho) = 0. \tag{3.47}
$$

We have assumed that the $x$- and $y$-components of the magnetic field in the forbidden region are zero and therefore, the $z$-component of the current density is zero. Setting $\tilde{J}_z(\rho) = 0$ in Equation (3.47), we solve for $\tilde{J}_x$ in terms of $\tilde{J}_y$,

$$
\tilde{J}_x = \frac{-\lambda}{\mu} \tilde{J}_y, \tag{3.48}
$$

where we have used the discrete formulation and our simplified notation. Substituting Equation (3.48) into Equation (3.46) yields the reconstruction equation for the B-zero
We change Equation (3.49) to look more like the matrix equation $\mathbf{M}\mathbf{F} = \mathbf{G}$ by letting

\begin{align*}
\mathbf{G}^\nu &= \lambda\nu C_x^\nu + 2\mu\nu C_y^\nu + \lambda\mu D^\nu \\
\mathbf{F}^m &= \mu_0\lambda J_x^m \\
\mathbf{M}^\nu_m &= \tilde{r}(\nu-m)
\end{align*}

(3.50)

For each $(\lambda, \mu)$ location, we calculate $\mathbf{G}$ from the known field and $\mathbf{M}$ from the spatial bounding function for the reconstruction region. We solve for the unknown vector $\mathbf{F}$ using Gaussian elimination. We then calculate $\tilde{J}_x^m$ from $J_x^m$ using Equation (3.48). Finally, the current density is inverse Fourier transformed and displayed as gray-levels.

This formulation has several interesting properties. The matrix $\mathbf{M}$ is calculated directly from the Fourier transform of the reconstruction region. The use of the Whittaker-Shannon sampling theorem is more obvious than in the Dallas Fourier-transform method; the $x$-component of the unknown current density is convolved with the interpolation function (note the similarity between the right-hand sides of Equations (3.11) and (3.49)). The sampling rate is determined by the size of the reconstruction region instead of the size of the forbidden region, because we are solving for the current that is confined to the reconstruction region instead of the unknown magnetic field.

The advantage of the B-zero method is that it accommodates reconstruction regions that are not planar. The disadvantages of this reconstruction method are that the $x$- and $y$-
components of the magnetic field change abruptly, and unrealistically, at the edge of the forbidden region and that the current density is assumed to have a zero \( z \)-component.

*The \( \nabla^2 B \)-zero Method*

The first variation of the original Fourier-transform method (the \( B \)-zero method) was able to solve for three dimensional current distributions, but the assumption that \( V_x \) and \( V_y \) are zero in the entire forbidden region was still too restrictive. Instead of placing more restrictions on the current density, we changed the assumptions we made about the magnetic field in the forbidden region. We assumed \( \nabla^2 V_x = 0 \) and \( \nabla^2 V_y = 0 \) in the forbidden region, instead of assuming that \( V_x = 0 \) and \( V_y = 0 \). We refer to this variation of the Dallas Fourier-transform method as the \( \nabla^2 B \)-zero method. This assumption about the behavior of the magnetic field in the forbidden region means a smooth variation across the boundary between the measurement region and the forbidden region. See Chapter 2 for a discussion about what the condition \( \nabla^2 \theta = 0 \) means about the function \( \theta \). The motivation for the assumption that \( \nabla^2 V_x = 0 \) and \( \nabla^2 V_y = 0 \) came from considering the curl of the Maxwell equation relating the magnetic field to the current density (Equation (1.7c))

\[
\nabla \times \nabla \times \mathbf{B}(r) = \nabla \times \mathbf{J}(r) .
\]

(3.51)

Using the vector property concerning the curl of the curl of a vector (Jackson, 1975) we rewrite Equation (3.51),

\[
\nabla \left[ \nabla \cdot \mathbf{B}(r) \right] - \nabla^2 \mathbf{B}(r) = \nabla \times \mathbf{J}(r) .
\]

(3.52)
The divergence of the magnetic field is zero, so the first term on the left-hand side of Equation (3.52) is zero. Writing out the remaining term in terms of the components yields

\[ \hat{i} \nabla^2 B_x(r) + \hat{j} \nabla^2 B_y(r) + \hat{k} \nabla^2 B_z(r) = \nabla \times J(r) . \]  

(3.53)

In a region where there is no current, \( J = 0 \), Equation (3.53) becomes

\[ \hat{i} \nabla^2 B_x(r) + \hat{j} \nabla^2 B_y(r) + \hat{k} \nabla^2 B_z(r) = 0 , \]  

(3.54)

and each component must be equal to zero. In regions where there is current, we can have one or two, but not all three, of the components in Equation (3.54) equal to zero. This observation led us to try using the assumption \( \nabla^2 V_x = 0 \) and \( \nabla^2 V_y = 0 \).

For the \( \nabla^2 B \)-zero method, we use a slab geometry and assume the reconstruction region is a volume, not a single plane. We can reconstruct either the entire forbidden region or a portion of it. We do not assume that the \( z \)-component of the current density is zero.

For the derivation of the reconstruction equations for this variation, we need to include the third curl equation that we previously discarded,

\[ C_k^\nabla = \mu_0 \sum_{m=1}^{32} \tilde{J}_m^\nabla \tilde{r}(\nu-m) - i2\pi \sum_{n=1}^{32} \left[ \lambda \tilde{V}_n^y - \mu \tilde{V}_n^x \right] \tilde{\sigma}(\nu-n) , \]  

(3.55)

where we have used the simplified notation

\[ C_k^\nabla = i2\pi \left[ \lambda \tilde{U}_y(\rho_d) - \mu \tilde{U}_x(\rho_d) \right] , \]  

(3.56)

and

\[ \tilde{J}_m^\nabla = \tilde{J}_m(\lambda, \mu, m) . \]  

(3.57)
Although the third curl equation does not contain any new information, we use it to get the combination of variables that we desire. We form the linear combinations

\[
\mu C'_x - \nu C'_y - \lambda D' = \mu_0 \sum_{m=1}^{32} \left[ \mu \vec{J}_x^m - \nu \vec{J}_y^m \right] \vec{r}(\nu-m) - \frac{i2\pi}{\nu} \sum_{n=1}^{32} (\lambda^2 + \mu^2 + \nu^2) \vec{V}_x^n \quad (3.58)
\]

and

\[
\nu C'_x - \lambda C'_y - \mu D' = \mu_0 \sum_{m=1}^{32} \left[ \nu \vec{J}_x^m - \lambda \vec{J}_y^m \right] \vec{r}(\nu-m) - \frac{i2\pi}{\nu} \sum_{n=1}^{32} (\lambda^2 + \mu^2 + \nu^2) \vec{V}_y^n \quad (3.59)
\]

We now assume \( \nabla^2 V_x = 0 \) and \( \nabla^2 V_y = 0 \). In Fourier space, this means

\[
(\lambda^2 + \mu^2 + \nu^2) \vec{V}_x(\rho_0) = 0 \quad (3.60)
\]

and

\[
(\lambda^2 + \mu^2 + \nu^2) \vec{V}_y(\rho_0) = 0 \quad (3.61)
\]

We use the fact that the total current must have zero divergence (Equation (3.47)) to solve for \( \vec{J}_x \) in terms of \( \vec{J}_x \) and \( \vec{J}_y \),

\[
\vec{J}_x^m = \left[ \frac{\lambda \vec{J}_x^m + \mu \vec{J}_y^m}{\nu} \right]. \quad (3.62)
\]

We include the assumptions that \( \nabla^2 V_x = 0 \) and \( \nabla^2 V_y = 0 \) (Equations (3.60) and (3.61)), and
substitute the expression for $\tilde{y}_m$ (Equation (3.62)) into Equations (3.58) and (3.59) to get

$$\mu C_x - \nu C_y - \lambda \nu^\nu = -\mu_0 \sum_{m=1}^{32} \left[ \frac{\lambda I}{\nu} \tilde{y}_m + \frac{\mu^2 + \nu^2}{\nu} \tilde{y}_m \right] \tilde{r}(\nu-m) \quad (3.63)$$

and

$$\nu C_x - \lambda C_y - \mu \nu^\nu = \mu_0 \sum_{m=1}^{32} \left[ \frac{\lambda I}{\nu} \tilde{y}_m + \frac{\lambda I}{\nu} \tilde{y}_m \right] \tilde{r}(\nu-m) \quad (3.64)$$

This leaves two equations and two unknowns. Before we define the terms in the matrix equation, $G = M F$, we simplify the left-hand sides of Equations (3.63) and (3.64). Substituting the expressions for $C_x$, $C_y$, $C_z$, and $D^\nu$ back into these two equations, we find

$$\mu C_x - \nu C_y - \lambda \nu^\nu = -i2\pi(\lambda^2 + \mu^2 + \nu^2) \tilde{U}_x \quad (3.65)$$

and

$$\nu C_x - \lambda C_y - \mu \nu^\nu = -i2\pi(\lambda^2 + \mu^2 + \nu^2) \tilde{U}_y \quad (3.66)$$

We now define two matrix equations. First, we eliminate $\tilde{y}_m$ from Equations (3.63) and (3.64) to yield the first reconstruction equation for the $\nabla^2 B$-zero method

$$-i2\pi \left[ \lambda \mu \tilde{U}_x + (\mu^2 + \nu^2) \tilde{U}_y \right] = \mu_0 \nu \sum_{m=1}^{32} \tilde{y}_m \tilde{r}(\nu-m) \quad (3.67)$$
We then define the matrices $G_1$, $F_1$, and $M_1$ by

$$
G_1^\nu = -i2\pi \left[ \lambda \mu U_\nu + (\mu^3 + \nu^2) \bar{U}_\nu \right]
$$

$$
F_1^m = \mu_\nu J^m_{2 \nu}
$$

and

$$
M_1^{\nu m} = \tilde{r}(\nu - m).
$$

We eliminate $J^m_{2 \nu}$ from Equations (3.63) and (3.64) to yield the second reconstruction equation

$$
i2\pi \left[ (\lambda^3 + \nu^2) U^\nu_x + \lambda \mu U^\nu_y \right] = \mu_\nu \sum_{m=1}^{32} \tilde{J}^m_{2 \nu} \tilde{r}(\nu - m),
$$

and define the matrices $G_2$, $F_2$, and $M_2$ by

$$
G_2^\nu = i2\pi \left[ (\lambda^3 + \nu^2) U^\nu_x + \lambda \mu U^\nu_y \right]
$$

$$
F_2^m = \mu_\nu J^m_{2 \nu}
$$

and

$$
M_2^{\nu m} = \tilde{r}(\nu - m).
$$

We use Gaussian elimination to solve the first matrix equation, $G_1 = M_1 F_1$, for the $x$-component of the current density and the second matrix equation, $G_2 = M_2 F_2$, for the $y$-component of the current density. Once we have $J^x$ and $J^y$, we use Equation (3.62) to find $J^x$. Note that we use the same definition of the matrix $M$ for both the B-zero method.
(Equation (3.55)) and the $\nabla^2 B$-zero method (Equations (3.68) and (3.70)). The definitions of the $F$ matrices are also very similar. The main difference between the $B$-zero method and the $\nabla^2 B$-zero method is the assumption about the behavior of the $x$- and $y$-components of the magnetic field in the reconstruction region. The advantage of the $\nabla^2 B$-zero method over both the Dallas Fourier-transform method and the $B$-zero method is that we are not restricted to currents that have a zero $z$-component.

**Conclusion**

We have presented a Fourier-transform method (and two variations of it) for processing magnetic field measurements in order to image the current distributions causing the magnetic fields. Results of the techniques are presented in Chapter 5. It should be noted that these results are not the only possible reconstructions because many different current distributions can give rise to the same magnetic field distribution. The problem is ill-posed, and we must make assumptions about the current density and or the magnetic field in the forbidden region. Once we have made these assumptions, the current is a unique solution of the equations.
Chapter 4

METHODS

In this chapter we describe the experimental set-up and equipment used for data collection. We also discuss the spatial-sampling requirements for biomagnetic measurements. All measurements were made between 19 June and 21 July 1989 at Philips Forschungslaboratorium Hamburg (PFH) in Hamburg, Germany.

Equipment and Measurement Procedures

The measurements were made in a specially built hut set in a field about 100 m from the nearest buildings. The hut was designed with great care so that all electronic equipment, anything that might create a magnetic field, could be isolated from the measurement location. The hut was built entirely out of non-magnetic materials: no metal nails, no metal hinges, no metal door or window handles. The magnetically 'quiet' location and the lack of magnetic materials in the hut meant that magnetic measurements could be made without an expensive magnetically shielded room. The only shielding in the hut was a Faraday cage. The Faraday cage enclosed a wooden bench and a wooden gantry holding the fiberglass cryostat. The hut consisted of a small entry-way with some storage cabinets, and two large rooms, one with all the electronic measurement equipment and the other with the Faraday cage and the SQUID system.

The magnetic fields were measured using a single-channel, thin film dc-SQUID manufactured by Biomagnetic Technologies Inc., San Diego. The magnetic fields were detected with a superconducting niobium (or Nb-alloy) wire-wound by second-order
The gradiometer with coil diameters of 1.98 cm and a baseline of 5.04 cm. The gradiometer coils (2 loops at 0.0 cm, 4 loops wound counterclockwise at a 5.04 cm distance, and 2 loops 10.08 cm above the first two windings) were mounted axially. For more information about SQUID systems, see Chapter 1.

The detecting equipment, consisting of the gradiometer coils and the SQUID, was enclosed in a fiberglass cryostat filled with liquid helium and isolated by vacuum chambers and superinsulation (Figure 4.1). The distance between the outer (room temperature) bottom of the dewar and the first 4.2 K cold detection loop was about 1.1 cm. The electronic equipment was at room temperature outside of the cryostat.

In order to make measurements at many locations with the single-channel SQUID, we used an x-y table to move the current source, instead of moving the heavy cryostat containing the SQUID. The x-y table was controlled by the same computer that collected the data from the SQUID. The computer programs controlling the x-y table and collecting the data were written by J. Kanzenbach. The voltage recorded by the computer was the average of 10 readings of the SQUID output at one location. After one set of measurements was made and recorded by the computer, the table was moved to a new position. At each position, the computer waited 10 seconds so that the magnetic field could stabilize, made 10 measurements in the next 10 seconds (one per second), recorded the average of the 10 measurements, then moved the x-y table to the next position.

The current flowing through the wire sources was alternating at frequency 10 - 12 Hz and was controlled by a lock-in amplifier. The resulting signal from the SQUID system was detected in synchronization with the current through the wire source. This frequency was chosen because alpha waves in the human brain are in the range 8 - 13 Hz. The strength of the current ranged from 2.8 μA to 500 μA (recall that a single neuron has a current of about 1 μA).
Figure 4.1
Detecting equipment for measuring biomagnetic fields.
The background noise (the reading when no current flowed through the wire loop) was -0.05 Volts ±0.005 V. The x, y, z distances and sizes of the loops are accurate to ±0.5mm. The voltage readings (output from the SQUID) are accurate to ±0.005 Volts. Several of the measurements were duplicated on different days to check the repeatability. The repeatability was found to be excellent; the values varied only by ±0.005 Volts, the accuracy of the measurement system.

We made measurements of the magnetic field generated by known sources in order to test the reconstruction algorithms. We did not measure unknown biological sources. We used seven different current sources (Figure 4.2): a current dipole immersed in a saline solution in a glass head phantom, a single square loop, two square loops, two different size round loops, and two different size wire loops bent into an "L" shape.

The current dipole (Figure 4.2a) was made of two platinum wires, twisted together except for the last 0.8 cm, where they were bent in opposite directions. The current flowed up one wire, out the end, through the saline solution and down the other wire. Thus the dipole consisted of a source and a sink in order to model the depolarization of an axon.

The wire loops had twisted-lead wires, and were made of copper wire. They were all planar loops, but were bent into different shapes (Figures 4.2b–f). Most of the measurements were made with horizontal loops so that the z-component of the current was approximately zero. We also made some measurements with a vertical loop and with a loop at approximately 40° from the horizontal. The shape, size, current strength, and measurement distance of the loops are listed in Table 4.1.

Twenty-two sets of measurements were made with the seven different current sources. Two different measurement grids were used, both of which were horizontal-planar. Each grid covered a different size area and used different sampling intervals. The spatial sampling requirements for biomagnetic measurements will be discussed later in this chapter.
Figure 4.2
Drawings of current sources used to generate magnetic fields measured for this dissertation.
(a) Current dipole
(b) Square loop
(c) Small round loop
(d) Large round loop
(e) Small "L"-shaped loop
(f) Large "L"-shaped loop
<table>
<thead>
<tr>
<th>Shape of loop</th>
<th>Size [cm]</th>
<th>Quantity</th>
<th>Current [μA]</th>
<th>Distance from Source Center to Cryostat Bottom [cm]</th>
<th>Measurement Grid</th>
<th>Comments</th>
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<td>2.8</td>
<td>7.8</td>
<td>A4 &amp; A3</td>
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</tr>
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<td>2</td>
<td>2.8</td>
<td>7.9 lower</td>
<td>A4 &amp; A3</td>
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(a)

<table>
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<tr>
<th>Location of Dipole in Head</th>
<th>Quantity</th>
<th>Current [μA]</th>
<th>Distance from Source Center to Cryostat Bottom [cm]</th>
<th>Measurement Grid</th>
<th>Measurement Plane w.r.t. Head</th>
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</thead>
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<td>A4</td>
<td>face</td>
</tr>
</tbody>
</table>

(b)

Table 4.1
Specifications of current sources used to generate the magnetic fields measured for this dissertation.
(a) Wire loops
(b) Current dipole in saline solution in glass head phantom
Eighteen sets were made using one of the grids, designated by the name grid-A4. Grid-A4 covered the largest area (18 cm × 18 cm), with 81 measurement points (sampling interval of 2 cm in both directions). The peak of the magnetic field for all of the current sources was approximately in the center of the grid.

The other four sets used the other grid, grid-A3. Grid-A3 covered a rectangular area near the peak of the magnetic field. It was 9 cm long with 8 measurement points in the x direction, and 6 cm wide with 5 measurement points in the y direction (1 cm sampling interval in both directions) for a total of 40 measurement points. As the pick-up coil diameter was 1.98 cm, the samples covered adjacent areas for grid-A4 and covered overlapping areas for grid-A3. The measurement distance was the vertical distance from the center of the loop or dipole to the bottom of the cryostat, and ranged from 2.3 cm to 10.5 cm.

Ten of the eighteen sets of data measured on the grid-A4 are displayed in the form of contour maps in Figure 4.3. The sets of data in Figures 4.3c and 4.3d are also plotted as surfaces in Figure 2.13. Reconstructions from nine of the ten sets of data are shown in Chapter 5 (the exception is the set of data in Figure 4.3d). Two of the four sets of data measured on the grid-A3 are shown in Figure 4.9. The remaining ten sets of measurements are not displayed because they do not add any new information.

Spatial Sampling Requirements for Biomagnetic Measurements

Even with large arrays (100 or more elements) of SQUIDs we do not measure the entire magnetic field external to the body we measure the magnetic field at a finite number of points on some surface and at a finite number of points in time. For our reconstruction algorithms (see Chapter 3), we need enough sample points to have a good representation of
Figure 4.3
Contour plots of measured magnetic fields used in this dissertation.
(a) Due to small "L"-shaped loop, 6.6cm from measurement plane
(b) Due to large "L"-shaped loop, 6.7cm from measurement plane
(c) Due to 2 square loops, upper loop 6.1cm from measurement plane
(d) Due to 2 square loops, upper loop 5.1cm from measurement plane
Figure 4.3 continued
Contour plots of measured magnetic fields used in this dissertation.
(e) Due to small round loop, 3.1cm from measurement plane
(f) Due to large round horizontal loop, 6.6cm from measurement plane
(g) Due to large round tilted loop, 4.1cm from measurement plane
(h) Due to large round vertical loop, 5.1cm from measurement plane
Figure 4.3 continued
Contour plots of measured magnetic fields used in this dissertation.
(l) Due to current dipole near side of glass head, measurement plane parallel to side of head
(j) Due to current dipole in middle of glass head, measurement plane on top of head
the spatial distribution of the z-component of the magnetic field in a plane. In Chapter 2, we discussed how to calculate all three components of the magnetic field from knowledge of the z-component in a single plane. The general nonideal sampling problem was discussed in Chapter 1. In this section we discuss the spatial sampling requirements for measuring the z-component of biomagnetic fields. Our goal is to determine the Nyquist rates for various sources, measured with single circular pick-up coils of various diameters.

In this section, we assume that the field is measured at a single instant in time. We consider the effects of the size of the pick-up coil and of the gradiometer as well as the effects of a finite number of sample points. The Nyquist sampling interval depends on a large number of factors including the size of the source, the distance from the source to the pick-up coil, the size of the pick-up coil, the order of the gradiometer, and the baseline of the gradiometer.

In many experiments, the sampling aperture is assumed to be small enough that it may be approximated by a delta function, and the aperture size is therefore ignored. In biomagnetic imaging, however, the sampling aperture is usually a circular coil a centimeter or two in diameter, which is not a negligible size compared to the spatial frequencies present in the magnetic field. In fact, gradiometers are most often used instead of just a single pick-up coil. The added complication of the gradiometer will be discussed after we cover the simpler case of a single circular pick-up coil.

Each sample value is the magnetic field averaged over the circular pick-up coil. As the pick-up coil gets larger, the Fourier transform of the circular coil gets narrower so the Fourier transform of the averaged field has fewer high frequencies. As the coil diameter decreases we get better representation of a given field, but the influence of the noise increases. Determining the size of the coil involves balancing the amount of smoothing with the acceptable noise levels, and a bit of black magic.
The influence of the pick-up coil on the measured magnetic field due to specific sources has been studied by several researchers. Williamson and Kaufman (1980) considered a single current dipole in a homogeneously conducting half-space, and used lead-field theory to calculate the magnetic field averaged by a single circular pick-up coil. They presented a plot of the averaged z-component of the magnetic field versus distance along the x-axis for five different a/d ratios ($a = $ coil radius, $d = $ source to coil distance). The plot illustrates how the field is smoothed by the coils and the strength is reduced. They also plotted correction factors for obtaining the correct strength and depth of a single current dipole in a half-space from the strength and depth calculated from the measurements of the averaged field.

This work was extended by Jazbinsek et al. (1989) to include quadrupolar sources in a homogeneously conducting half-space. They considered four different a/d ratios and published correction factors for the depth of the quadrupolar source. The combined influence of the finite size pick-up coil and the gradiometer was studied by Cuffin and Cohen (1983). They considered a single current dipole in a homogeneous spherical conductor (9cm radius), and varied the depth of the current dipole. For a magnetometer and a planar gradiometer, they considered four coil radii and dipole depths from 1 cm to 6 cm. Instead of deriving correction factors, they plotted the peak strength versus dipole depth, the peak separation as measured by the magnetometer versus dipole depth, and the null separation as measured by the planar gradiometer versus dipole depth for the different sized coils. For a first-order axial gradiometer, they assumed equal sized coils and a 6 cm baseline and concluded that the effect of the back coil will be small compared to the effect of the finite size of the front coil for sources near the surface of the brain. For deeper sources, the effect of the back coil is large compared to that of the front coil, but correction factors for the effect can be calculated.
The combined effects of the finite sampling aperture and the gradiometer were also studied by Jeffs et al. (1987). They considered a second-order gradiometer with 2 cm pick-up coils, and the magnetic field due to a single current dipole (at various depths) in a homogeneous half-space. All of these researchers considered just the effects of the finite size coil or the combined effects of the finite coil size and the gradiometer, they did not investigate sample-spacing requirements, the effects of a finite number of samples, or current sources other than a single current dipole.

Sampling with a Single Pick-up Coil

The magnetic field is a three-dimensional vector field, so we must sample in three dimensions. A magnetometer measures the component of the magnetic field normal to the plane of the pick-up coil at one \((x,y,z)\) location. For our work, we measured the \(z\)-component, \(B_z(x,y,z)\), in a single plane. Thus, we sample the field at many \((x,y)\) locations but only one \(z\) location. Sampling \(B_z(x,y,z)\) in a single plane is equivalent to multiplying \(B_z(x,y,z)\) by a delta function at \(z = z_m\), \(\delta(z-z_m)\), and reduces the three-dimensional sampling problem to a two-dimensional problem. To describe the sampled magnetic field we apply Equation (1.50), where the original function, \(f(x,y)\), corresponds to \([B_z(x,y,z) \delta(z-z_m)]\), the sampling aperture is a cylinder function, the sample spacing is the same in the \(x\)- and \(y\)-directions, and there are a finite number of samples,

\[
B_{\text{measured}} = \left[ B_z(x,y,z)\delta(z-z_m) \right] \ast \ast \text{cyl}\left( \frac{r}{d} \right) \text{comb}\left( \frac{x}{s}, \frac{y}{s} \right) \text{rect}\left( \frac{x}{2N}, \frac{y}{2M} \right)
\]

(4.1)

where \(r = \sqrt{x^2 + y^2} \). The Nyquist sampling interval, \(s\), depends on the spectral extent of the
averaged magnetic field.

As noted in Chapter 1, the effect of sampling a function at a finite number of sample points is important because the Fourier transform of the function is convolved with a regularly spaced array of sinc functions instead of an array of delta functions. If only a few samples are taken, this effect can be large. We know that biomagnetic fields fall off as $r^{-2}$ or $r^{-3}$ which is rather slow. Although the $z$-component of the magnetic field does not go exactly to zero at a finite distance from the peak value, it is important to measure the signal over a large enough area that the signal becomes negligible compared to the peak value. The definition of negligible is dictated by the size of our equipment (both the sampling aperture and the size of the gantry holding the SQUID system) and the time a patient is able to remain stationary. The best we can do is to measure out to where the signal is lost in the noise. The importance of measuring the entire magnetic field so that the edges as well as the peaks are measured is well-known (Romani and Leoni (1985), Knuutila, et al., (1985)). Bruno and Costa Ribeiro (1989) considered the extent of the area over which the magnetic field should be measured. They defined the 'total energy' of the signal as proportional to the integral over all space of the magnitude-squared of the $z$-component of the magnetic field. They then calculated the area of interest as the area that contained 99% of the energy of the field. In the calculation, they did not consider effects due to the finite size pick-up coil or noise. For our experiments, we took measurements (with grid-A4) out to where the signal was lost in the noise. For the rest of this discussion we will neglect the effects due to the finite number of samples.
Sampling with an Axial Gradiometer

In biomagnetism, the magnetic field is usually measured with a gradiometer. Thus, we have two or more pick-up coils, and we record the difference between the fields at the coils. In the general case of an $N$th order axial gradiometer that has circular pick-up coils of diameters $d = d_1, d_2, ..., d_{N+1}$ located at $z = z_{m1}, z_{m2}, ..., z_{mN+1}$, the measured field is

$$B_{\text{measured}} = \sum_{i=1}^{N+1} \left[ k_i B_k(x, y, z) \delta(z-z_{m_i}) \ast \ast \text{cyl} \left( \frac{r}{d_i} \right) \right] \text{comb} \left( \frac{x}{s}, \frac{y}{s} \right) \quad (4.2)$$

where $k_i$, the number of turns of the $i$th coil, is positive if the coil is wound in the same sense as the bottom coil, and is negative if the coil is wound in the opposite sense. The sample spacing, $s$, is assumed to be the same in the $x$- and $y$-directions and is required to be the same for all of the pick-up coils because they are connected axially. For a first-order axial gradiometer with baseline $b$ and pick-up coils of equal area the measured field is

$$B_{\text{measured}} = B_{\text{bottom-coil}} - B_{\text{top-coil}} \quad (4.3)$$

$$= \left\{ \left[ B_k(x, y, z) \delta(z-z_m) - B_k(x, y, z) \delta(z-z_m-b) \right] \ast \ast \text{cyl} \left( \frac{r}{d} \right) \right\} \text{comb} \left( \frac{x}{s}, \frac{y}{s} \right)$$

If we write the total measured magnetic field as the sum of the fields due to biomagnetic sources, the earth, and urban noise,

$$B_k^{\text{total}} = B_k^{\text{biomagnetic}} + B_k^{\text{earth}} + B_k^{\text{urban}} \quad (4.4)$$
then it is easy to see why the gradiometer output is proportional to just the field of interest. We assume $B_{z\text{\,\,\,earth}}$ and $B_{z\text{\,\,\,urban}}$ are approximately constant over the baseline so the difference between the field at $z = z_m$ and the field at $z = z_m + b$ is

$$B_{z\text{\,\,\,total}}(z = z_m) - B_{z\text{\,\,\,total}}(z = z_m + b) \approx B_{z\text{\,\,\,biomagnetic}}(z = z_m) - B_{z\text{\,\,\,biomagnetic}}(z = z_m + b). \quad (4.5)$$

If the magnetic field due to biomagnetic sources is not zero at the upper coil then the output is not proportional to just the field threading the lower coil in contrast to that which is often assumed.

Katila (1980) studied the influence of the gradiometer baseline on the measured magnetic field due to a magnetic dipole pointing in the $z$-direction. He assumed an infinitesimal pick-up coil size. He found that the strength and the zero-crossings of the magnetic field are reduced as the order of the gradiometer is increased. Bruno, et al., (1984) defined a transfer function for an $N^{th}$-order axial gradiometer and noted that gradiometers are high-pass filters, and the higher-order gradiometers get rid of more of the lower frequencies. Thus, the effect of the gradiometer is opposite to the effect of the finite coil size: the gradiometer acts as a high-pass filter while the finite size coil acts as a low-pass filter.

*Examples of Sampling Rates for Specific Magnetic Fields*

We now discuss several specific magnetic fields and their sampling intervals as a function of the pick-up coil diameter, the distance from the source to the pick-up coil, and the size of the current source. For this section, we considered eight coil diameters: 0, 1, 2, 5, 10, 15, 20, and 25 mm. We analytically calculated $H_z(\xi, \eta, z)$ for four current sources: a
magnetic dipole pointing in the x-direction, a magnetic dipole pointing in the z-direction, a square loop in the x-y plane with side length \( \ell \), and a current dipole in a homogeneous conducting half-space. Having an analytic expression for \( \mathbf{B}_s(x, \eta, z) \) allowed us to easily vary the distance between the source and the pick-up coil. The averaged Fourier transform was calculated by multiplying the analytic Fourier transform of the magnetic field by a somb function; the width of the somb function depended on the diameter of the pick-up coil. We determined the Nyquist sampling interval using two criteria for the widths of the averaged Fourier transforms: the cutoff frequency where the averaged Fourier transform was 1) 10% of the peak value (full-width-at-10%-max or FW10%M) and 2) 1% of the peak value (full-width-at-1%-max or FW1%M). These two criterion are arbitrary, and were chosen because 1\% is a strict requirement to meet and will reduce the chance of significant aliasing, while the relaxed level of 10\% will have acceptable amounts of aliasing if the noise levels are high.

For a given source-to-pick-up-coil distance, the magnetic field due to a smaller source will have more high frequencies than the field due to a larger source. Thus, a magnetic dipolar field will have more high frequencies and a smaller Nyquist sampling interval than the field due to a large square loop. The magnetic field due to a current dipole in a homogeneous half-space will also have a narrower Fourier transform than the magnetic dipolar field because the return currents occupy a large region of space.

The Fourier transform of the z-component of the magnetic field (that would be measured by a infinitesimal pick-up coil) due to a current dipole 20 mm from the pick-up coil is shown in Figure 4.4. It is difficult to tell where the 10\% and 1\% levels are on the two-dimensional plot, so we also plotted the Fourier transform of the magnetic field for the frequency values of \( \eta = 0, \xi \geq 0 \) in order to find the cutoff frequency in the \( \xi \) direction. Figure 4.5 shows the Fourier transform of the z-component of the magnetic fields due to a current dipole in a homogeneous half-space, a magnetic dipole pointing in the x-direction, a
Figure 4.4
Surface plot of the Fourier transform of the averaged $B_z$ due to a current dipole in a homogeneous half-space.
Pick-up coil diameter = 0.001mm
Source to detector distance = 20mm
Figure 4.5
Plots of the Fourier transform of the averaged $B_z$ used to determine $\xi_{\text{cutoff}}$. Each plot was normalized so that the peak value was one. Two cutoff frequencies were chosen: 10% of the peak value and 1% of the peak value.

Figure 4.6
Plots of the Fourier transform of the averaged $B_z$ used to determine $\eta_{\text{cutoff}}$. Each plot was normalized so that the peak value was one. Two cutoff frequencies were chosen: 10% of the peak value and 1% of the peak value.
magnetic dipole pointing in the z-direction, and a square loop. The Fourier transforms of the fields due to the magnetic dipole pointing in the x-direction and the magnetic dipole pointing in the z-direction are identical along the line $\eta = 0$, so they have identical sampling requirements in the x-direction.

To find the cutoff frequency in the $\eta$ direction, we plotted the Fourier transforms of the $z$-component for $\eta \geq 0$. For two of the sources we considered (the magnetic dipole pointing in the z-direction and the square loop), the cutoff frequencies are the same in the $\eta$ and $\xi$ directions, so the sampling intervals for these fields are the same in the x and y-directions. The Fourier transforms of the fields due to a magnetic dipole pointing in the z-direction and a square loop are plotted in Figure 4.6 for $\xi = 0$. The other two fields have cutoff frequencies that are different in the $\eta$ and $\xi$ directions. In addition, the Fourier transforms of the fields are zero for $\xi = 0$, so we plotted the Fourier transforms along the line $\xi = (2\pi z)^{-1}$. This value of $\xi$ was chosen because that is where the peak is located. We produced plots similar to Figures 4.5 and 4.6 for all the pick-up coil diameters in order to estimate the Nyquist sampling interval as a function of the pick-up coil diameter.

Figure 4.7 shows the Nyquist sampling interval versus pick-up coil diameter at both the FW10%M and FW1%M levels for the magnetic dipole pointing in the z-direction 20 mm away from the pick-up coil. At the FW1%M level, the field needs to be sampled at intervals less than the pick-up coil diameter for coils greater than 7 mm diameter. This represents the worst case that biomagnetic researchers will probably encounter. The sources will usually be larger and farther from the pick-up coil. However, we don’t know a priori what the Fourier transform of the magnetic field will be, so we don’t know what sampling scheme to use. The conservative approach is to sample at whatever rate is needed for the worst case, and then all the other cases will be oversampled. Oversampling may waste time but it insures that the sampled field is a good representation of the actual field.
Figure 4.7
Sampling interval versus pick-up coil diameter for magnetic dipole 20 mm from detector. Cutoff frequencies determined from Figures 4.5 and 4.6.

Figure 4.8
Plots of the Fourier transform of the averaged B_z used to determine ε_{cutoff} for measured data. Each plot was normalized so that the peak value was one. Two cutoff frequencies were chosen: 10% of the peak value and 1% of the peak value.
Romani and Leoni (1985) considered, one at a time, the effects of a finite pick-up coil area, sample spacing, length of the gradiometer baseline, and a finite number of samples. They considered a current dipole in a semi-infinite medium with homogeneous conductivity and recommended that for a second-order gradiometer with coil diameters of 15 mm and a measurement distance of 30 mm, a sample spacing of 20 mm was appropriate (at a level of 10% of the peak value of the Fourier transform). This value for the sampling interval has become dogma in the biomagnetic community, and is used even if the source is not a current dipole in a homogeneous half-space 30 mm from the pick-up coil, and even if the field is not being measured with a second-order gradiometer with 15 mm diameter pick-up coils. Bruno and Costa Ribeiro (1989) considered sampling requirements for a magnetometer, first-order planar and second-order planar gradiometers. They assumed the pick-up coils had a finite size, but assumed the magnetic field was constant over the area of the pick-up coil. The current source, a single current dipole in a semi-infinite homogeneous conductor, was located 30 mm below the pick-up coil. They determined the Nyquist sampling interval by defining the total energy of the Fourier transform as being proportional to the integral over all space of the magnitude-squared of the Fourier transform. They then chose the cutoff frequency, F, such that 99% of the energy of the Fourier transform was contained between F and -F. They found that, for some source-to-pick-up-coil distances, the Nyquist sampling interval was less than the spacing between the pick-up coils of the planar gradiometers. For the planar gradiometers and the specific source they considered, a sampling interval of 20 mm would result in undersampling.
Undersampling

To illustrate the effects of undersampling, we consider the magnetic field due to a planar, square loop and compare the results of a computer simulation to measured data. For the computer simulations, we analytically calculated the z-component of the magnetic field in Fourier space (using continuous Fourier transforms) in order to avoid the pitfalls involved in taking the discrete Fourier transform of the magnetic field.

Consider a planar square loop centered about the origin, where the length of the sides is \( \ell \) and the width of the sides is a delta function. The Fourier transform of \( B_z \) is

\[
\mathcal{B}_z(\xi, \eta, z) = -\pi \ell^2 \text{sinc}(\xi \ell) \text{sinc}(\eta \ell) \sqrt{\xi^2 + \eta^2} e^{-2\pi |z| \sqrt{\xi^2 + \eta^2}}.
\]  

(4.6)

If we consider the Fourier transform of \( B_z \) without any sampling or smoothing effects we see that as \( z \) increases, \( \exp(-2\pi |z| \sqrt{\xi^2 + \eta^2}) \) gets narrower, so the Fourier transform of \( B_z \) has fewer high frequencies. This observation is the Fourier space counterpart of the observation in real space that as we go away from the source the field gets smoother. As the length of the side, \( \ell \), increases, \( \text{sinc}(\xi \ell) \text{sinc}(\eta \ell) \) gets narrower, so it also has fewer high frequencies. The magnetic field due to a magnetic dipole has more high frequencies, for a given distance from the source, than the magnetic field due to a large loop.

For the examples in this section, we assumed a coil diameter of 20 mm and sample spacings of 20 mm and 10 mm. We considered two different current distributions: a square loop 8.5 mm on a side at a measurement distance of 31 mm, and a square loop 55 mm on a side at a measurement distance of 66 mm. The general statements above predict that the magnetic field due to the smaller loop with the smaller measurement distance will have a wider Fourier transform than the field due to the larger loop with the larger measurement.
distance, and will therefore need to be sampled more frequently. In order to determine the
Nyquist rates for the two different cases, we determined the width of the Fourier transform
of the averaged field (Figure 4.8) at both the FW10%M and FW1%M levels. The field due to
the smaller loop has a Nyquist sampling interval of 50 mm at the FW10%M level and
33.3 mm at the FW1%M level. The field due to the larger loop has a Nyquist sampling
interval of 21.7 mm at the FW10%M level and 14.7 mm at the FW1%M level.

In order to test the sampling rate of the measured data we compared a sinc
interpolation of the data measured on a coarse grid to data actually measured on a fine grid.
When we collected data, we used 10 mm (grid-A3, fine), as well as 20 mm (grid-A4, coarse)
intervals. If sampling at 20 mm is often enough, then we should be able to sinc-interpolate
the data measured at 20 mm intervals and calculate the same data that we measured at
10 mm intervals. To do the sinc-interpolation, we put the data, measured at 20 mm
intervals, into every other point of an array. The Fourier transform of the sampled data
consisted of two orders (recall that a sampled function in one space is periodic in the other
space), so we picked out just the lowest order of the Fourier transform by multiplying by a
rect function of the appropriate size. Thus, in real space the intermediate points were filled
in by interpolating with sinc functions. Finally, the interpolated values were compared with
the measured values (Figure 4.9). As predicted by the computer simulation, the larger loop is
adequately sampled at 20 mm intervals, and the smaller loop needs to be sampled at smaller
intervals to avoid aliasing. The differences between the measured and interpolated fields are
particularly evident near the peaks. The interpolated field due to the small loop (Figure
4.9b) has a smaller peak than the measured field (Figure 4.9a). In contrast, the interpolated
and measured fields due to the large loop appear almost identical. This is an example of why
it is important to measure known sources. We wouldn't have had any way of telling that the
data were aliased if we hadn't known what and where the source was. If we reconstruct the
Figure 4.9
Magnetic fields measured at 1cm and 2cm intervals.
(a) Due to small round loop, 3.1cm from measurement plane, measurement interval = 1cm
(b) Due to small round loop, 3.1cm from measurement plane measurement interval = 2cm, sinc-interpolated to 1cm
(c) Due to large round loop, 6.6cm from measurement plane measurement interval = 1cm
(d) Due to large round loop, 6.6cm from measurement plane measurement interval = 2cm, sinc-interpolated to 1cm
current from the undersampled data, the result will be wrong (see Chapter 5).

In conclusion, we can’t calculate a single Nyquist rate that will work for all biomagnetic fields. There are just too many different magnetic fields, some very smooth and others that change rapidly with location. We can say some general things such as for highly spatially localized current sources close to the lower pick-up coil, we need to sample more frequently than for large or deep sources. For larger pick-up coils we can sample at larger intervals than for smaller coils, but the samples may measure overlapping areas. We can also generate graphs of the sampling interval versus pick-up coil diameter (such as Figure 4.7) for specific current sources.
RESULTS

In this chapter we present reconstructions of current sources using the two Fourier-transform techniques described in Chapter 3. The first technique (called the B-zero method) uses reconstruction equation (3.48), which assumes that the $x$- and $y$-components of the magnetic field in the forbidden region are zero, and that the $z$-component of the current density is zero. The second technique (the $\nabla^2 B$-zero method) uses reconstruction equations (3.66) and (3.68), which assume that $\nabla^2 B_x$ and $\nabla^2 B_y$ are zero in the forbidden region. We first present reconstructions of current sources from computer-simulated magnetic fields and then from measured magnetic fields generated by known current sources.

The reconstruction algorithms that we developed for this dissertation use a slab geometry, so it is natural to divide the regions into planes and to display the vector field on each plane. In this chapter we shall use grey-level pictures to display the reconstructions. Each picture is scaled individually from the maximum to the minimum value. For all of the pictures, except Figure 4.3, white represents the maximum and black represents zero. In Figure 4.3 white represents the maximum positive value and black represents the maximum negative value.

The total volume was assumed to be $32 \times 32 \times 32$ pixels and was divided into three regions (Figure 3.3): the measurement region, the forbidden region, and the reconstruction region. The three regions are described in Chapter 3. The size of the total volume was restricted by the available memory on our computer (a VAX8600). The size of the measurement region depends on the size of the forbidden region, and can be found by subtracting the number of planes in the forbidden region from 32. We assumed that the
magnetic field is nonzero on only one side (the top) of the forbidden region, because with real data we have measurements on only one side of the current source, and we wish to compare the reconstructions from computer-simulated data to those from real data. In this chapter, we shall describe the regions in terms of the number of planes in the region, because the x-y size of all three regions was always 32x32 pixels. The reconstruction region and the forbidden region were always an even number of planes.

Reconstructions From Simulated Data

In this section we present reconstructions of current densities from computer-simulated magnetic fields. In all cases, we started with an analytic expression for the z-component of the magnetic field at 32x32 points in a single plane. We did not include any effects due to a finite-size sampling aperture, a gradiometer, or noise. The magnetic field was extrapolated (see Chapter 2) because our reconstruction methods require knowledge of all three components of the magnetic field in the measurement region. For computer simulations the original current sources were chosen from a limited object class; they were all square loops confined to an x-y plane (so that the z-component of the current was zero). The width of the sides of the square loop was always 1 pixel, but the length of the sides varied. The original current loops used for the reconstructions presented in this section are shown in Figure 5.1. The indices run from 1 to 32, where the pixel (1, 1) is in the upper left corner of the grey-level pictures.

We present the effects on the reconstruction due to the form of the differentiation operator (see Chapter 1 for a discussion of approximations to the differentiation operator), the size of the original loop, the distance between the original loop and the measurement
Figure 5.1
Original square current loops for reconstructions from computer-simulated data. Indices run from 1 to 32, where (1,1) is in the upper left corner.
(a) side length of 3 pixels, upper left corner at (12,12,2)
(b) side length of 7 pixels, upper left corner at (8,8,2)
(c) side length of 3 pixels, upper left corner at (3,3,2)
plane, the lateral location of the original loop, the size of the forbidden region, and the size of the reconstruction region. Although most of the reconstructions were of a single current loop, the ability of the algorithms to reconstruct more than one loop is also demonstrated.

*Effect of Approximating the Differentiation Operator*

For the reconstructions, we defined the differentiation operator in three different ways: an asymmetric-finite-difference operator (equation (1.41)), a symmetric-finite-difference operator (equation (1.40)), and a sampled periodic version of the continuous-derivative operator \((i2\pi\lambda, \text{ in Fourier space})\). The effects of the various approximations to the differentiation operators on the reconstruction of a single square loop are shown in Figures 5.2a-f and 5.3a-f. The parameters for the reconstructions shown in Figures 5.2a-f and 5.3a-f are the following: a forbidden region of 8 planes, a reconstruction region of 8 planes, each side of the square loop was 3 pixels long, and the upper left corner of the loop was at \((12, 12, 1)\) (Figure 5.1a).

Figure 5.2a-f shows the magnitude of the reconstructed current, the left-hand column is the current reconstructed with the B-zero method and the right-hand column is the current reconstructed with the \(\nabla^2 B\)-zero method. The reconstructions in the top row were done by approximating the differentiation operator with asymmetric finite differences, the middle row used symmetric finite differences, and the bottom row used the sampled version of the continuous differentiation operator. The form of the differentiation operator resulted in dramatically different reconstructed currents for the B-zero method, but made only a slight difference for the \(\nabla^2 B\)-zero method. For all three approximations to the differentiation operator, the currents reconstructed with the \(\nabla^2 B\)-zero method had fewer artifacts than when reconstructed with the B-zero method. The artifacts from the B-zero
Figure 5.2
Magnitude of reconstructed current density from B-zero method and $V^2B$-zero method, using three different approximations to the differentiation operator. Reconstructions should look like Figure 5.1a.
Left-hand column: B-zero method
Right-hand column: $V^2B$-zero method
Top row: Asymmetric finite differences
Middle row: Symmetric finite differences
Bottom row: Sampled periodic version of continuous operator
method are probably due to the assumption that the x- and y-components of the magnetic field are zero in the forbidden region. The x- and y-components of the magnetic field are nonzero in the measurement region and the abrupt change of the magnetic field at the boundary between the forbidden region and measurement region introduces artificial high frequencies. In contrast, the $\nabla^2 B$-zero method allows the x- and y-components of the magnetic field to vary smoothly across the boundary.

The magnitude of the current reconstructed with the $B$-zero method and the asymmetric finite differences (Figure 5.2a) appears to be an asymmetric loop and has fewer artifacts than the current in Figure 5.2b. The magnitude of the current reconstructed with the $B$-zero method and the symmetric finite differences (Figure 5.2b) is shaped more like a loop than the current in Figure 5.2c, but there are many artifacts, including what could be interpreted as a second, larger, current loop. The magnitude of the current reconstructed with the $B$-zero method and the sampled-continuous-derivative operator (Figure 5.2c) does not look like a loop; it looks like four separated current dipoles. Of the three reconstructions with the $B$-zero method, the one with the asymmetric finite differences is shaped most like a loop and has the fewest artifacts.

The magnitude of the current reconstructed with the $\nabla^2 B$-zero method and the asymmetric finite differences (Figure 5.2d) looks like an asymmetric loop. The magnitude of the current reconstructed with the $\nabla^2 B$-zero method and the sampled-continuous-derivative operator (Figure 5.2f) seems like a loop although it is slightly more smeared out than the current reconstructed with the symmetric finite differences (Figure 5.2e). The asymmetry of the reconstructed loops in Figures 5.2a and 5.2d is due to the ambiguity of where the result of the asymmetric finite difference operation should go.

Figures 5.3a–f show the real and imaginary parts of the x- and y-components of the reconstructed current. The z-component of the reconstructed current is zero for both the
Figure 5.3
Real and imaginary parts of x- and y-components of the reconstructed current from B-zero method and $V^2$B-zero method, using three different approximations to the differentiation operator.

Left-hand column: B-zero method
Top row: Asymmetric finite differences
Middle row: Symmetric finite differences
Bottom row: Sampled periodic version of continuous operator

Right-hand column: $V^2$B-zero method
B-zero method and the $\nabla^3 B$-zero method. Figures 5.3a–c show the real and imaginary parts of the components of the current reconstructed with the B-zero method. Figures 5.3d–f show the real and imaginary parts of the components of the current reconstructed with the $\nabla^3 B$-zero method. The grey-level values for the real and imaginary parts of each reconstruction are scaled separately, with white representing the maximum positive value and black representing the maximum negative value. The reconstructions in the top row used the asymmetric finite differences, the middle row used the symmetric finite differences, and the bottom row used the sampled periodic version of the continuous differentiation operator. In each figure, the top pictures represent the real part, the bottom the imaginary part; the left column is the x-component while the right column is the y-component.

For the current reconstructed with the B-zero method and the sampled-continuous-derivative (Figure 5.3c), the maximum of the real part of the x-component is an order of magnitude smaller than the maximum of the real part of the y-component ($10^{-1}$ versus 10), and similarly with the imaginary parts ($10^{-3}$ versus $10^{-2}$). In contrast, the currents reconstructed with the B-zero method and the symmetric finite differences (Figure 5.3b) or the asymmetric finite differences (Figure 5.3a) have the maxima of the real parts of the x- and y-components approximately equal (both are on the order of $10^{-3}$), and the maxima of the real parts of both components are six orders of magnitude larger than the maxima of the imaginary part.

For the current reconstructed with the $\nabla^3 B$-zero method and the sampled-continuous-derivative (Figure 5.3f), the maxima of the real and imaginary parts of both the x- and y-components are on order the order of $10^{-1}$. For the currents reconstructed with the $\nabla^3 B$-zero method and the symmetric (Figure 5.2e) and the asymmetric (Figure 5.3d) finite differences, the maxima of the real parts of the x- and y-components are approximately
equal (both are on the order of $10^{-3}$), and the maxima of the real part of both components are six orders of magnitude larger than the maxima of the imaginary part. Thus, for both the B-zero method and the $\nabla^2 B$-zero method, the imaginary part of the current reconstructed with the sampled-continuous-derivative is significant, but the imaginary parts of the reconstructed currents for the finite differences are negligible compared to the real parts. The difference in the order of magnitudes for the real and imaginary parts indicates that the reconstructions with the sampled-continuous-derivative result in a current that is not physical; currents must be purely real.

Our conclusion is that the sampled-continuous-derivative is unacceptable because of the large imaginary parts of the reconstructed currents. In addition, the reconstructed current from the B-zero method with the sampled-continuous-derivative is not shaped like a loop. The B-zero method with the symmetric finite differences is inferior to that method using asymmetric finite differences because the artifacts are more pronounced. The $\nabla^2 B$-zero method produces equally good results with the symmetric and asymmetric finite differences. For the rest of this chapter, we shall present results of the B-zero method and the $\nabla^2 B$-zero method with asymmetric finite differences. Reconstructions using the other two approximations to the differentiation operator were performed, but are not presented here. The conclusions about the performance of the reconstruction algorithms based on the other reconstructions are the same as those based on the reconstructions using asymmetric finite differences.

Effect of the Size of the Original Source

One of the disadvantages of the reconstruction methods based on the concept of an equivalent current dipole (see Chapter 3) is that they are not able to distinguish between
Figure 5.4
Magnitude of reconstructed current; original source: square loop, side length of 7 pixels, upper left corner at (8,8,1). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) $\nabla^2$B-zero method

Figure 5.5
Magnitude of reconstructed current; original source: square loop, side length of 7 pixels, upper left corner at (8,8,3). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) $\nabla^2$B-zero method

Figure 5.6
Magnitude of reconstructed current; original source: square loop, side length of 3 pixels, upper left corner at (3,3,3). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) $\nabla^2$B-zero method
highly localized sources and widely distributed sources. Our Fourier-transform reconstruction methods do not have this problem; small loops are reconstructed smaller than large loops for the same distance from the measurement plane to the original source. In Figure 5.4, we show the magnitude of the reconstructed current when the original current (Figure 5.1b) was a square loop 7 pixels on a side with the upper left corner located at (8,8,1) (the reconstruction region and the forbidden region were both 8 planes). The original loop for the reconstruction shown in Figure 5.2 was 3 pixels on a side and the same distance from the measurement plane as the original loop for the reconstruction in Figure 5.4. Comparing Figure 5.4a to Figure 5.2a and Figure 5.4b to Figure 5.2d, we see that the reconstructed loops are the appropriate sizes, for both the B-zero method and the $V^3B$-zero method.

Effect of the Distance From the Measurement Plane to the Original Source

Measuring the magnetic field as close to the source as possible is critical due to the rapid loss of high frequencies in the magnetic field with increased distance from the source. The high frequency information provides a better lateral localization of the reconstructed current source. Figure 5.5 shows the magnitude of the reconstructed current when the original current (Figure 5.1b) was a square loop 7 pixels on a side and the upper left corner was located at (8,8,3) (the reconstruction region and the forbidden region were both 8 planes). Thus, the original loop for Figure 5.5 was 2 planes closer to the measurement plane than the original loop for Figure 5.4. Comparing Figure 5.4a to Figure 5.5a and Figure 5.4b to Figure 5.5b, we see that when the original loop is closer to the measurement plane the reconstructed loops are sharper, for both the B-zero method and the $V^3B$-zero method, than when the original loops are further from the measurement plane.
Lateral localization of the current source is excellent for both the $B$-zero method and the $\nabla^2 B$-zero method. In Figure 5.6 we show the magnitude of the reconstructed current when the original source (Figure 5.1c) was a square current loop 3 pixels on a side and the upper left corner was located at (3,3,3) (the reconstruction region and the forbidden region were both 8 planes). Thus, the original loop for Figure 5.6 was 2 planes closer to the measurement plane, and in a different ($x,y$) location than the original loop for Figure 5.2. Comparing Figure 5.6a to Figure 5.2a and Figure 5.6b to Figure 5.2d, we see that the reconstructed loop has the same displacement as the original loop, for both the $B$-zero method and the $\nabla^2 B$-zero method. The reconstructed loops in Figure 5.6 are sharper than the loops in Figure 5.2 because the original loop was closer to the measurement plane.

The effects of the sizes of the forbidden region and reconstruction region are related. The size of the forbidden region provides an upper limit to the size of the reconstruction region because the reconstruction region can consist of all or part of the forbidden region. The size of the forbidden region is important because it limits how close to the source we can make measurements. The size of the reconstruction region is important because it limits the number of possible sources to the set of sources that are confined to that region. One of our goals is to localize the current source in the $z$-direction as well as in the $x$- and $y$-directions, so for a large reconstruction region, we would like to be able to discriminate between deep and shallow sources. We shall first discuss the case where the reconstruction region is the same size as the forbidden region and then discuss the case when the...
reconstruction region is smaller than the forbidden region.

We investigated three different forbidden-region sizes: 2, 8, and 16 planes. For a reconstruction region the same size as the forbidden region, we expect the B-zero method to provide very little information about the depth of the source because it assumes that the x- and y-components of the magnetic field are zero in the entire forbidden region so the depth information can come only from the z-component of the magnetic field. On the other hand, we expect the \( \nabla^2 B \)-zero method to provide some depth information because the x- and y-components of the magnetic field are not constant in the forbidden region. What we find is that both the B-zero method and the \( \nabla^2 B \)-zero method put the strongest reconstructed current on the surface of the reconstruction region and weaker reconstructed currents in the other planes (Figure 5.7).

In Figure 5.7 the magnitudes of the reconstructed currents for each plane of the 8 planes in the reconstruction region are shown. In each figure, the reconstructed current in the bottom plane of the reconstruction region (the plane farthest from the measurement plane) is in the top left-hand corner. The distance to the measurement plane decreases as we go from left to right across each row. The reconstructed current in the top plane of the reconstruction region (the plane directly below the measurement plane) is shown in the middle of the bottom row. For the B-zero method, the plane farthest from the measurement plane has the strongest reconstructed current, and it is this plane that is shown in Figure 5.2c. For the \( \nabla^2 B \)-zero method, the plane directly under the measurement plane has the strongest reconstructed current (also shown in Figure 5.2f). The change in source strength with depth was consistent for each reconstruction method and independent of the size, location or number of the original sources, the size of the forbidden region, and the form of the differentiation operator. The disappointing lack of depth discrimination is an area for future research.
Figure 5.7
Magnitude of reconstructed current in all 8 planes of the reconstruction region. In each figure, the bottom plane of reconstruction region is in the top left corner and the distance to the measurement plane decreases from right to left across each row. Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method 
(b) $\nabla^2$B-zero method

Figure 5.8
Magnitude of reconstructed current in top 9 planes of the 16-plane reconstruction region. In each figure, the top plane of reconstruction region is in the bottom right corner and the distance to the measurement plane increases from left right across each row. Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method 
(b) $\nabla^2$B-zero method
For the $\nabla^2 B$-zero method, the same original current (a square loop: upper left corner at (12,12,1), each side 3 pixels long, Figure 5.1a) resulted in a reconstructed current that was most smeared out when the forbidden region was 16 planes (Figure 5.8) and was sharpest when the forbidden region was 2 planes (Figure 5.9). When the forbidden region and reconstruction were 2 planes, the current reconstructed with the B-zero method is difficult to interpret due to the artifacts. The different reconstructions for the different forbidden region sizes is related to the distance from the measurement plane to the original source, not the size of the forbidden region itself. When the distance between the measurement plane and the original source is kept constant as the size of the forbidden region is increased, the reconstructions retain the same shape. Figure 5.10 shows the magnitude of the reconstructed current for the same conditions leading to the reconstruction in Figure 5.6, except for the sizes of the forbidden region and the reconstruction region. For Figure 5.10, the reconstruction region and forbidden region were both 16 planes, and for Figure 5.6 the reconstruction region and forbidden region were both 8 planes.

When the reconstruction region was smaller than the forbidden region, the depth discrimination didn't improve. Figure 5.11 shows the magnitude of the reconstructed current (a square loop: upper left corner at (12,12,1), each side 3 pixels long) when the forbidden region was 8 planes and the reconstruction region was 2 planes. The reconstructed current for both the B-zero method (Figure 5.11a) and the $\nabla^2 B$-zero method (Figure 5.11b) is slightly fuzzier than for the cases when the forbidden region and reconstruction region were the same size (Figure 5.2a and f, where the forbidden region and reconstruction region were both 8 planes, or Figure 5.8a and b, where the forbidden region and reconstruction region were both 2 planes). The current reconstructed with the $\nabla^2 B$-zero method appears to be a loop in all three cases (Figures 5.11b, 5.2f, and 5.8b), but the current reconstructed with the
Figure 5.9
Magnitude of reconstructed current in both planes of the 2-plane reconstruction region. In each figure, the bottom plane is on the left and the top plane is on the right. Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) \( \nabla^2 \)B-zero method

Figure 5.10
Magnitude of reconstructed current; original source: square loop, side length of 3 pixels, upper left corner at (3,3,7). Reconstruction region and forbidden region were 16 planes. Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) \( \nabla^2 \)B-zero method

Figure 5.11
Magnitude of reconstructed current in both planes of the 2-plane reconstruction region, when the forbidden region was 8 planes. In each figure, the bottom plane is on the left and the top plane is on the right. Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) \( \nabla^2 \)B-zero method
B-zero method doesn’t look like a loop for the two cases when the reconstruction region is 2 planes (Figures 5.11a, and 5.8a).

In summary, the size of the reconstruction region has very little influence on the reconstructions, and the size of the forbidden region is important only because it limits the size of the reconstruction region. The important variable is the distance between the original source and the measurement plane. For the rest of this chapter, we shall consider only reconstruction regions and forbidden regions of 8 planes.

Reconstructions of More Than One Original Loop

Our reconstruction methods are able to reconstruct adequately (with some exceptions) a linear combination of current sources because the Maxwell equations and the Fourier transforms, on which the reconstruction methods are based, are linear. For example, consider a source that consists of 2 square loops, one with 7 pixels on a side with the upper left corner located at (8,8,1), and the other 3 pixels on a side with the upper left corner located at (3,3,3). The original source is a linear combination of the original sources used in the reconstructions shown in Figures 5.4 and 5.6. We expect the reconstructed current to be a linear combination of the reconstructions shown in Figures 5.4 and 5.6. Indeed it is, as shown in Figure 5.12, for both the B-zero method and the $\nabla^2B$-zero method. When the two original loops are in the same plane (one with 7 pixels on a side with the upper left corner located at (8,8,1), and the other 3 pixels on a side with the upper left corner located at (3,3,1)), the reconstructed current seems to be two separated loops (Figure 5.13). In Figure 5.13, the smaller loop is not as strong as the larger loop because its contribution to the measured magnetic field is not as strong.
Figure 5.12
Magnitude of reconstructed current; original source: 2 square loops, side length of 3 pixels, upper left corner at (3,3,3) and side length of 7 pixels, upper left corner at (8,8,1). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) $\nabla^2$B-zero method

Figure 5.13
Magnitude of reconstructed current; original source: 2 square loops, side length of 3 pixels, upper left corner at (3,3,1) and side length of 7 pixels, upper left corner at (8,8,1). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method  
(b) $\nabla^2$B-zero method
Figure 5.14
Magnitude of reconstructed current; original source: 2 square loops, side length of 3 pixels, upper left corner at (12,12,1) and side length of 7 pixels, upper left corner at (8,8,3). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method (b) $\nabla B$-zero method

Figure 5.15
Magnitude of reconstructed current; original source: 2 square loops, side length of 3 pixels, upper left corner at (12,12,3) and side length of 7 pixels, upper left corner at (8,8,1). Differentiation operator approximated by asymmetric finite differences.
(a) B-zero method (b) $\nabla B$-zero method
Unfortunately, our reconstruction methods are not always able to reconstruct adequately two current sources. Figure 5.14 shows the reconstruction when the original source consisted of 2 square loops, one with 7 pixels on a side with the upper left corner located at (8,8,3), and the other 3 pixels on a side with the upper left corner located at (12,12,1). The original source is a linear combination of the original sources used in the reconstructions shown in Figures 5.5 and 5.2. In Figure 5.14, the smaller loop is hidden by the larger loop, which is also closer to the measurement plane. When the original sources are switched so that the smaller loop is closer to the measurement plane, both loops become evident in the reconstruction (Figure 5.15). The inability to reconstruct two current loops is due to the lack of depth discrimination between sources at different depths.

Summary of Reconstructions From Simulated Data

The reconstructions of computer simulated data provide us with a way of testing the new Fourier-transform reconstruction algorithms discussed in Chapter 3. In general, we found that the $V^2B$-zero method was better than the $B$-zero method because it generates fewer artifacts. The effects on the reconstructions of the form of the differentiation operator, the size of the original source, and the location of the original source were similar for both the $B$-zero method and the $V^2B$-zero method. Approximating the differentiation operator in Fourier space as a sampled version of the continuous operator is unacceptable because it produces reconstructed currents that have significant imaginary parts, while currents must be purely real. The symmetric and asymmetric finite differences both produce physically acceptable reconstructed currents. The reconstructions of small loops are proportionally smaller than large loops when the original loops are the same distance from the measurement plane. Loops closer to the measurement plane have sharper reconstructions.
than loops further away. The lateral resolution of both the B-zero method and the \( \nabla^2 B \)-zero method is excellent. The inability of either reconstruction method to provide any information about the depth of the current was disappointing.

When the original source consisted of two current loops of different sizes and locations, both reconstruction methods reconstructed currents that appeared as two loops. If the original loops were in different planes, then the loop closer to the measurement plane was reconstructed stronger and sharper than the loop further away. If the two loops are too close together and/or too far from the measurement plane, then the reconstructed current looks like a single loop.

We would like to stress that these conclusions are valid only for a limited object class. The original sources we considered in this section were all planar, square loops. The results of the reconstruction methods may be entirely different for other current sources (i.e. current sources that have a nonzero z-component). In the next section, we extend the object class to include current loops that are not square, currents loops that have a nonzero z-component, and current dipoles.

Reconstructions From Measured Data

The results of the computer simulations are encouraging, but, it is important to test the reconstruction methods on real measured data. In this section we present reconstructions from measured data using asymmetric finite differences as the approximation to the differentiation operator. The original current sources and the measurement techniques are described in Chapter 4. The data used for the reconstructions presented in this section consist of the z-component of the magnetic field measured at 81 points in a plane (grid-A4). The measured data were extrapolated before the reconstructions were performed. For the
following reconstructions, we assumed the total volume was 32×32×32 pixels and the forbidden region and the reconstruction region were both 8 planes. The pixel dimensions for all the grey-level pictures in this section are 1 cm × 1 cm, and the total area covered is 32 cm × 32 cm.

Reconstructions of Round Loops

The first reconstructions we present are of small and large round loops. Figure S.16 shows the magnitude of the reconstructed current when the original source was a small round loop (diameter = 0.85 cm) 3.1 cm from the measurement plane. Figure S.17 shows the reconstruction when the original source was a large round loop (diameter = 5.5 cm) 6.6 cm from the measurement plane. There are many similarities between the reconstructions from measured data and the reconstructions from simulated data; in particular, the reconstructions from the B-zero method have similar artifacts. The artifacts evident in the reconstructions from the V²B-zero method are due to the finite number of magnetic field sample points; the field was measured on a grid that was 9×9 points (and interpolated onto a grid that was 18×18 points) but the reconstruction algorithms used a grid that was 32×32 points. The similarities indicate that the effects of a finite sampling aperture, gradiometer and noise (which were neglected for the computer simulations), are small.

Computer simulations indicated that for original sources that are the same distance from the measurement plane, smaller sources will have smaller reconstructions than larger sources. This holds true for reconstructions from measured data; the reconstructed currents are different sizes, corresponding to the different sizes of the original sources. The reconstructed large round loop has a diameter of about 5 cm (1 pixel corresponds to 1 cm), which is approximately the same as the original-source diameter of 5.5 cm. Although the
Figure 5.16
Magnitude of current reconstructed from measured data; original source: round loop, diameter = 0.85cm, 3.1cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method 
(b) $\nabla^2 B$-zero method

Figure 5.17
Magnitude of current reconstructed from measured data; original source: horizontal round loop, diameter = 5.5cm, 6.6cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method 
(b) $\nabla^2 B$-zero method
reconstruction of the small round loop is shaped like a loop, it has a much larger diameter than the original source: about 3 cm for the reconstruction, instead of 0.85 cm. In Chapter 4, we found that the magnetic field due to the small round loop was undersampled at 2 cm intervals, while the magnetic field due to the large round loop was adequately sampled. The incorrect size of the reconstructed small loop is in part due to the undersampling of the field. In Figure 5.18, we show the reconstruction (from computer simulated data) of a square loop (side length of 1 cm, distance between the source and measurement plane of 3 cm) where the z-component of the magnetic field was sampled at 1 cm intervals. The diameter of the reconstruction in Figure 5.18, although more than 1 cm, is a more accurate representation of the original source than the reconstruction of the undersampled data in Figure 5.16. In Figure 5.19, we show the reconstruction from computer simulated data of a square loop that is approximately the same size and distance from the measurement plane (side length of 6 cm, distance between the source and the measurement plane of 7 cm) as the large round loop. In this case, the reconstructions from real data and from computer-simulated data are almost identical.

The last point we would like to make about the reconstructions of these small and large loops is that the reconstructions of the round loops (from measured data) and the reconstructions of the square loops (from simulated data) all appear to be loops, but they don't look distinctly round or square.

**Reconstructions of "L"-Shaped Loops**

The next reconstructions we present are of small and large "L"-shaped loops. Figure 5.20 shows the magnitude of the reconstructed current when the original source was a small "L"-shaped loop 6.6 cm from the measurement plane, and Figure 5.21 shows the
Figure 5.18
Magnitude of current reconstructed from computer simulated data; original source: square loop, side length of 1 cm, 3 cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method
(b) $\nabla^2$B-zero method

Figure 5.19
Magnitude of current reconstructed from computer simulated data; original source: square loop, side length of 6 cm, 7 cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method
(b) $\nabla^2$B-zero method
Figure 5.20
Magnitude of current reconstructed from measured data; original source: small 'L'-shaped loop, 6.6cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method
(b) $\nabla^2 B$-zero method

Figure 5.21
Magnitude of current reconstructed from measured data; original source: large 'L'-shaped loop, 6.7cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method
(b) $\nabla^2 B$-zero method

Figure 5.22
Magnitude of current reconstructed from measured data; original source: 2 square loops, upper loop 5.1cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method
(b) $\nabla^2 B$-zero method
reconstruction when the original source was a large "L"-shaped loop 6.7 cm from the measurement plane. As with the round loops discussed above, the reconstructed currents are of different sizes, but we cannot discern the true shape of the loop. The inability to reconstruct the correct shape of the loops is due to the large distance between the original sources and the measurement plane. The computer simulations predict that loops closer to the measurement plane will have more detail about the location of the edges in the reconstructions.

*Reconstruction of Two Square Loops*

In Figure 5.22, we show the reconstructed magnitude when the original source consisted of two square loops. The loops were 1.2 cm on a side, and were separated by 1.8 cm in the z-direction and 3.3 cm in the x-direction. Unfortunately, the reconstructed current appears to be only one loop. We can't distinguish the two loops because they were too small, too close together and too far from measurement plane.

*Reconstructions of Sources with Nonzero Z-Component*

All of the original sources for the computer simulations had zero z-component, and the reconstructed currents also had zero z-component. The current reconstructed with the B-zero method should always have a zero z-component because the method was developed with the assumption that \( J_z \) was zero. The \( \nabla^2 B \)-zero method, on the other hand, was not restricted to reconstructing planar currents and should reconstruct all three components of the current. In order to investigate the effect of a current source with a nonzero z-component, we measured the magnetic field due to a horizontal round loop (5.5 cm
Figure 5.23
Magnitude of current reconstructed from measured data; original source: horizontal round loop, diameter = 5.5cm, 6.6cm from measurement plane. Same reconstruction as in Figure 5.17. 1 pixel = 1 cm.
(a) B-zero method  (b) $\nabla B$-zero method

Figure 5.24
Magnitude of current reconstructed from measured data; original source: round loop tilted 40° from horizontal, diameter=5.5cm, 4.1cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method  (b) $\nabla B$-zero method

Figure 5.25
Magnitude of current reconstructed from measured data; original source: vertical round loop, diameter = 5.5cm, 5.1cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method  (b) $\nabla B$-zero method
diameter), and then tilted the loop so that the current was partially in the z-direction. The magnitude of the reconstructed current for the horizontal loop is shown in Figure 5.17, and is repeated here in Figure 5.23 for easier comparison to the following two reconstructions. Figure 5.24 shows the reconstruction when the original loop was tilted approximately 40° from horizontal. The intensity distribution of the reconstruction shifts as the original loop was tilted. Figure 5.25 shows the reconstruction when the original loop was vertical. Again, the part of the original loop that was closest to the measurement plane is the strongest part of the reconstructed current. Unfortunately, the z-component of the current reconstructed with the $\nabla^2 B$-zero method was zero even when the original source had a nonzero z-component.

For the loops that were not parallel to the measurement plane the assumptions in the B-zero method (the x- and y-components of the magnetic field are zero in the forbidden region and the z-component of the current is zero) are violated. In spite of this violation, the B-zero method was able to reconstruct the lateral location of the current.

Reconstructions of Current Dipoles

The final reconstructions we shall present are of a current dipole immersed in saline solution in a glass head phantom. We have shown that our Fourier-transform reconstruction methods can reconstruct widely distributed sources, but we also need to be able to reconstruct highly localized currents. In Figure 5.26, we show the reconstruction of a single current dipole located near the side of the glass head. The head was positioned so that the measurement plane was parallel to the side of the head. The current reconstructed with the $\nabla^2 B$-zero method is highly laterally localized, but the current reconstructed with the B-zero method is not as sharp. Figure 5.27 shows the magnitude of the reconstructed current when
Figure 5.26
Magnitude of current reconstructed from measured data; original source: current dipole off to side in glass head phantom, 2.3cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method  
(b) $\nabla^2$B-zero method

Figure 5.27
Magnitude of current reconstructed from measured data; original source: current dipole in middle of glass head phantom, 4.3cm from measurement plane. Differentiation operator approximated by asymmetric finite differences. 1 pixel = 1 cm.
(a) B-zero method  
(b) $\nabla^2$B-zero method
the current dipole was located in the center of the glass head and the measurement plane was on the top of the head. Again, the current reconstructed with the $\nabla^2 B$-zero method is highly localized, but the current reconstructed with the $B$-zero method has many artifacts. Note that the orientation of the reconstructed current in Figure 5.27 is different than reconstruction of current dipole with measurement plane on side of head. Both reconstructions of the current dipole are very similar to the reconstruction of the vertical loop because the measured magnetic fields are similar.

**Summary of Reconstructions From Measured Data**

The reconstructions from measured data are very similar to the reconstructions from simulated magnetic fields. The similarities indicate that neglecting the effects of the pick-up coil size, the gradiometer, and noise in the computer simulations was an acceptable approximation. Just as with the computer simulations, the size of the reconstructed loop is related to the size of the original source, the lateral localization is excellent, there is no depth discrimination, and sources closer to the measurement plane have sharper reconstructions than sources further away. The results of the $B$-zero method and the $\nabla^2 B$-zero method were mixed for sources that were not planar loops parallel to the measurement plane. We believe that the inability to discriminate between the round, square, and "L"-shaped loops is due to the large distances between the original sources and the measurement plane. One negative result was that the $\nabla^2 B$-zero method was unable to correctly reconstruct the $z$-component of the current (the $z$-component of the current is assumed to be zero for the $B$-zero method). On the positive side, both reconstruction methods were able to reconstruct current dipoles.

We did not present reconstructions from the other sets of measured data discussed in Chapter 4 for several reasons. Most importantly, the other reconstructions do not provide
additional insight into the behavior of the reconstruction algorithms; most of the other magnetic fields were measured too far from the source to produce good reconstructions. We also did not present reconstructions using the other approximations to the differentiation operator because the differences are small between reconstructions from computer simulated data and from measured data.
Biomagnetism provides a noninvasive technique for investigating the spatial distribution of electric current in the body. The knowledge gained from studying the body’s magnetic fields will aid in our understanding of the function of the organs. Although the magnetic fields due to bioelectric currents are very weak, approximately six orders of magnitude weaker than the earth’s field, equipment sensitive enough to measure biomagnetic fields is available. The object of biomagnetic imaging is to estimate the strength and spatial distribution of bioelectric currents from measurements of the magnetic field. Unfortunately, the inverse problem is ill-posed; an infinite number of currents could cause the same measured magnetic field. To overcome the ill-posed nature of the inverse problem, we restrict the allowed object class.

The goal of this dissertation was to develop a Fourier-transform technique for estimating current sources from magnetic field measurements. The general technique, a noniterative method, consists of forming a set of linear equations from the Fourier-transformed Maxwell equations. The set of equations is sampled according to the Whittaker-Shannon sampling theorem and solved by matrix methods. Assumptions about the current distribution and measurement geometry are included in the reconstruction technique by means of the sampling theorem. Two variations of the technique, presented in Chapter 3, are extensions of a Fourier-transform method developed by Dallas. The first technique (called the B-zero method) assumes that the x- and y-components of the magnetic field in the forbidden region are zero, and that the z-component of the current density is zero. The second technique (the $\nabla^2 B$-zero method) assumes that $\nabla^2 B_x$ and $\nabla^2 B_y$ are zero in the
forbidden region, and is not restricted to reconstructing currents with zero z-component.

We tested the Fourier-transform reconstruction methods on both computer-simulated magnetic fields and measured data. In general, we found that the $\nabla^3 B$-zero method was superior to the $B$-zero method because it generates fewer artifacts. The effects on the reconstructions of the form of the differentiation operator, the size of the original source, and the location of the original source were similar for both the $B$-zero method and the $\nabla^3 B$-zero method. Approximating the differentiation operator in Fourier space as a sampled version of the continuous operator is unacceptable because it produces reconstructed currents that have significant imaginary parts, while currents must be purely real. The symmetric and asymmetric finite differences both produce acceptable reconstructed currents. The reconstructions of smaller sources are proportionally smaller than large sources for the same distance from the measurement plane. Sources closer to the measurement plane have sharper reconstructions than sources further away. The lateral resolution of both the $B$-zero method and the $\nabla^3 B$-zero method is excellent. The inability of either reconstruction method to provide any information about the depth of the current is disappointing.

When the original source consisted of two current loops of different sizes and locations, both reconstruction methods reconstructed currents that appeared to be two loops. If the original loops were in different planes, then the loop that was closer to the measurement plane was reconstructed stronger and sharper than the loop that was further away. If the two loops are too close together and/or too far from the measurement plane, then the reconstructed current looks like a single loop.

The results of the $B$-zero method and the $\nabla^3 B$-zero method were mixed for sources that were not planar square loops parallel to the measurement plane. We believe that the inability to discriminate between the round, square, and "L"-shaped loops is due to the large distances between the original sources and the measurement plane. One negative result was
that the $\nabla^2 B$-zero method was unable to correctly reconstruct the z-component of the current (the z-component of the current is assumed to be zero for the B-zero method). On the positive side, both reconstruction methods were able to reconstruct current dipoles.

There are several aspects of the reconstruction algorithms that need to be studied further. The algorithms should provide information about the depth of the current source and be able to reconstruct the z-component of the current density. In addition, the spatial resolution needs to be improved so that the reconstructions are sharper.
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